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# TWISTING MODULI FOR $GL(2)$

BENJAMIN BEDERT, GEORGE COOPER\*, THOMAS OLIVER, AND PENGCHENG ZHANG

**Abstract.** We prove various converse theorems for automorphic forms on  $\Gamma_0(N)$ , each assuming fewer twisted functional equations than the last. We show that no twisting at all is needed for holomorphic modular forms in the case that  $N \in \{18, 20, 24\}$  - these integers are the smallest multiples of 4 or 9 not covered by earlier work of Conrey–Farmer. This development is a consequence of finding generating sets for  $\Gamma_0(N)$  such that each generator can be written as a product of special matrices. As for real-analytic Maass forms of even (resp. odd) weight we prove the analogous statement for  $1 \leq N \leq 12$  and  $N \in \{16, 18\}$  (resp.  $1 \leq N \leq 12$ ,  $14 \leq N \leq 18$  and  $N \in \{20, 23, 24\}$ ).

## 1. INTRODUCTION

The use of twisted functional equations in the characterisation of automorphic representations dates back to Weil’s archetypal converse theorem for holomorphic modular forms [Wei67]. It is a problem of long-standing interest to limit the number of required twists [PS71], [Raz77], [Li81], [CF95], [DPZ02]. In [PS71] it is shown that, for any fixed prime  $p$ , it suffices to assume the analytic properties of twists by primitive characters modulo  $p^r$  for all  $r \geq 0$ . On the other hand, the main result of [DPZ02] states that there exists a prime  $q$  such that the analytic properties of twists by primitive characters modulo  $q$  are sufficient to capture the holomorphic modular forms on  $\Gamma_0(N)$ . In [BBB<sup>+</sup>18] it is demonstrated that there is a density 1 subset of the primes from which we can choose  $q$ . Still in the holomorphic case it was shown in [CF95] and [CFOS07] that, for  $1 \leq N \leq 17$  and  $N = 23$ , no twisting at all is required. This result generalises earlier work of Hecke which applies to the cases  $1 \leq N \leq 4$  [Hec36].

In Theorem 2.1 we prove a converse theorem for Maass forms on  $\Gamma_0(N)$  assuming twisted functional equations for primitive Dirichlet characters with the same moduli as Weil [Wei67]. For context, note that a converse theorem for Maass forms of small level was established by Maass [Maa49] and the converse theorem of Jacquet–Langlands [JL70] applies to Maass forms on  $GL_2(\mathbb{A}_F)$ . Nevertheless, generalising Weil’s classical approach to real-analytic forms had been regarded as a difficult problem, see for example [GM04, Section 3.4], with two different approaches having appeared only recently in [NO20, Section 3] and [MSSU]. We note that the results of [MSSU] apply to half-integer weight, though they make the additional assumption of analytic properties for twists by imprimitive characters. Theorem 2.1 uses an extension of the method used in [NO20, Section 3] to Maass forms of arbitrary integer weight  $k$ . In particular, we develop further the method of “two circles” implemented in [NO20]. By the theory of weight raising and lowering operators, it is enough to work with the cases

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$k = 0, 1$ . As [NO20, Theorem 3.1] solves the problem when  $k = 0$ , it remains to solve the case  $k = 1$ . In Theorem 2.2 we show that it is in fact sufficient to assume only twists by primitive Dirichlet characters modulo a single prime  $q$  constrained by congruence conditions as in [DPZ02] (equation (2.11)). In Theorem 2.3, we show that no twisting at all is required for  $1 \leq N \leq 12$  and  $N \in \{16, 18\}$  when  $k = 0$  and  $1 \leq N \leq 12$ ,  $14 \leq N \leq 18$  and  $N \in \{20, 23, 24\}$  when  $k = 1$ . In Theorem 2.4, we deduce from the same argument that the main result of [CF95] can be extended to the cases  $N \in \{18, 20, 24\}$ . The numbers 20 and 24 (resp. 18) are the smallest multiples of  $2^2$  (resp.  $3^2$ ) not covered by Conrey–Farmer. Our proof works by writing generating sets for  $\Gamma_0(N)$  such that each generator can be written as a product of special matrices. Whilst the matrices in  $\Gamma_0(N)$  have integer coefficients, we sometimes allow matrices with non-integer coefficients amongst the factors. In each of the Theorems established here, it would be interesting to allow for the various  $L$ -functions to have poles as in [BK13], [NO20, Theorem 1.1], and [HO, Theorem 1.2].

**Notation.** By  $\mathcal{H}$  we denote the upper half-plane, the smooth functions on which are denoted by  $C^\infty(\mathcal{H})$ . The weight- $k$  Laplace-Beltrami operator on  $C^\infty(\mathcal{H})$  is

$$(1.1) \quad \Delta_k = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + ik y \frac{\partial}{\partial x}.$$

A weight- $k$  Maass form on  $\Gamma_0(N)$  is a real-analytic eigenfunction of  $\Delta_k$  with polynomial growth at the cusps of  $\Gamma_0(N)$  satisfying a weight- $k$  automorphic transformation law with respect to  $\Gamma_0(N)$ . Given two complex numbers  $\kappa$  and  $\mu$ , the Whittaker function  $W_{\kappa, \mu} : \mathbb{R}_{>0} \rightarrow \mathbb{C}$  is annihilated by the differential operator:

$$(1.2) \quad \frac{d^2}{dy^2} + \left( -\frac{1}{4} + \frac{\kappa}{y} + \frac{1/4 - \mu^2}{y^2} \right),$$

and satisfies the following asymptotic formula as  $y \rightarrow \infty$ :

$$(1.3) \quad W_{\kappa, \mu}(y) \sim e^{-\frac{y}{2}} y^\kappa.$$

We use the following variants of the gamma function:

$$(1.4) \quad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) = (2\pi)^{-s} \Gamma(s).$$

If  $\psi$  is a Dirichlet character mod  $q$ , then the associated Gauss sum is:

$$(1.5) \quad \tau(\psi) = \sum_{a \bmod q} \psi(a) \exp(2\pi i a/q).$$

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## 2. THEOREMS

In the Theorems below, a function  $F(s)$  on  $\mathbb{C}$  is called *EBV* if it is entire and bounded on every vertical strip.

