Explicit evaluations of log–log integrals

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ORIGINAL RESEARCH PAPER

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Explicit evaluations of log–log integrals

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Abstract
By investigating a family of log-log type integrals on the unit domain and on the positive half line, we produce a substantial number of new identities, representing the value of the integral with the aid of Euler sums. A new family of Euler sum identities will also be given, thereby extending the current knowledge.

Keywords Euler sums · Definite integral · Log functions · Clausen functions · Riemann zeta functions

Mathematics Subject Classification Primary 11M06 · 11M35 · 26B15 · Secondary 33B15 · 42A70 · 65B10

1 Introduction preliminaries and notation

There exists a vast literature in which an exceptionally large number of integral formula have been developed, refer to [2, 4–6, 10, 12, 13, 23, 26] There are also many research papers dealing with specific evaluations and analysis of representations of log-log type integrals, refer to [1, 3, 7, 9, 14, 25, 28] . In this paper the intention is to extend the knowledge and application of these log-log type integrals by examining families of the type,

$$I(a,m,p,q) = \int_x \frac{x^a \ \log^p x \ \log(1-x^q)}{(1-x)^{m+1}} dx, \tag{1.1}$$

where $(p,q) \in \mathbb{N},\ m \in \mathbb{N}_0, a \in \mathbb{R}$ in both the unit interval $x \in (0,1)$ and in the positive half line $x \geq 0$. We shall represent the resulting integral (1.1) in closed form

Communicated by S Ponnusamy.

Dedicated to Christian Kara on a brilliant beginning.

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in terms of special functions including the Riemann zeta function, Clausen functions and harmonic numbers. For example we obtain some difficult to evaluate results of the form

$$\int_{x=0}^{1} \frac{\log^3 x \ \log(1-x^3)}{(1-x)^3} dx = \frac{895}{108}\zeta(4) - 6\left(\frac{3}{4}+\frac{2}{3}\ln 3\right)\zeta(2) - \frac{2}{27\sqrt{3}}\pi^3 + \left(\frac{91}{6}-\frac{13}{3}\ln 3\right)\zeta(3) - \left(\pi + 3\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\right)\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) - \sqrt{3}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right), \tag{1.2}$$

and on the positive half line

$$\int_{x=0}^{\infty} \frac{\log^3 x \ \log(1-x^3)}{(1-x)^3} dx = \frac{812}{27}\zeta(4) - (9+8\ln 3)\zeta(2) - \frac{4}{27\sqrt{3}}\pi^3 + \frac{1}{3}\zeta(3) - 2\pi\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) + 3i\pi(\zeta(3)-\zeta(2)), \tag{1.3}$$

where $\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)$ and $\mathrm{Cl}_4\left(\frac{2\pi}{3}\right)$ are the Clausen functions. Some special cases of (1.1) have been considered in [8] and [14] but only for the case $m = 0$. The $m \neq 0$ case adds some unexpected complexities in the analysis and the need to develop some new Euler sum identities of the form

$$W_{1,p}^{++}(\beta,\alpha) = \sum_{n\geq 1} \frac{n^\beta \ H_n}{(n+\alpha)^p} \tag{1.4}$$

for real parameters (α, β, p) such that $W_{1,p}^{++}(\beta,\alpha)$ converges and the harmonic numbers

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi(n+1)$$

where γ is the familiar Euler Mascheroni constant and for complex values of z, $z \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$, $\psi(z)$ is the digamma (or psi) function defined by

$$\psi(z) : -\frac{d}{dz}(\log\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z)$ is the Gamma function, see [24] . In this paper we denote $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{Z}$ and $\mathbb{N}$ as the sets of complex numbers, real numbers, positive real numbers integers and positive integers respectively and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \mathbb{Z}\backslash\mathbb{N}_0$. In (1.2) we have the appearance of the Clausen function where the generalized Clausen functions are defined for $z \in \mathbb{C}$ with $\Re(z) > 1$ as,

$$S_z(x) = \sum_{k\geq 1} \frac{\sin(kx)}{k^z}$$
$$C_z(x) = \sum_{k\geq 1} \frac{\cos(kx)}{k^z}$$

and may be extended to all the complex plane through analytic continuation. When z is replaced by a non negative integer n, the standard Clausen functions are defined by the Fourier series

$$\mathrm{Cl}_n(x) = \begin{cases} \sum_{k\geq 1} \frac{\sin(kx)}{k^n}, & \text{for } n \text{ even} \\ \sum_{k\geq 1} \frac{\cos(kx)}{k^n}, & \text{for } n \text{ odd} \end{cases}.$$

The polylogarithm function $\mathrm{Li}_p(z)$ is, for $|z| \leq 1$

$$\mathrm{Li}_p(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^p} \tag{1.5}$$

and in terms of the Polylogarithm,

$$\mathrm{Cl}_n(\theta) = \begin{cases} \frac{i}{2}\left(\mathrm{Li}_n\left(e^{-i\theta}\right) - \mathrm{Li}_n\left(e^{i\theta}\right)\right), & \text{for even } n \\ \frac{1}{2}\left(\mathrm{Li}_n\left(e^{-i\theta}\right) + \mathrm{Li}_n\left(e^{i\theta}\right)\right), & \text{for odd } n. \end{cases}$$