**Theorem 2.1.** Let  $\varepsilon \in \{\pm 1\}$  and  $\nu \in \mathbb{C} \setminus \{0\}$ . Say  $N \in \mathbb{Z}_{>0}$  and let  $\chi$  be a Dirichlet character mod  $N$ . Assume that  $(a_n)_{n=-\infty}^{\infty}, (b_n)_{n=-\infty}^{\infty}$  are complex sequences such that  $a_n, b_n = O(n^\sigma)$  for some  $\sigma \in \mathbb{R}_{>0}$  and, for  $n > 0$ ,  $a_{-n} = \varepsilon \nu a_n$  and  $b_{-n} = \varepsilon \nu b_n$ . Given a Dirichlet character  $\psi$  modulo  $p$  (either 1 or a prime number), define:

$$(2.1) \quad L_f(s, \psi) = \sum_{n=1}^{\infty} \psi(n) a_n n^{-s}, \quad L_g(s, \psi) = \sum_{n=1}^{\infty} \bar{\psi}(n) b_n n^{-s}, \quad \operatorname{Re}(s) > \sigma + 1.$$

and

$$(2.2) \quad \begin{aligned} \Lambda_f(s, \psi) &= \Gamma_{\mathbb{R}} \left( s + \frac{1 + \psi(-1)\varepsilon}{2} + \nu \right) \Gamma_{\mathbb{R}} \left( s + \frac{1 - \psi(-1)\varepsilon}{2} - \nu \right) L_f(s, \psi), \\ \Lambda_g(s, \psi) &= \Gamma_{\mathbb{R}} \left( s + \frac{1 + \psi(-1)\varepsilon}{2} + \nu \right) \Gamma_{\mathbb{R}} \left( s + \frac{1 - \psi(-1)\varepsilon}{2} - \nu \right) L_g(s, \psi). \end{aligned}$$

If  $\psi = \mathbf{1}$  is the trivial Dirichlet character we omit it from the notation. For a finite set  $\mathcal{S}$  of primes including 2 and those dividing  $N$ , define  $\mathcal{P}$  to be the complement of  $\mathcal{S}$  in the set of all primes. Assume the following:

- (1) The functions  $\Lambda_f(s)$  and  $\Lambda_g(s)$  continue to holomorphic functions on  $\mathbb{C} - \{\varepsilon\nu, 1 - \varepsilon\nu\}$  with at most simple poles in the set  $\{\varepsilon\nu, 1 - \varepsilon\nu\}$ , are uniformly bounded on every vertical strip outside of a small neighbourhood around each pole and satisfy the functional equation

$$(2.3) \quad \Lambda_f(s) = N^{\frac{1}{2}-s} \Lambda_g(1-s).$$

- (2) For all primitive Dirichlet characters  $\psi$  of conductor  $q \in \mathcal{P}$ , continue to entire functions which are EBV and satisfy the functional equation

$$(2.4) \quad \Lambda_f(s, \psi) = \psi(N) \chi(q) \frac{\tau(\psi)}{\tau(\bar{\psi})} (q^2 N)^{\frac{1}{2}-s} \Lambda_g(1-s, \bar{\psi}).$$

Define  $f_0, g_0 : \mathbb{R}_{>0} \rightarrow \mathbb{C}$  by

$$(2.5) \quad f_0(y) = -\operatorname{Res}_{s=\varepsilon\nu} \Lambda_f(s) y^{\frac{1}{2}-\varepsilon\nu}, \quad g_0(y) = \operatorname{Res}_{s=\varepsilon\nu} \Lambda_g(s) y^{\frac{1}{2}-\varepsilon\nu},$$

define  $\tilde{f}, \tilde{g} : \mathcal{H} \rightarrow \mathbb{C}$  by

$$(2.6) \quad \begin{aligned} \tilde{f}(x+iy) &= \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\pi n}} \left( W_{\frac{1}{2}, \nu}(4\pi n y) e(nx) + \varepsilon \nu W_{-\frac{1}{2}, \nu}(4\pi n y) e(-nx) \right), \\ \tilde{g}(x+iy) &= \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{\pi n}} \left( W_{\frac{1}{2}, \nu}(4\pi n y) e(nx) + \varepsilon \nu W_{-\frac{1}{2}, \nu}(4\pi n y) e(-nx) \right), \end{aligned}$$

and define  $f, g : \mathcal{H} \rightarrow \mathbb{C}$  by

$$(2.7) \quad f(x+iy) = f_0(y) + \tilde{f}(x+iy), \quad g(x+iy) = g_0(y) + \tilde{g}(x+iy).$$

Then  $f$  and  $g$  are weight-1 Maass forms on  $\Gamma_0(N)$  satisfying

$$(2.8) \quad f(z) = \frac{iz}{|z|} g\left(-\frac{1}{Nz}\right).$$

In the above Theorem, we do not assume that the  $L$ -functions have an Euler product. Including this assumption, we may restrict the set of twisting moduli.

**Theorem 2.2.** Let  $\varepsilon, \nu, N, \chi, (a_n)_{n=-\infty}^{\infty}$ , and  $(b_n)_{n=-\infty}^{\infty}$  be as in Theorem 2.1 and let  $k \in \{0, 1\}$ . Given a Dirichlet character  $\psi \pmod p$  (either 1 or a prime) define  $L_f(s, \psi), L_g(s, \bar{\psi})$  as in equation (2.1) and define

$$(2.9) \quad \begin{aligned} \Lambda_f(s, \psi) &= \Gamma_{\mathbb{R}} \left( s + \frac{1 - (-1)^k \psi(-1)\varepsilon}{2} + \nu \right) \Gamma_{\mathbb{R}} \left( s + \frac{1 - \psi(-1)\varepsilon}{2} - \nu \right) L_f(s, \psi), \\ \Lambda_g(s, \psi) &= \Gamma_{\mathbb{R}} \left( s + \frac{1 - (-1)^k \psi(-1)\varepsilon}{2} + \nu \right) \Gamma_{\mathbb{R}} \left( s + \frac{1 - \psi(-1)\varepsilon}{2} - \nu \right) L_g(s, \psi). \end{aligned}$$

Assume the following:

- (1) For  $\operatorname{Re}(s) > \sigma + 1$ , the functions  $L_f(s)$  and  $L_g(s)$  have an Euler product expansion of the form

$$(2.10) \quad \begin{aligned} L_f(s) &= \prod_{p \nmid N} (1 - a_p p^{-s} + p^{-2s})^{-1} \prod_{\substack{p|N \\ p^2 \nmid N}} (1 - p^{-s})^{-1}, \\ L_g(s) &= \prod_{p \nmid N} (1 - b_p p^{-s} + p^{-2s})^{-1} \prod_{\substack{p|N \\ p^2 \nmid N}} (1 - p^{-s})^{-1}. \end{aligned}$$

- (2) The functions  $\Lambda_f(s)$  and  $\Lambda_g(s)$  admit extensions to all of  $\mathbb{C}$  which are EBV and satisfy the functional equation (2.3).