The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}} \tag{1.6}$$

has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}$$

and can be connected to the Clausen function in the following way. The Clausen function of rational argument and even integer order $\mathrm{Cl}_{2m}\left(\frac{\pi p}{q}\right) = \sum_{k\geq 1} \frac{\sin\left(\frac{k\pi p}{q}\right)}{k^{2m}}$, then, for p an odd integer

$$(2m-1)!(2q)^{2m}\mathrm{Cl}_{2m}\left(\frac{\pi p}{q}\right) = \sum_{j=1}^{q} \sin\left(\frac{j\pi p}{q}\right)\left(\psi^{(2m-1)}\left(\frac{j}{2q}\right) - \psi^{(2m-1)}\left(\frac{j+q}{2q}\right)\right) \tag{1.7}$$

and for p an even integer

$$(2m-1)!(q)^{2m}\mathrm{Cl}_{2m}\left(\frac{\pi p}{q}\right)=\sum_{j=1}^{q}\sin\left(\frac{j\pi p}{q}\right)\psi^{(2m-1)}(\frac{j}{q}). \tag{1.8}$$

From (1.5) we can define the polylogarithm for all non positive integer order, where $\mathrm{Li}_1(z)=-\log(1-z),\mathrm{Li}_0(z)=\frac{z}{1-z}$and, for $n\in\mathbb{N}_0$

$$\mathrm{Li}_{-n}(z)=\left(z\frac{\partial}{\partial z}\right)^n\frac{z}{1-z}=\sum_{j=0}^{n}j!S(n+1,j+1)\left(\frac{z}{1-z}\right)^{j+1}, \tag{1.9}$$

where $S(n+1,j+1)$ are Stirling numbers of the second kind. Equivalently we can write

$$\mathrm{Li}_{-n}(z)=\frac{1}{(1-z)^{n+1}}\sum_{j=0}^{n-1}\left\langle\begin{matrix}n\\j\end{matrix}\right\rangle z^{n-j}$$

where the Eulerian numbers

$$\left\langle\begin{matrix}n\\j\end{matrix}\right\rangle=\sum_{r=0}^{j+1}(-1)^r\binom{n+1}{r}(j+1-r)^n$$

Some other pertinent papers dealing with Euler sums are [15–17] and the excellent books [23, 26]. We expect that integrals of the type (1.1) may be represented by Euler sums and therefore in terms of special functions such as the Riemann zeta function, the Clausen function and the polygamma functions. A search of the current literature has found some examples for the representation of the log-log integrals in terms of Euler sums, see [27]. The following papers [11, 18–21] and [22] also examined some integrals in terms of Euler sums. The two examples (1.2) and (1.3) will be considered in detail, moreover, these integrals are not amenable to a computer mathematical package.

2 Analysis of integrals

Consider the following.

Theorem 1 *Let $a\in\mathbb{R}\geq-2$ and $(m,p,q)\in\mathbb{N}_0$, the following integral,*

$$I(a,m,p,q)=\int_0^1\frac{x^a\ln^p(x)\ln(1-x^q)}{(1-x)^{m+1}}dx \tag{2.1}$$

$$= \int_0^1 \ln^p(x) \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \frac{q^{\mu+1-\lambda} s(\mu, \mu+1-\lambda)\ \mathrm{Li}_{\lambda-\mu}(x^q)}{\mu! x^{\mu-a}(1-x)^{m+1-\mu}} dx$$
$$+ (-1)^{p+1} p! \sum_{n\geq 1} H_n \sum_{j=1}^{q} \frac{\binom{qn+j-1}{m}}{(qn+j+a-m)^{p+1}} \tag{2.2}$$

where H_n are harmonic numbers, $s(\mu, \mu+1-\lambda)$ are signed Stirling numbers of the first kind and $\mathrm{Li}_{\lambda-\mu}(x^q)$ are the polylogarithms at zero or negative integer.

Proof From the definition of the polyogarithm $\mathrm{Li}_1(x^q) = -\log(1-x^q)$, a Taylor series expansion, for $x \in (0, 1)$ produces

$$\frac{\mathrm{Li}_1(x^q)}{1-x} = -\sum_{n\geq 1} H_n \sum_{j=1}^{q} x^{qn+j-1}\ . \tag{2.3}$$

The m^{th} derivative

$$\frac{d^m}{dx^m}\left(\frac{\mathrm{Li}_1(x^q)}{1-x}\right) = \left(\frac{\mathrm{Li}_1(x^q)}{1-x}\right)^{(m)} = \sum_{\mu=0}^{m} \binom{m}{\mu} (\mathrm{Li}_1(x^q))^{(\mu)} \left(\frac{1}{(1-x)}\right)^{(m-\mu)}$$
$$= \sum_{\mu=0}^{m} \binom{m}{\mu} (\mathrm{Li}_1(x^q))^{(\mu)} \frac{(m-\mu)!}{(1-x)^{m+1-\mu}}$$
$$= m! \frac{\mathrm{Li}_1(x^q)}{(1-x)^{m+1}} + m! \sum_{\mu=1}^{m} \frac{(\mathrm{Li}_1(x^q))^{(\mu)}}{\mu!(1-x)^{m+1-\mu}}$$
$$= m! \frac{\mathrm{Li}_1(x^q)}{(1-x)^{m+1}} + m! \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \frac{q^{\mu+1-\lambda} s(\mu, \mu+1-\lambda) \mathrm{Li}_{\lambda-\mu}(x^q)}{\mu! x^{\mu}(1-x)^{m+1-\mu}}.$$

Therefore

$$x^a \log^p(x) \left(\frac{\mathrm{Li}_1(x^q)}{1-x}\right)^{(m)} = m! \frac{x^a \log^p(x) \mathrm{Li}_1(x^q)}{(1-x)^{m+1}}$$
$$+ m! \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \frac{q^{\mu+1-\lambda} x^a \log^p(x) s(\mu, \mu+1-\lambda) \mathrm{Li}_{\lambda-\mu}(x^q)}{\mu! x^{\mu}(1-x)^{m+1-\mu}}$$

and integrating both sides for $x \in (0, 1)$, we have

$$\int_0^1 x^a \log^p(x) \left(\frac{\mathrm{Li}_1(x^q)}{1-x}\right)^{(m)} dx = m! \int_0^1 \frac{x^a \log^p(x)\mathrm{Li}_1(x^q)}{(1-x)^{m+1}} dx \\ + m! \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \frac{q^{\mu+1-\lambda} s(\mu, \mu+1-\lambda)}{\mu!} \int_0^1 \frac{x^a \log^p(x)\mathrm{Li}_{\lambda-\mu}(x^q)}{x^\mu (1-x)^{m+1-\mu}} dx. \tag{2.4}$$

From (2.3),

$$\left(\frac{\mathrm{Li}_1(x^q)}{1-x}\right)^{(m)} = m! \sum_{n\geq 1} H_n \sum_{j=1}^{q} \binom{qn+j-1}{m} x^{qn+j-1-m},$$

consequently

$$\frac{1}{m!} \int_0^1 x^a \log^p(x) \left(\frac{\mathrm{Li}_1(x^q)}{1-x}\right)^{(m)} dx \\ = \sum_{n\geq 1} H_n \sum_{j=1}^{q} \binom{qn+j-1}{m} \int_0^1 x^{qn+j+a-1-m} \log^p x dx \\ = \sum_{n\geq 1} H_n \sum_{j=1}^{q} \binom{qn+j-1}{m} \frac{(-1)^p p!}{(qn+j+a-m)^{p+1}}. \tag{2.5}$$

Here we require the exclusion of all terms of the form $qn+j+a-m=0$ and for convergence requirements we put $p \geq m+1$. From the equivalent expressions (2.4) and (2.5)

$$m! \int_0^1 \frac{x^a \log^p(x)\mathrm{Li}_1(x^q)}{(1-x)^{m+1}} dx \\ + m! \sum_{\mu=1}^{m} \sum_{\lambda=1}^{\mu} \frac{q^{\mu+1-\lambda} s(\mu, \mu+1-\lambda)}{\mu!} \int_0^1 \frac{x^a \log^p(x)\mathrm{Li}_{\lambda-\mu}(x^q)}{x^\mu (1-x)^{m+1-\mu}} dx \\ = m! \sum_{n\geq 1} H_n \sum_{j=1}^{q} \binom{qn+j-1}{m} \frac{(-1)^p p!}{(qn+j+a-m)^{p+1}}$$

and theorem (1) is proved. □

The next corollary deals with the degenerate case of $m=0$ for the representation of the integral (2.1).

Corollary 1 *Let* $a \in \mathbb{R} \geq -2, m=0$ *and* $(p,q) \in \mathbb{N}_0$, *the following integral,*

$$I(a,0,p,q)=\int_0^1 \frac{x^a \ln^p(x)\ln(1-x^q)}{1-x}dx \tag{2.6}$$

$$\begin{aligned}
&=(-1)^{p+1}p!\sum_{n\geq 1}H_n\sum_{j=1}^{q}\frac{1}{(qn+j+a)^{p+1}}\\
&=\frac{(-1)^{p+1}p!}{q^{p+1}}\sum_{j=1}^{q}\sum_{n\geq 1}\frac{H_n}{\left(n+\frac{j+a}{q}\right)^{p+1}}\\
&=\frac{(-1)^{p+1}p!}{q^{p+1}}\sum_{j=1}^{q}S_{1,p+1}^{++}\left(0,\frac{j+a}{q}\right)
\end{aligned} \tag{2.7}$$

for $p\geq 1$. We utilize the following notation

$$S_{p,q}^{++}(\alpha,\beta)=\sum_{n\geq 1}\frac{H_n^{(p)}(\alpha)}{(n+\beta)^q}$$

where

$$\zeta(p,\alpha)=H_n^{(p)}(\alpha)=\sum_{j=1}^{n}\frac{1}{(n+\alpha)^p},n\in\mathbb{N},p\in\mathbb{C},\alpha\in\mathbb{C}\backslash\{-1,-2,-3,\ldots.\}$$