- (3) Given a generating set  $\left\{ \begin{pmatrix} A_j & B_j \\ C_j N & D_j \end{pmatrix} \in \Gamma_1(N) : j = 1, \dots, h \right\}$  for  $\Gamma_1(N)$ , there is a prime  $q$  satisfying

$$(2.11) \quad q \equiv A_j \pmod{q|C_j|}, \quad j = 1, \dots, h,$$

such that for all primitive Dirichlet characters  $\psi \pmod q$  the functions  $\Lambda_f(s, \psi)$  and  $\Lambda_g(s, \bar{\psi})$  admit extensions which are EBV and satisfy the functional equation:

$$(2.12) \quad \Lambda_f(s, \psi) = \varepsilon^{1-k} \psi(N) \chi(q) \frac{\tau(\psi)}{\tau(\bar{\psi})} (q^2 N)^{\frac{1}{2}-s} \Lambda_g(1-s, \bar{\psi}).$$

Then  $\Lambda_f(s, \psi)$  and  $\Lambda_g(s, \bar{\psi})$  satisfy functional equation (2.4) for all primitive Dirichlet characters  $\psi \pmod p \in \mathcal{P}$ .

An upper bound for the smallest prime  $q$  satisfying equation (2.11) is given in [DPZ02, Section 3].

**Theorem 2.3.** Let  $a_n, \sigma$  be as in Theorem 2.1, and let  $\nu \in \mathbb{C}$ . Given  $k = 0$  (resp.  $k = 1$ ), let  $N \in \mathbb{Z}_{>0}$  satisfy  $1 \leq N \leq 12$  and  $N \in \{16, 18\}$  (resp.  $1 \leq N \leq 12, 14 \leq N \leq 18$  and  $N \in \{20, 23, 24\}$ ), and let  $\chi$  be a Dirichlet character mod  $N$ . For  $\operatorname{Re}(s) > \sigma + 1$ , define  $L_f(s)$  as in equation (2.1) and suppose that  $L_f(s)$  has an Euler product as in equation (2.10). Define  $\Lambda_f(s)$  as in equation (2.9), and suppose that  $\Lambda_f(s)$  admits an extension to  $\mathbb{C}$  which is EBV and satisfies the functional equation

$$(2.13) \quad \Lambda_f(s) = N^{\frac{1}{2}-s} \Lambda_f(1-s).$$

Then  $f$  is a weight- $k$  Maass form for  $\Gamma_0(N)$ .

**Theorem 2.4.** *Let  $a_n, \sigma$  be as in Theorem 2.1, let  $k \in \mathbb{Z}_{>0}$  be an even integer, and  $N \in \{18, 20, 24\}$ . For  $\operatorname{Re}(s) > \sigma + 1$ , define  $L_f(s)$  by equation (2.1). Suppose that  $L_f(s)$  has an Euler product expansion*

$$(2.14) \quad L_f(s) = \prod_{p \mid N} (1 - a_p p^{-s} + p^{k+1-2s})^{-1} \prod_{\substack{p \mid N \\ p^2 \nmid N}} (1 - p^{k/2-1-s})^{-1}.$$

Define

$$(2.15) \quad \Lambda_f(s) = \Gamma_{\mathbb{C}}(s) L_f(s),$$

and suppose that  $\Lambda_f(s)$  admits an extensions to  $\mathbb{C}$  which is EBV and satisfies the functional equation:

$$(2.16) \quad \Lambda_f(s) = (-1)^{k/2} N^{\frac{k}{2}-s} \Lambda_f(k-s)$$

If,

$$(2.17) \quad f(z) = \sum_{n=1}^{\infty} a_n \exp(2\pi i n z),$$

then  $f$  is a weight- $k$  modular form on  $\Gamma_0(N)$ .

The conclusion of Theorem 2.4 is weaker than that in [CF95], in which  $f$  is shown to be cuspidal. In both Theorems 2.3 and 2.4, cuspidality of  $f$  would be a consequence of the convergence of  $L_f(s)$  in  $\operatorname{Re}(s) > 1 - \delta$  for any  $\delta > 0$ . For a given  $N$ , this assumption may or may not be necessary (cf. [CF95, Section 5]).

### 3. PROOFS

**3.1. Weil's Lemma.** Let  $\operatorname{GL}_2^+(\mathbb{R})$  denote the subgroup of  $\operatorname{GL}_2(\mathbb{R})$  consisting of matrices with positive determinant, which contains  $\operatorname{SL}_2(\mathbb{R})$  as a subgroup. For  $k \in \mathbb{Z}_{\geq 0}$ , the weight- $k$  action of  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2^+(\mathbb{R})$  on  $f : \mathcal{H} \rightarrow \mathbb{C}$  is given by

$$(3.1) \quad (f|_k \gamma)(z) = \exp(-ik \arg(cz + d)) f\left(\frac{az + b}{cz + d}\right).$$

For  $r \in \mathbb{Q}$  write

$$(3.2) \quad P_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in \operatorname{SL}_2(\mathbb{R}).$$

For  $N \in \mathbb{Z}_{>0}$  set

$$(3.3) \quad H_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \operatorname{GL}_2^+(\mathbb{R}).$$

Equation (2.3) is equivalent to

$$(3.4) \quad g = i^k f|_k H_N.$$

Extending (3.1) by linearity, we equip  $C^\infty(\mathcal{H})$  with a right  $\mathbb{C}[\operatorname{GL}_2^+(\mathbb{R})]$ -module structure. For  $f \in C^\infty(\mathcal{H})$ , we introduce the following right-ideal of  $\mathbb{C}[\operatorname{GL}_2^+(\mathbb{R})]$ :

$$(3.5) \quad \Omega_f = \{W \in \mathbb{C}[\operatorname{GL}_2^+(\mathbb{R})] : f|_k W = 0\}.$$

For elements  $\gamma_1, \gamma_2$  of  $\mathbb{C}[\mathrm{GL}_2^+(\mathbb{R})]$  we will write  $\gamma_1 \equiv \gamma_2$  to mean  $\gamma_1 - \gamma_2 \in \Omega_f$ . If  $f$  has an expansion as in equation (2.7), then

$$(3.6) \quad 1 - P_r \equiv 0, \quad r \in \mathbb{Z}.$$

For a prime number  $p$ , recall the Hecke operator

$$(3.7) \quad T_p = \frac{1}{\sqrt{p}} \left( \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{p-1} \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} \right) \in \mathbb{C}[\mathrm{GL}_2^+(\mathbb{R})],$$

and the operator

$$(3.8) \quad U_p = \sum_{a=0}^{p-1} \begin{pmatrix} p & a \\ 0 & p \end{pmatrix} \in \mathbb{C}[\mathrm{GL}_2^+(\mathbb{R})].$$

If  $L_f(s)$  has an Euler product as in equation (2.10), then, for  $p \nmid N$ ,

$$(3.9) \quad a_p - T_p \equiv 0.$$

On the other hand, if  $p|N$ , then

$$(3.10) \quad U_p \equiv \begin{cases} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, & p^2 \nmid N, \\ 0, & p^2 | N. \end{cases}$$

We say a matrix  $E \in \mathrm{SL}_2(\mathbb{R})$  is elliptic if it has a unique fixed point in  $\mathcal{H}$ .