This integral (2.6) is completely determined by (2.7), in terms of special functions, since it depends on the known value of the Euler sums

$$S_{1,p+1}^{++}\left(0,\frac{j+a}{q}\right)=\sum_{n\geq 1}\frac{H_n(0)}{\left(n+\frac{j+a}{q}\right)^{p+1}}$$

The following required Euler sum identity appears in [18]. Let α be a real number $\alpha\neq -1,-2,-1,\ldots$, and assume that $p\in\mathbb{N}\backslash\{1\}$. Then

$$\sum_{n\geq 1}\frac{H_n}{(n+\alpha)^p}=S_{1,p}^{++}(0,\alpha)$$
$$=\frac{(-1)^p}{(p-1)!}\begin{pmatrix}(\psi(\alpha)+\gamma)\psi^{(p-1)}(\alpha)\\ -\frac{1}{2}\psi^{(p)}(\alpha)+\sum_{j=1}^{p-2}\binom{p-2}{j}\psi^{(j)}(\alpha)\psi^{(p-j-1)}(\alpha)\end{pmatrix} \tag{2.8}$$

where γ is the Euler Mascheroni constant.

Proof From the Taylor series (2.3)

$$
\begin{aligned}
I(a,0,p,q) &= \int_0^1 \frac{x^a \ln^p(x)\ln(1-x^q)}{1-x}dx \\
&= -\sum_{n\geq 1} H_n \sum_{j=1}^{q} \int_0^1 x^{qn+j+a-1}\ln^p(x)dx \\
&=(-1)^{p+1}p!\sum_{n\geq 1} H_n \sum_{j=1}^{q} \frac{1}{(qn+j+a)^{p+1}} \\
&=\frac{(-1)^{p+1}p!}{q^{p+1}}\sum_{j=1}^{q} S_{1,p+1}^{++}\left(0,\frac{j+a}{q}\right)
\end{aligned}
$$

and proof is complete. □

Two examples are now given.

Example 1 For $a=0, m=0, p=3$ and $q=3$

$$
\begin{aligned}
I(0,0,3,3) &= \int_0^1 \frac{\ln^3(x)\ln(1-x^3)}{1-x}dx \\
&=\frac{3!}{3^4}\sum_{j=1}^{3} S_{1,4}^{++}\left(0,\frac{j}{3}\right) \\
&=\frac{2}{27}\left(S_{1,4}^{++}\left(0,\frac{1}{3}\right)+S_{1,4}^{++}\left(0,\frac{2}{3}\right)+S_{1,4}^{++}(0,1)\right).
\end{aligned}
$$

Now we can apply (2.8) then simplify with the aid of (1.8) and (1.7) and eventually are led to the identity

$$
\begin{aligned}
\int_0^1 \frac{\ln^3(x)\ln(1-x^3)}{1-x}dx =& 36\zeta(5)-\frac{35}{27}\pi^2\zeta(3)-\frac{8}{81}\pi^4\ln 3 \\
&-\frac{8}{27}\pi^3\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)-2\pi\mathrm{Cl}_4\left(\frac{2\pi}{3}\right),
\end{aligned}
$$

here $\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)$ and $\mathrm{Cl}_4\left(\frac{2\pi}{3}\right)$ are the Clausen functions.

In this next example we shall require Euler sum identities of the form (1.4), the next proposition will be essential.

Proposition 1 *Let* α *be a real number* $\alpha \neq -1,-2,-1,\ldots,$ *and assume that* $p \in \mathbb{N}\backslash\{1\}$. *Then*

$$
\begin{aligned}
W^{++}_{1,p+1}(1,\alpha) &= \sum_{n\geq 1}\frac{n\,H_n}{(n+\alpha)^{p+1}} = S^{++}_{1,p}(0,\alpha)\\
&+\frac{\alpha}{p}\left((\gamma+\psi(\alpha))\psi^{(p)}(\alpha)+\psi^{(p-1)}(\alpha)\psi^{(\prime)}(\alpha)-\frac{1}{2}\psi^{(p+1)}(\alpha)\right)\\
&+\frac{\alpha}{p}\sum_{j=1}^{p-2}\binom{p-2}{j}\left(\psi^{(j+1)}(\alpha)\psi^{(p-j-1)}(\alpha)+\psi^{(j)}(\alpha)\psi^{(p-j)}(\alpha)\right),
\end{aligned}
\tag{2.9}
$$

the sum $S^{++}_{1,p}(0,\alpha)$ is given by (2.8) and $\psi^{(p)}(\alpha)$ are the polygamma functions.

Proof In (2.8) we put $\alpha=\frac{1}{y}, y\neq 0$, differentiate with respect to y and then rename y as α so that (2.9) follows. Similar analysis allows us to evaluate $W^{++}_{1,p+1}(\beta,\alpha)$ for $\beta\in\mathbb{N}$. □

Example 2 For this example consider the case $a=0, m=2, p=3$ and $q=3$

$$
\begin{aligned}
I(0,2,3,3) &= \int_0^1 \frac{\ln^3(x)\ln(1-x^3)}{(1-x)^3}dx = 6\sum_{n\geq 1}H_n\sum_{j=1}^{3}\frac{\binom{3n+j-1}{2}}{(3n+j-2)^4}\\
&+\int_0^1 \ln^3(x)\sum_{\mu=1}^{2}\sum_{\lambda=1}^{\mu}\frac{3^{\mu+1-\lambda}s(\mu,\mu+1-\lambda)\,\mathrm{Li}_{\lambda-\mu}(x^3)}{\mu!x^{\mu}(1-x)^{3-\mu}}dx\\
&=\int_0^1 \ln^3(x)\left(\frac{3\mathrm{Li}_0(x^3)}{x(1-x)^2}-\frac{3\mathrm{Li}_0(x^3)}{2x^2(1-x)}+\frac{9\mathrm{Li}_{-1}(x^3)}{2x^2(1-x)}\right)dx\\
&+6\sum_{n\geq 1}H_n\left(\frac{\binom{3n}{2}}{(3n-1)^4}+\frac{\binom{3n+1}{2}}{(3n)^4}+\frac{\binom{3n+2}{2}}{(3n+1)^4}\right).
\end{aligned}
$$

We have put in the values of the Stirling numbers of the first kind,

$$
s(1,1)=1, s(2,2)=1, s(2,1)=-1 \tag{2.10}
$$

and evaluating

$$
Li_{-1}\left(x^3\right)=\frac{x^3}{(1-x^3)^2},\; Li_0\left(x^3\right)=\frac{x^3}{1-x^3} \tag{2.11}
$$

we obtain

$$I(0,2,3,3)=6\sum_{n\geq 1}H_n\left(\frac{3n}{2(3n-1)^3}+\frac{3n+1}{2(3n)^3}+\frac{3n+2}{2(3n+1)^3}\right)$$
$$+\int_0^1 \ln^3(x)\left(\frac{9x}{2(1-x)(1-x^3)^2}-\frac{3x}{2(1-x)(1-x^3)}+\frac{3x^2}{(1-x^3)(1-x)^2}\right)dx. \tag{2.12}$$

The Euler sums in (2.12) can be evaluated from (2.8) and (2.9). The integrals in (2.12) can be evaluated by standard techniques and we evaluate the first integral to give a hint of the method used. Consider

$$\begin{aligned}\int_0^1 \frac{x\ln^3(x)}{(1-x)(1-x^3)^2}dx &= \sum_{n\geq 1}\frac{n(n+1)}{2}\sum_{j=1}^{3}\int_0^1 x^{3n-3+j}\ln^3(x)dx\\ &=-3\sum_{n\geq 1}n(n+1)\sum_{j=1}^{3}\frac{1}{(3n-3+j+1)^4}\\ &=-3\sum_{n\geq 1}\left(\frac{n(n+1)}{(3n-1)^4}+\frac{n(n+1)}{(3n)^4}+\frac{n(n+1)}{(3n+1)^4}\right),\end{aligned}\tag{2.13}$$

upon expansion and simplification using (1.8) and (1.7) we obtain

$$\int_0^1 \frac{x\ln^3(x)}{(1-x)(1-x^3)^2}dx=\frac{2}{\sqrt{3}}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right)-\frac{1}{3}\zeta(2)-\zeta(3)+\frac{8\pi^3}{243\sqrt{3}}-\frac{80}{243}\zeta(4).$$

Finally we obtain the identity (1.2),

$$\int_0^1 \frac{\ln^3(x)\ln(1-x^3)}{(1-x)^3}dx=\frac{895}{108}\zeta(4)-6\left(\frac{3}{4}+\frac{2}{3}\ln 3\right)\zeta(2)-\frac{2}{27\sqrt{3}}\pi^3$$
$$+\left(\frac{91}{6}-\frac{13}{3}\ln 3\right)\zeta(3)-\left(\pi+3\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\right)\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)-\sqrt{3}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right),$$

3 The positive half line $x \geq 0$

In this section we analyze the integral

$$J(a,m,p,q)=\int_{x\geq 0}\frac{x^a\ln^p(x)\ln(1-x^q)}{(1-x)^{m+1}}dx$$

and show that, without loss of generality the integral

$$J(0,m,p,q) = J(m,p,q) = \int\limits_{x\geq 0} \frac{\ln^p(x)\ln(1-x^q)}{(1-x)^{m+1}}dx, \tag{3.1}$$

in certain cases of the parameters (m,p,q) depends on the representation of the integral (2.2).