**Lemma 3.1.** *Let  $k = 0$  and let  $c : \mathcal{H} \rightarrow \mathbb{C}$  be a continuous function. Say there exist  $E_1, E_2 \in \mathrm{SL}_2(\mathbb{R})$  such that  $E_1$  is an elliptic matrix of infinite order with fixed point  $a \in \mathcal{H}$ ,  $a$  is not a fixed point of  $E_2$ , and  $c|_0 E_1 = c|_0 E_2 = c$ , then  $c$  is constant on  $\mathcal{H}$ .*

*Proof.* The matrix  $E_3 = E_2 E_1 E_2^{-1}$  is elliptic, because  $\mathrm{tr}(E_2 E_1 E_2^{-1}) = \mathrm{tr}(E_1)$  and  $|\mathrm{tr}(E_1)| < 2$  since  $E_1$  is elliptic. Since  $E_1$  has infinite order, so does  $E_3$ . Moreover,  $E_3$  fixes  $E_2 a \neq a$  by assumption. Therefore  $c$  is preserved by the two infinite order elliptic matrices with distinct fixed points and hence the result follows from [NO20, Theorem 3.10].  $\square$

**Lemma 3.2.** *Let  $k = 1$  and let  $c : \mathcal{H} \rightarrow \mathbb{C}$  be a continuous function. If there exists an elliptic matrix  $E$  of infinite order such that  $c|_1 E = c$ , then  $c = 0$ .*

*Proof.* Let  $a \in \mathcal{H}$  denote the unique fixed point of  $E$ . The elliptic matrix  $E$  is diagonalised by the Cayley transform:

$$K = \frac{1}{\sqrt{a - \bar{a}}} \begin{pmatrix} 1 & -a \\ 1 & -\bar{a} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}),$$

which takes  $\mathcal{H}$  to the open unit disc  $\mathcal{D}$  and maps the point  $a$  to 0. Because  $E$  is of infinite order, we have

$$K E K^{-1} = \begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix}, \quad \theta \in \mathbb{R} \setminus \mathbb{Q}.$$

Although  $K$  is not a matrix in  $\mathrm{GL}_2^+(\mathbb{R})$ , we define the function  $\tilde{c} = (c|_1 K) : \mathcal{D} \rightarrow \mathbb{C}$  by equation (3.1) with  $k = 1$  and  $\gamma = K$ . For all  $z \in \mathcal{D}$  and  $m \in \mathbb{Z}$  we have

$$\tilde{c}(z) = e^{\pi i m \theta} \tilde{c}(e^{2\pi i m \theta} z).$$

Denote by  $S^1$  the unit circle. Fix  $z \in \mathcal{D}$  and consider the continuous function

$$\begin{aligned} S^1 &\rightarrow \mathbb{C} \\ \omega &\mapsto \omega \tilde{c}(\omega^2 z). \end{aligned}$$

Given two topological spaces  $X$  and  $Y$ , if  $Y$  is Hausdorff and  $c_1, c_2 : X \rightarrow Y$  are continuous functions which agree on some dense subset of  $X$  then  $c_1 = c_2$ . In particular, because  $\theta$  is irrational, we deduce the following for all  $\omega \in S^1$ :

$$\tilde{c}(z) = \omega \tilde{c}(\omega^2 z).$$

Taking  $\omega = -1$  gives  $\tilde{c}(z) = -\tilde{c}(z)$  for all  $z \in \mathcal{D}$ , so  $\tilde{c} = 0$  and therefore  $c = 0$ .  $\square$

**3.2. Proof of Theorem 2.1.** In this Section we follow certain conventions from [BCK19]. Let  $\sigma, \varepsilon, \nu, a_n, b_n$  be as in Theorem 2.1. The Fourier expansions of  $f$  and  $g$  immediately imply that they are eigenfunctions of  $\Delta_1$ . Given  $\mathcal{P}$  be as in Theorem 2.1, let  $q \in \mathcal{P}$  and  $a \in \mathbb{Z}$  satisfy  $(a, q) = 1$  or  $a = 0$ . Denote by  $\alpha$  the quotient  $a/q$ . For  $m \in \mathbb{Z}_{\geq 0}$ , define:

$$(3.11) \quad L_f(s, \alpha, \cos^{(m)}) = \sum_{n=1}^{\infty} \cos^{(m)}(2\pi n \alpha) a_n n^{-s},$$

where  $\cos^{(m)}$  the  $m$ -th derivative of the cosine function. We have:

$$(3.12) \quad L_f(s, \alpha, \cos^{(m)}) = \frac{i^m}{q-1} \sum_{\substack{\psi \bmod q \\ \psi(-1)=(-1)^m}} \tau(\bar{\psi}) \psi(a) L_f(s, \psi) + \begin{cases} (-1)^{\frac{k}{2}} \left( L_f(s) - \frac{q}{q-1} L_f(s, \mathbf{1}_q) \right), & m \text{ even,} \\ 0, & m \text{ odd,} \end{cases}$$

where the summation on the right-hand side is over the set of all primitive Dirichlet characters of conductor  $q$ , and  $\mathbf{1}_q$  denotes the trivial character modulo  $q$ . For  $m \in \mathbb{Z}_{>0}$  we let  $\langle m \rangle$  denote  $+$  if  $m$  is even and  $-$  if  $m$  is odd, and define:

$$(3.13) \quad \Lambda_f(s, \alpha, \cos^{(m)}) = \gamma_f^{\langle m \rangle}(s) L_f(s, \alpha, \cos^{(m)}),$$

where

$$(3.14) \quad \gamma_f^{\pm}(s) = \Gamma_{\mathbb{R}}\left(s + \nu + \frac{1 \mp (-1)^k \varepsilon}{2}\right) \Gamma_{\mathbb{R}}\left(s - \nu + \frac{1 \mp \varepsilon}{2}\right).$$

Equation (3.12) implies:

$$(3.15) \quad \Lambda_f(s, \alpha, \cos^{(m)}) = \frac{i^m}{q-1} \sum_{\substack{\psi \bmod q \\ \psi(-1)=(-1)^k}} \tau(\bar{\psi}) \psi(a) \Lambda_f(s, \psi) + \begin{cases} (-1)^{\frac{m}{2}} \left( \Lambda_f(s) - \frac{q}{q-1} \Lambda_f(s, \mathbf{1}_q) \right), & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$



Applying equation (2.4), we deduce:

$$(3.16) \quad \Lambda_f(s, \alpha, \cos^{(m)}) = \frac{(-i)^m \chi(q) (q^2 N)^{\frac{1}{2}-s}}{q-1} \sum_{\substack{\psi \bmod q \\ \psi(-1)=(-1)^m}} \tau(\psi) \psi(Na) \Lambda_g(1-s, \bar{\psi}) \\ + \begin{cases} (-1)^{\frac{m}{2}} \left( \Lambda_f(s) - \frac{q}{q-1} \Lambda_f(s, \mathbf{1}_q) \right), & m \text{ even,} \\ 0, & m \text{ odd.} \end{cases}$$

It follows from [GR15, (6.699.2), (6.699.3), (9.234.1), (9.234.2), (9.235.2)] that, for  $\ell \in \{0, 1\}$ ,  $w \in \mathbb{R}$ , and  $\text{Re}(s) \gg 0$ :

$$(3.17) \quad \int_0^\infty f(wy + iy + \alpha) y^{s-\frac{1}{2}} \frac{dy}{y} \\ = \sum_{n=1}^\infty \frac{a_n}{n^s} \sum_{\ell \in \{0,1\}} (-i)^\ell \cos^{(\ell)}(2\pi n\alpha) \left( \gamma_f^{(\ell)}(s) {}_2F_1 \left( \frac{s+\nu+(1+(-1)^\ell \varepsilon)/2}{2}, \frac{s-\nu+(1-(-1)^\ell \varepsilon)/2}{2} \middle| -w^2 \right) \right. \\ \left. + 2\pi i w \gamma_f^{(\ell+1)}(s+1) {}_2F_1 \left( \frac{s+\nu+(3+(-1)^{\ell+1} \varepsilon)/2}{2}, \frac{s-\nu+(3-(-1)^{\ell+1} \varepsilon)/2}{2} \middle| -w^2 \right) \right).$$

Defining, for  $w \in \mathbb{R}$  and  $\ell \in \{0, 1\}$ ,

$$(3.18) \quad H_f^\ell(s, w) = (-i)^\ell \left( {}_2F_1 \left( \frac{s+\nu+(1+(-1)^\ell \varepsilon)/2}{2}, \frac{s-\nu+(1-(-1)^\ell \varepsilon)/2}{2} \middle| -w^2 \right) \right. \\ \left. + 2\pi i w \frac{\gamma_f^{(\ell+1)}(s+1)}{\gamma_f^{(\ell)}(s)} {}_2F_1 \left( \frac{s+\nu+(3+(-1)^{\ell+1} \varepsilon)/2}{2}, \frac{s-\nu+(3-(-1)^{\ell+1} \varepsilon)/2}{2} \middle| -w^2 \right) \right),$$

we may rewrite equation (3.17) as

$$(3.19) \quad \int_0^\infty f(wy + iy + \alpha) y^{s-\frac{1}{2}} \frac{dy}{y} = \sum_{\ell \in \{0,1\}} H_f^\ell(s, w) \Lambda_f(s, \alpha, \cos^{(\ell)}).$$

Applying Mellin inversion to equation (3.19) with  $\alpha = 0$ , we obtain for  $c > 1 + |\text{Re}(\nu)|$ :

$$(3.20) \quad \tilde{f}(wy + iy) = \frac{1}{2\pi i} \int_{(c)} \Lambda_f(s) H_f^0(s, w) y^{\frac{1}{2}-s} ds,$$

where  $(c)$  is the vertical line  $\text{Re}(s) = c$ . Note that  $H_f^0(s, w)$  is an entire function of  $s$ , and so the only possible singularities of the integrand in equation (3.20) are simple poles at the points  $\varepsilon\nu$  and  $1 - \varepsilon\nu$ . As a consequence of [NO20, Lemma 4.1], the integrand in (3.20) decays to zero in vertical strips. Shifting the line of integration to  $(1 - c)$  gives by Cauchy's residue theorem:

$$(3.21) \quad \frac{1}{2\pi i} \int_{(1-c)} \Lambda_f(s) H_f^0(s, w) y^{\frac{1}{2}-s} ds + \sum_{p \in \{\varepsilon\nu, 1-\varepsilon\nu\}} \text{Res}_{s=p} \Lambda_f(s) H_f^0(s, w) y^{\frac{1}{2}-s} \\ = \frac{1}{2\pi i} \int_{(c)} \Lambda_f(1-s) H_f^0(1-s, w) y^{s-\frac{1}{2}} ds + \sum_{p \in \{\varepsilon\nu, 1-\varepsilon\nu\}} \text{Res}_{s=p} \Lambda_f(s) H_f^0(s, w) y^{\frac{1}{2}-s}.$$

Using the identities

$$(3.22) \quad H_f^0(1-s, w) = i \left( \frac{|w+i|}{w+i} \right) (1+w^2)^{s-\frac{1}{2}} H_f^0(s, -w), \quad H_f^0 = H_g^0,$$

and applying equation (2.4), we compute

$$(3.23) \quad \begin{aligned} \tilde{f}(wy+iy) - R(z) &= i \left( \frac{|w+i|}{w+i} \right) \cdot \frac{1}{2\pi i} \int_{(c)} (N(1+w^2)y)^{s-\frac{1}{2}} \Lambda_g(s) H_g^0(s, -w) ds \\ &= i \left( \frac{|w+i|}{w+i} \right) \tilde{g} \left( -\frac{w}{N(w^2+1)y} + \frac{i}{N(w^2+1)y} \right) = i \left( \frac{|w+i|}{w+i} \right) \tilde{g} \left( -\frac{1}{N(wy+iy)} \right), \end{aligned}$$

where

$$(3.24) \quad \begin{aligned} R(z) &= \sum_{p \in \{\varepsilon\nu, 1-\varepsilon\nu\}} \text{Res}_{s=p} \Lambda_f(s) H_f^0(s, w) y^{\frac{1}{2}-s} \\ &= H_f^0(\varepsilon\nu, w) \text{Res}_{s=\varepsilon\nu} \Lambda_f(s) y^{\frac{1}{2}-\varepsilon\nu} + i \left( \frac{|w+i|}{w+i} \right) (N(1+w^2)y)^{-\frac{1}{2}+\varepsilon\nu} H_g^0(\varepsilon\nu, -w) \text{Res}_{s=\varepsilon\nu} \Lambda_g(s) \\ &= H_f^0(\varepsilon\nu, w) \text{Res}_{s=\varepsilon\nu} \Lambda_f(s) \cdot \text{Im}(z)^{\frac{1}{2}-\varepsilon\nu} + i \left( \frac{|z|}{z} \right) H_g^0(\varepsilon\nu, -w) \text{Res}_{s=\varepsilon\nu} \Lambda_g(s) \cdot \text{Im} \left( -\frac{1}{Nz} \right)^{\frac{1}{2}-\varepsilon\nu}. \end{aligned}$$

Therefore

$$(3.25) \quad \begin{aligned} \tilde{f}(z) - H_f^0(\varepsilon\nu, w) \text{Res}_{s=\varepsilon\nu} \Lambda_f(s) \cdot \text{Im}(z)^{\frac{1}{2}-\varepsilon\nu} \\ = i \left( \frac{|z|}{z} \right) \cdot \left( \tilde{g} \left( -\frac{1}{Nz} \right) + H_g^0(\varepsilon\nu, -w) \text{Res}_{s=\varepsilon\nu} \Lambda_g(s) \cdot \text{Im} \left( -\frac{1}{Nz} \right)^{\frac{1}{2}-\varepsilon\nu} \right). \end{aligned}$$

Since

$${}_2F_1 \left( \frac{\varepsilon\nu+\nu+(1+\varepsilon)/2}{2}, \frac{\varepsilon\nu-\nu+(1-\varepsilon)/2}{2} \middle| -w^2 \right) = 1, \quad \frac{\gamma_f^-(\varepsilon\nu+1)}{\gamma_f^+(\varepsilon\nu)} = 0,$$

we know that  $H_f^0(\varepsilon\nu, w) = H_g^0(\varepsilon\nu, -w) = 1$  and so equation (3.25) becomes equation (2.8).