Theorem 2 *Let* $(m,p,q)\in\mathbb{N}$ *and put* $p\geq m+1$, *then*

$$J(m,p,q) = \int\limits_{x\geq 0} \frac{\ln^p(x)\ln(1-x^q)}{(1-x)^{m+1}}dx = I(0,m,p,q) + (-1)^{m+p+1}I(m-1,m,p,q)$$
$$+ (-1)^{m+1}i\pi p!\sum_{n\geq 0}\frac{(n+1)_m}{m!(n+m)^{p+1}} - q(-1)^m(p+1)!\sum_{n\geq 0}\frac{(n+1)_m}{m!(n+m)^{p+2}} \tag{3.2}$$

where $I(0,m,p,q)$ and $I(m-1,m,p,q)$ are given by (2.2), $(n+1)_m$ is the Pochhammer symbol and $i=\sqrt{-1}$.

Proof Let us put

$$\Lambda(m,p,q;x) = \frac{\ln^p(x)\ln(1-x^q)}{(1-x)^{m+1}}$$

and notice that $\lim\limits_{x->0^+}\Lambda(m,p,q;x)=0$, $\lim\limits_{x->1^-}\Lambda(m,p,q;x)=0$, we can write

$$J(m,p,q) = \int\limits_{x\geq 0} \frac{\ln^p(x)\ln(1-x^q)}{(1-x)^{m+1}}dx = \int\limits_{x=0}^{1}\Lambda(m,p,q;x)dx + \int\limits_{x=1}^{\infty}\Lambda(m,p,q;x)dx,$$

upon making the transformation $xt=1$ (and renaming t as x) in the third integral we obtain

$$\begin{aligned} J(m,p,q) &= \int\limits_{x=0}^{1}\Lambda(m,p,q;x)dx + \int\limits_{x=1}^{0}\Lambda\left(m,p,q;\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)dx \\ &= \int\limits_{x=0}^{1}\Lambda(m,p,q;x)dx + (-1)^{m+p+1}\int\limits_{x=0}^{1}\frac{x^{m-1}\ln^p(x)\ln\left(\frac{x^q-1}{x^q}\right)}{(1-x)^{m+1}}dx \\ &= I(0,m,p,q) + (-1)^{m+p+1}I(m-1,m,p,q) \\ &+ i(-1)^{m+p+1}\pi\int\limits_{x=0}^{1}\frac{x^{m-1}\ln^p(x)}{(1-x)^{m+1}}dx - q(-1)^{m+p+1}\int\limits_{x=0}^{1}\frac{x^{m-1}\ln^{p+1}(x)}{(1-x)^{m+1}}dx. \end{aligned}$$

Expanding the last two integrands in a Taylor series form and then integrating in the interval $x\in(0,1)$, we have

$$J(m,p,q) = I(0,m,p,q) + (-1)^{m+p+1} I(m-1,m,p,q) + (-1)^{m+1} i\pi p! \sum_{n\geq 0} \frac{\binom{n+m}{m}}{(n+m)^{p+1}} - q(-1)^m (p+1)! \sum_{n\geq 0} \frac{\binom{n+m}{m}}{(n+m)^{p+2}}.$$

The Pochhammer symbol $(\lambda)_\omega$ for $(\lambda, \omega) \in \mathbb{C}$ can be defined in terms of the Gamma function $\Gamma(\cdot)$, by

$$(\lambda)_\omega = \frac{\Gamma(\lambda+\omega)}{\Gamma(\lambda)} = \begin{cases} 1, & \omega = 0, \lambda \in \mathbb{C}\backslash\{0\} \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \omega = n \in \mathbb{N}, \lambda \in \mathbb{C} \end{cases}$$

and conventionally understood that $(0)_0 = 1$. Finally we obtain

$$J(m,p,q) = I(0,m,p,q) + (-1)^{m+p+1} I(m-1,m,p,q) + (-1)^{m+1} i\pi p! \sum_{n\geq 0} \frac{(n+1)_m}{m!(n+m)^{p+1}} - q(-1)^m (p+1)! \sum_{n\geq 0} \frac{(n+1)_m}{m!(n+m)^{p+2}}$$

and (3.2) is proved. □

The particular case of $m = 1$ has a complete representation in terms of polygamma functions and the details are developed in the following corollary.

Corollary 2 *Let* $m = 1, (p,q) \in \mathbb{N}$ *and put* $p \geq 2$, *then*

$$J(1,p,q) = \int_{x\geq 0} \frac{\ln^p(x)\ln(1-x^q)}{(1-x)^2} dx$$
$$= i\pi p!\zeta(p) + q(p+1)!\zeta(p+1) + \frac{(-1)^{p+1} p!}{q^p} \sum_{j=1}^{q} S^{++}_{1,p}\left(0, \frac{j-1}{q}\right) \tag{3.3}$$
$$+ \frac{(1+(-1)^p)p!}{q^{p+1}} \sum_{j=1}^{q} \left((j-1)\psi^{(p)}\left(\frac{j-1}{q}+1\right) + pq\psi^{(p-1)}\left(\frac{j-1}{q}+1\right)\right)$$

where $\psi^{(p)}\left(\frac{j-1}{q}\right)$ are the polygamma functions (1.6) and $S^{++}_{1,p}\left(0, \frac{j-1}{q}\right)$ are the Sofo–Cvijović–Euler sums (2.8).