Analogously to [NO20, Section 3.3], for a primitive Dirichlet character  $\psi$  of conductor  $q \in \mathcal{P}$  we define

$$f_\psi(x+iy) = \sum_{n=1}^{\infty} \frac{\psi(n)a_n}{\sqrt{\pi n}} \left( W_{\frac{1}{2}, \nu}(4\pi ny) e(nx) + \psi(-1)\varepsilon\nu W_{-\frac{1}{2}, \nu}(4\pi ny) e(-nx) \right),$$

and similarly for  $g_\psi$ . Replacing  $a_n$  by  $\psi(n)a_n$  in equation (3.17) we know, for  $\text{Re}(s)$  sufficiently large:

$$(3.26) \quad H_f^{\frac{1-\psi(-1)}{2}}(s, w) \Lambda_f(s, w) = (-i)^{\frac{1-\psi(-1)}{2}} \int_0^\infty f_\psi(wy+iy) y^{s-\frac{1}{2}} \frac{dy}{y}.$$

For  $\ell \in \{0, 1\}$ , we have

$$(3.27) \quad H_f^\ell(1-s, w) = i \left( \frac{|w+i|}{w+i} \right) (1+w^2)^{s-\frac{1}{2}} H_f^\ell(s, -w).$$

Given  $a, b \in \mathbb{Z}$  such that if  $c \in \{a, b\}$  is non-zero then  $(c, q) = 1$ , set  $\alpha = \frac{a}{q}$  and  $\beta = \frac{b}{q}$ . Since  $\alpha, \beta \in \mathbb{R}$ , we have

$$f(z + \alpha) - f(z + \beta) = \tilde{f}(z + \alpha) - \tilde{f}(z + \beta) = \frac{1}{q-1} \sum_{\psi \bmod q} \tau(\bar{\psi})(\psi(a) - \psi(b)) f_{\psi}(z).$$

Applying Mellin inversion to equation (3.17) gives

$$f(z + \alpha) - f(z + \beta) = \sum_{\ell \in \{0,1\}} \frac{1}{2\pi i} \int_{(c)} (\Lambda_f(s, \alpha, \cos^{(\ell)}) - \Lambda_f(s, \beta, \cos^{(\ell)})) H_f^{\ell}(s, w) y^{\frac{1}{2}-s} ds.$$

By equation (3.15),  $\Lambda_f(s, \alpha, \cos^{(\ell)}) - \Lambda_f(s, \beta, \cos^{(\ell)})$  is a linear combination of functions of the form  $\Lambda_f(s, \psi)$  where  $\psi$  is a primitive Dirichlet character of conductor  $q$ , and so  $\Lambda_f(s, \alpha, \cos^{(\ell)}) - \Lambda_f(s, \beta, \cos^{(\ell)})$  is an entire function of  $s$ . Moreover each  $H_f^{\ell}(s, w)$  is an entire function. Therefore we may use Cauchy's theorem to shift the path of integration to  $\text{Re}(s) = 1 - c$  to obtain

$$\begin{aligned} (3.28) \quad & \int_{(c)} (\Lambda_f(s, \alpha, \cos^{(\ell)}) - \Lambda_f(s, \beta, \cos^{(\ell)})) H_f^{\ell}(s, w) y^{\frac{1}{2}-s} ds \\ &= \int_{(1-c)} (\Lambda_f(s, \alpha, \cos^{(\ell)}) - \Lambda_f(s, \beta, \cos^{(\ell)})) H_f^{\ell}(s, w) y^{\frac{1}{2}-s} ds \\ &= \int_{(c)} (\Lambda_f(1-s, \alpha, \cos^{(\ell)}) - \Lambda_f(1-s, \beta, \cos^{(\ell)})) H_f^{\ell}(1-s, w) y^{s-\frac{1}{2}} ds. \end{aligned}$$

From equation (3.13), we deduce

$$\begin{aligned} & \Lambda_f(1-s, \alpha, \cos^{(\ell)}) - \Lambda_f(1-s, \beta, \cos^{(\ell)}) \\ &= \frac{i^{\ell} \chi(q) (q^2 N)^{s-\frac{1}{2}}}{q-1} \sum_{\substack{\psi \bmod q \\ \psi(-1) = (-1)^{\ell}}} \psi(-N) \tau(\psi) (\psi(a) - \psi(b)) \Lambda_g(s, \bar{\psi}). \end{aligned}$$

Using this and equation (3.27) we obtain

$$\begin{aligned} & \int_{(c)} (\Lambda_f(s, \alpha, \cos^{(\ell)}) - \Lambda_f(s, \beta, \cos^{(\ell)})) H_f^{\ell}(s, w) y^{\frac{1}{2}-s} ds \\ &= \frac{i^{\ell} \chi(q)}{q-1} \cdot i \left( \frac{|w+i|}{w+i} \right) \cdot \sum_{\substack{\psi \bmod q \\ \psi(-1) = (-1)^{\ell}}} \psi(-N) \tau(\psi) (\psi(a) - \psi(b)) \\ & \quad \cdot \int_{(c)} \Lambda_g(s, \bar{\psi}) H_g^{\ell}(s, -w) \left( \frac{1}{q^2 N (1+w^2) y} \right)^{\frac{1}{2}-s} ds. \end{aligned}$$

Applying Mellin inversion to equation (3.26) gives

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(c)} (\Lambda_f(s, \alpha, \cos^{(\ell)}) - \Lambda_f(s, \beta, \cos^{(\ell)})) H_f^\ell(s, w) y^{\frac{1}{2}-s} ds \\ &= \frac{i^\ell \chi(q)}{q-1} \cdot i \left( \frac{|w+i|}{w+i} \right) \cdot \sum_{\substack{\psi \bmod q \\ \psi(-1)=(-1)^\ell}} \psi(-N) \tau(\psi) (\psi(a) - \psi(b)) \cdot (-i)^\ell g_{\bar{\psi}} \left( -\frac{1}{q^2 N z} \right), \end{aligned}$$

from which we deduce:

$$f(z + \alpha) - f(z + \beta) = \frac{\chi(q)}{q-1} \cdot i \left( \frac{|w+i|}{w+i} \right) \sum_{\psi \bmod q} \psi(-N) \tau(\psi) (\psi(a) - \psi(b)) g_{\bar{\psi}} \left( -\frac{1}{q^2 N z} \right).$$

For all  $a, b \not\equiv 0 \pmod{q}$  we conclude that:

$$\begin{aligned} & \sum_{\psi \bmod q} \psi(a) (\tau(\bar{\psi}) f_\psi - i \chi(q) \psi(-N) \tau(\psi) g_{\bar{\psi}} |_1 W_{q^2 N}) \\ &= \sum_{\psi \bmod q} \psi(b) (\tau(\bar{\psi}) f_\psi - i \chi(q) \psi(-N) \tau(\psi) g_{\bar{\psi}} |_1 W_{q^2 N}). \end{aligned}$$

The expression on the left hand side is therefore independent of  $a \not\equiv 0 \pmod{q}$ , so we have a linear combination of primitive characters producing the principal character modulo  $q$ . On the other hand, the set of Dirichlet characters mod  $q$  is linearly independent over  $\mathbb{C}$  so the coefficients of this linear combination must all vanish. Therefore, for any primitive Dirichlet character  $\psi \bmod q$ :

$$(3.29) \quad f_\psi = i \psi(-N) \chi(q) \frac{\tau(\psi)}{\tau(\bar{\psi})} g_{\bar{\psi}} |_1 W_{q^2 N}.$$

Let  $\psi$  be a primitive Dirichlet character of conductor  $q \in \mathcal{P}$ . We have

$$f_\psi = \frac{1}{\tau(\bar{\psi})} \sum_{\substack{a \bmod q \\ (a,q)=1}} \bar{\psi}(a) f |_1 P_{\frac{a}{q}}.$$

Using  $f = i g |_1 H_N$  and equation (3.29) we obtain

$$\begin{aligned} (3.30) \quad -i \chi(q) \psi(-N) \tau(\psi) g_{\bar{\psi}} &= \tau(\bar{\psi}) f_\psi |_1 W_{q^2 N} = i \sum_{\substack{a \bmod q \\ (a,q)=1}} \bar{\psi}(a) g |_1 H_N P_{\frac{a}{q}} W_{q^2 N} \\ &= i \sum_{\substack{a \bmod q \\ (a,q)=1}} \bar{\psi}(a) g |_1 \begin{pmatrix} -q^2 N & 0 \\ qaN^2 & -N \end{pmatrix} = -i \sum_{\substack{a \bmod q \\ (a,q)=1}} \bar{\psi}(a) g |_1 \begin{pmatrix} q^2 & 0 \\ -qaN & 1 \end{pmatrix} \\ &= -i \psi(-N) \sum_{\substack{a \bmod q \\ (a,q)=1}} \psi(a) g |_1 \begin{pmatrix} q & -a \\ -\tilde{a}N & \frac{Na\tilde{a}+1}{q} \end{pmatrix} P_{\frac{a}{q}}. \end{aligned}$$

Here  $\tilde{a}$  is an integer that is inverse to  $-aN \pmod{q}$ ; notice  $\overline{\psi}(a) = \psi(-N)\psi(\tilde{a})$ . Therefore for all primitive characters  $\psi$  we have

$$(3.31) \quad \chi(q) \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \psi(a)g \mid_1 P_{\frac{a}{q}} = \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \psi(a)g \mid_1 \begin{pmatrix} q & -a \\ -\tilde{a}N & \frac{Na\tilde{a}+1}{q} \end{pmatrix} P_{\frac{a}{q}}.$$

The argument from now on is very similar to [NO20, Section 3.3] and [Bum98, Section 1.5]. As the primitive Dirichlet characters of conductor  $q$  span the complex vector space

$$(3.32) \quad V = \left\{ \theta : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C} : \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \theta(a) = 0 \right\},$$

we can replace  $\psi$  in equation (3.31) with any function  $\theta \in V$ . Take  $s \in \mathcal{P} \setminus \{q\}$  and write  $qs = 1 + r\tilde{r}N$  for some integers  $r, \tilde{r}$ . Define  $\theta_1 \in V$  by

$$\theta_1(n) = \begin{cases} \pm 1, & n \equiv \pm r \pmod{q}, \\ 0 & \text{otherwise.} \end{cases}$$

Equation (3.31) then becomes

$$\chi(q) \left( g \mid_1 P_{\frac{r}{q}} - g \mid_1 P_{-\frac{r}{q}} \right) = g \mid_1 \begin{pmatrix} q & -r \\ -\tilde{r}N & s \end{pmatrix} P_{\frac{r}{q}} - g \mid_1 \begin{pmatrix} q & r \\ \tilde{r}N & s \end{pmatrix} P_{-\frac{r}{q}}.$$

We define

$$A_{\pm} = \begin{pmatrix} q & \pm r \\ \pm \tilde{r}N & s \end{pmatrix},$$

which satisfies

$$A_{\pm}^{-1} = \begin{pmatrix} s & \mp r \\ \mp \tilde{r}N & q \end{pmatrix}.$$

We have

$$(A_+ - \chi(q))P_{-\frac{r}{q}} \equiv (A_- - \chi(q))P_{\frac{r}{q}}, \quad (A_-^{-1} - \chi(s))P_{-\frac{r}{s}} \equiv (A_+^{-1} - \chi(s))P_{\frac{r}{s}},$$

and so  $(A_+ - \chi(q))(1 - M(q, s, r)) \in \Omega_1$  where

$$M(q, s, r) = A_+^{-1} P_{\frac{2r}{s}} A_- P_{\frac{2r}{q}} = \begin{pmatrix} 1 & \frac{2r}{q} \\ -\frac{2\tilde{r}N}{s} & -3 + \frac{4}{qs} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

The matrix  $M(q, s, r) \in \mathrm{SL}_2(\mathbb{R})$  is an elliptic matrix of infinite order, since its eigenvalues are not roots of unity and the absolute value of its trace is  $< 2$ . Let  $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}$  be an arbitrary element of  $\Gamma_0(N)$ . Choose distinct primes  $q, s \in \mathcal{P}$  with  $q = a - ucN$  and  $s = d - vcN$  for integers  $u$  and  $v$ . Let  $r = b - av + uvcN - ud$ , so that

$$g \mid_1 \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = g \mid_1 P_u \begin{pmatrix} q & r \\ cN & s \end{pmatrix} P_v = g \mid_1 \begin{pmatrix} q & r \\ cN & s \end{pmatrix} P_v.$$

We have  $G_1 \mid_1 M(q, s, r) = G_1$  where  $G_1 = g \mid_1 \begin{pmatrix} q & r \\ cN & s \end{pmatrix} - \chi(a)g$  (note  $\chi(q) = \chi(a)$ ). By Lemma 3.2 we must have  $G_1 = 0$ , which establishes the weight-1 modular transformation

law. To complete the proof of the proof of Theorem 2.1, we note that the correct growth of  $f(z)$  and  $g(z)$  at the cusps of  $\Gamma_0(N)$  is a consequence of the assumption that  $L_f(s)$  and  $L_g(s)$  converge in some right half-plane<sup>1</sup>.