Proof Let $m = 1, (p,q) \in \mathbb{N}$,

$$J(1,p,q) = \int_{x\geq 0} \frac{\ln^p(x)\ln(1-x^q)}{(1-x)^2} dx = (1+(-1)^p) I(0,1,p,q) + i\pi p! \sum_{n\geq 0} \frac{1}{(n+1)^p} + q(p+1)! \sum_{n\geq 0} \frac{1}{(n+1)^{p+1}} \tag{3.4}$$
$$= (1+(-1)^p) I(0,1,p,q) + i\pi p!\zeta(p) + q(p+1)!\zeta(p+1).$$

The required evaluation of $I(0,1,p,q)$ can be done by applying (2.2) so that

$$I(0,1,p,q) = \int_{x=0}^{1} \frac{\ln^p(x)\ln(1-x^q)}{(1-x)^2}dx$$
$$= qs(1,1)\int_{x=0}^{1} \frac{\ln^p(x)\mathrm{Li}_0(x^q)}{x(1-x)}dx + (-1)^{p+1}p!\sum_{n\geq 1} H_n \sum_{j=1}^{q} \frac{1}{(qn+j-1)^p},$$

since the Stirling numbers of the first kind $s(1,1)=1$ and $\mathrm{Li}_0(x^q)=\frac{x^q}{1-x^q}$, then

$$I(0,1,p,q) = q\int_{x=0}^{1} \frac{x^{q-1}\ln^p(x)}{(1-x)(1-x^q)}dx + \frac{(-1)^{p+1}p!}{q^p}\sum_{j=1}^{q} S_{1,p}^{++}\left(0,\frac{j-1}{q}\right).$$

A similar evaluation of that used for the integrals in (2.12) yields the result

$$I(0,1,p,q) = \frac{(-1)^{p+1}p!}{q^p}\sum_{j=1}^{q}\sum_{n\geq 1}\frac{n}{\left(n+\frac{j-1}{q}\right)^{p+1}} + \frac{(-1)^{p+1}p!}{q^p}\sum_{j=1}^{q} S_{1,p}^{++}\left(0,\frac{j-1}{q}\right)$$
$$= \frac{1}{q^{p+1}}\sum_{j=1}^{q}\left((j-1)\psi^{(p)}\left(\frac{j-1}{q}+1\right) + pq\psi^{(p-1)}\left(\frac{j-1}{q}+1\right)\right)$$
$$+ \frac{(-1)^{p+1}p!}{q^p}\sum_{j=1}^{q} S_{1,p}^{++}\left(0,\frac{j-1}{q}\right).$$

Substituting into (3.4) we have

$$J(1,p,q) = i\pi p!\zeta(p) + q(p+1)!\zeta(p+1) + \frac{(-1)^{p+1}p!}{q^p}\sum_{j=1}^{q} S_{1,p}^{++}\left(0,\frac{j-1}{q}\right)$$
$$+ \frac{(1+(-1)^p)p!}{q^{p+1}}\sum_{j=1}^{q}\left((j-1)\psi^{(p)}\left(\frac{j-1}{q}+1\right) + pq\psi^{(p-1)}\left(\frac{j-1}{q}+1\right)\right)$$

which is the result (3.3). It is evident from (3.4) that in the case of odd integer p, say $2p-1$ then

$$J(1,2p-1,q) = \int_{x=0}^{\infty} \frac{\ln^{2p-1}(x)\ln(1-x^q)}{(1-x)^2}dx$$
$$= i\pi(2p-1)!\zeta(2p-1) + q(2p)!\zeta(2p).$$

□

Some examples follow.

Example 3 For $m=2, p=3, q=3$ and from (3.2)

$$J(2,3,3) = \int\limits_{x\geq 0} \frac{\ln^3(x)\ln(1-x^3)}{(1-x)^3} dx = I(0,2,3,3) + I(1,2,3,3) \\ -3i\pi \sum_{n\geq 0} \frac{(n+1)}{(n+2)^3} - 36 \sum_{n\geq 0} \frac{(n+1)}{(n+2)^4} \tag{3.5}$$

The evaluation of $I(0,2,3,3)$ is detailed in (2.12), therefore we now indicate the evaluation of $I(1,2,3,3)$, from theorem 1

$$I(1,2,3,3) = \int\limits_{x=0}^{1} \frac{x\ln^3(x)\ln(1-x^3)}{(1-x)^3} dx$$

$$= \int\limits_0^1 \ln^3(x) \sum_{\mu=1}^{2} \sum_{\lambda=1}^{\mu} \frac{3^{\mu+1-\lambda} s(\mu,\mu+1-\lambda)\ \mathrm{Li}_{\lambda-\mu}(x^3)}{\mu! x^{\mu-a}(1-x)^{m+1-\mu}} dx$$

$$+6\sum_{n\geq 1} H_n \sum_{j=1}^{q} \frac{\binom{3n+j-1}{2}}{(3n+j+1-2)^4}.$$

Using the Stirling numbers (2.10), the values of $\mathrm{Li}_0(x^3)$ and $\mathrm{Li}_{-1}(x^3)$ given in (1.9) we have

$$I(1,2,3,3) = 3\sum_{n\geq 1} H_n \left(\frac{3n-1}{(3n)^3} + \frac{3n}{(3n+1)^3} + \frac{3n+1}{(3n+2)^3} \right)$$

$$+ \int\limits_0^1 \ln^3(x) \left(\frac{3x^3}{(1-x)^2(1-x^3)} + \frac{9x^2}{2(1-x)(1-x^3)^2} - \frac{3x^2}{2(1-x)(1-x^3)} \right) dx.$$

The Euler sums can be evaluated with the aid of (2.8) and (2.9), the integrals here can be evaluated by the same method as described by (2.13). After some calculations we find,

$$I(1,2,3,3) = \frac{71}{6}\zeta(3) - \frac{1}{486}\left(\psi^{(3)}\left(\frac{1}{3}\right) - 4\psi^{(3)}\left(\frac{2}{3}\right) \right) - \frac{40\pi^4}{729} - \frac{2\pi^3}{27\sqrt{3}} - \frac{3\pi^2}{4}$$
$$+ \frac{4\gamma\pi^2}{9} - \frac{\pi^4}{8} + \frac{28}{3}\zeta(3) + \frac{13}{3}\zeta(3)\ln 3 + \frac{28}{3}\zeta(3) + \frac{13}{3}\zeta(3)\ln 3$$
$$+ \left(\psi^{(1)}\left(\frac{1}{3}\right) - \frac{1}{2}\ln 3 - \frac{\pi}{6\sqrt{3}} - \frac{\gamma}{3} \right) \psi^{(1)}\left(\frac{1}{3}\right) + \left(\psi^{(1)}\left(\frac{2}{3}\right) \right.$$
$$\left. - \frac{1}{2}\ln 3 + \frac{\pi}{6\sqrt{3}} - \frac{\gamma}{3} \right) \psi^{(1)}\left(\frac{2}{3}\right).$$

If we know simplify by employing (1.8) and (1.7), and specifically using

$$\psi^{(1)}\left(\frac{1}{3}\right) = \frac{2\pi^2}{3} + 3\sqrt{3}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right), \psi^{(1)}\left(\frac{2}{3}\right) = \frac{2\pi^2}{3} - 3\sqrt{3}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right),$$
$$\psi^{(3)}\left(\frac{1}{3}\right) = \frac{8\pi^4}{3} + 162\sqrt{3}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right), \psi^{(3)}\left(\frac{2}{3}\right) = \frac{8\pi^4}{3} - 162\sqrt{3}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right),$$

we get

$$I(1,2,3,3) = \sqrt{3}\mathrm{Cl}_4\left(\frac{2\pi}{3}\right) + \left(3\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) - \pi\right)\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) - \frac{2\pi^3}{27\sqrt{3}}$$
$$+ \frac{127}{6}\zeta(3) + \frac{13}{3}\zeta(3)\ln 3 - \frac{307\pi^4}{1944} - \frac{2\pi^2}{3}\ln 3.$$

Replacing these values in (3.5) we have

$$J(2,3,3) = \int_{x\geq 0} \frac{\ln^3(x)\ln(1-x^3)}{(1-x)^3}dx = I(0,2,3,3) + I(1,2,3,3)$$
$$+ 3i\pi(\zeta(3) - \zeta(2)) - 36(\zeta(3) - \zeta(4))$$

and simplifying we arrive at (1.3).

Example 4 For $m = 2, p = 4, q = 4$ and employing similar calculations as above, provides the identity

$$J(2,4,4) = \int_{x\geq 0} \frac{\ln^4(x)\ln(1-x^4)}{(1-x)^3}dx = 12\pi\beta(4) + \frac{3}{2}\pi^3 G - \frac{12\pi^4}{5} + \frac{5\pi^5}{64}$$
$$+ \frac{25\pi^4}{32}\ln 2 + \frac{179\pi^2}{32}\zeta(3) - \frac{3}{64}\zeta(5) + i\left(\frac{2\pi^5}{15} - 12\pi\zeta(3)\right)$$

here, the Catalan constant

$$G = \beta(2) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2} \approx 0.91597$$

is a special case of the Dirichlet beta function

$$\beta(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^z}, \text{ for } \mathrm{Re}(z) > 0$$
$$= \frac{1}{(-2)^{2z}(z-1)!}\left(\psi^{(z-1)}(\frac{1}{4}) - \psi^{(z-1)}(\frac{3}{4})\right),$$

with functional equation

$$\beta(1-z)=\left(\frac{2}{\pi}\right)^{z}\sin\left(\frac{\pi z}{2}\right)\Gamma(z)\beta(z)$$

extending the Dirichlet Beta function to the left hand side of the complex plane $\mathrm{Re}(z)\leq 0$.

Concluding Remarks We have studied of a family of integrals having log – log and polynomial functions in terms of Euler sums, which themselves incorporate special functions such as Beta functions, Clausen functions and Zeta functions. For higher values of the parameters m, p and q our results are new in the literature. We have evaluated four specific examples which are not amenable to a mathematical computer package. Further work examining integral families containing Polylogarithmic functions will be presented in the near future.

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