**3.3. Proof of Theorem 2.2.** Let  $k \in \{0, 1\}$  and say  $q$  is a prime such that  $q \nmid N$ . For all primitive  $\psi \pmod q$ , we have

$$(3.33) \quad \sum_{\substack{a \pmod q \\ (a,q)=1}} \psi(a) f|_k P_{\frac{a}{q}} = \sum_{\substack{a \pmod q \\ (a,q)=1}} \psi(a) f|_k \begin{pmatrix} q & -a \\ -\tilde{a}N & \frac{Na\tilde{a}+1}{q} \end{pmatrix} P_{\frac{a}{q}}.$$

Indeed, when  $k = 0$  this follows from [NO20, equation (3.16)] and when  $k = 1$  this is equation (3.31). Though the argument in [DPZ02] is written for a group action slightly different to (3.1), analysing the proof we see that it may be modified to our situation using equations (3.6) and (3.4) and the axiomatic properties of group actions.

**3.4. Proof of Theorem 2.3.** The Fourier expansion of  $f$  immediately implies it is an eigenfunction of  $\Delta_k$ . Below we will establish the modular transformation laws with respect to  $\Gamma_0(N)$ . Given that, the correct growth property of  $f(z)$  at the cusps of  $\Gamma_0(N)$  is a consequence of the convergence of  $L_f(s)$  in some right half-plane.

*Proof when  $1 \leq N \leq 4$ .* In this case  $\Gamma_0(N)$  is generated by the matrices  $P_1, H_N P_{-1} H_N^{-1}$ , and so the result follows from equations (3.6) and (3.4).  $\square$

*Proof when  $N \in \{5, 6, 7, 8, 9, 10, 12, 16\}$ .* The proof given in [CF95] does not require any elliptic matrices and so can be modified to real-analytic setting using the action (3.1) and equations (3.6), (3.4), and (3.9).  $\square$

For  $N \in \mathbb{Z}$ , introduce

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W_N = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}.$$

In particular,

$$(3.34) \quad Q \equiv 1, \quad W_N = H_N P_{-1} H_N^{-1} \equiv 1.$$

*Proof when  $N = 11$ .* When  $k = 1$  the result follows from [CF95] and Lemma 3.2. When  $k = 0$ , analysing the proof given in [CF95] it suffices to show that  $f$  is invariant under the matrix  $\begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}$ . From [loc. cit.], we know that

$$1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} = \left( 1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \right) \begin{pmatrix} 1 & -2/3 \\ 11/2 & -8/3 \end{pmatrix},$$

in which the final matrix on the right-hand side is elliptic of infinite order. On the other hand, we know that

$$W_{11} = \begin{pmatrix} 2 & -1 \\ 11 & 5 \end{pmatrix} P \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 11 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}^{-1} \equiv 1,$$

<sup>1</sup>The proof given in [Ogg69, V-15] can be modified to real-analytic forms.

and that  $\begin{pmatrix} 2 & -1 \\ 11 & 5 \end{pmatrix} \equiv 1$  by [loc. cit.]. Combining these two facts, we deduce that

$$1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} = \left( 1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \right) \begin{pmatrix} 2 & -1 \\ 11 & 5 \end{pmatrix}^{-1}.$$

The final matrix on the right-hand side does not have the same fixed point as  $\begin{pmatrix} 1 & -2/3 \\ 11/2 & -8/3 \end{pmatrix}$  and so we are done by Lemma 3.2.  $\square$

We introduce

$$A = \begin{pmatrix} 7 & -1 \\ 36 & -5 \end{pmatrix} \in \Gamma_0(18), \quad B = \begin{pmatrix} 13 & -8 \\ 18 & -11 \end{pmatrix} \in \Gamma_0(18),$$

$$C = \begin{pmatrix} 13 & -2 \\ 20 & -3 \end{pmatrix} \in \Gamma_0(20), \quad D = \begin{pmatrix} 19 & -4 \\ 24 & -5 \end{pmatrix} \in \Gamma_0(24).$$

One may check that the table below records a set of generators for  $\Gamma_0(N)$  when  $N \in \{18, 20, 24\}$ .

$N$	Generators for $\Gamma_0(N)$
18	$P_1, Q, A, B, \begin{pmatrix} 71 & -15 \\ 90 & -9 \end{pmatrix}, \begin{pmatrix} 55 & -13 \\ 72 & -17 \end{pmatrix}, \begin{pmatrix} 7 & -2 \\ 18 & -5 \end{pmatrix}, \begin{pmatrix} 31 & -25 \\ 36 & -29 \end{pmatrix}.$
20	$P_1, Q, C, \begin{pmatrix} 49 & -9 \\ 60 & -11 \end{pmatrix}, \begin{pmatrix} 31 & -7 \\ 40 & -9 \end{pmatrix}, \begin{pmatrix} 29 & -8 \\ 40 & -11 \end{pmatrix},$ $\begin{pmatrix} 31 & -9 \\ 100 & -29 \end{pmatrix}, \begin{pmatrix} 17 & -6 \\ 20 & -7 \end{pmatrix}.$
24	$P_1, Q, D, \begin{pmatrix} 19 & -2 \\ 48 & -5 \end{pmatrix}, \begin{pmatrix} 61 & -7 \\ 96 & -11 \end{pmatrix}, \begin{pmatrix} 59 & -8 \\ 96 & -13 \end{pmatrix}, \begin{pmatrix} 13 & -2 \\ 72 & -11 \end{pmatrix},$ $\begin{pmatrix} 17 & -5 \\ 24 & -7 \end{pmatrix}, \begin{pmatrix} 61 & -25 \\ 144 & -5 \end{pmatrix}, \begin{pmatrix} 13 & -6 \\ 24 & -11 \end{pmatrix}, \begin{pmatrix} -5 & -2 \\ 48 & -19 \end{pmatrix}.$

For  $N \equiv 0 \pmod{2}$ , define

$$J_N = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 & 1 \\ N & -\frac{N+2}{2} \end{pmatrix} \in SL_2(\mathbb{R}).$$

Though the argument in [CF95, Lemma 3] is written for a group action slightly different to (3.1), analysing the proof we see that it may be modified to show that  $J_{18} \equiv -1$  provided  $2^2$  does not divide  $N$ . On the other hand, for  $N \equiv 0 \pmod{4}$  define

$$L_N = \begin{pmatrix} \frac{N}{4} - 1 & -\frac{1}{2} \\ \frac{N}{2} & -1 \end{pmatrix} \in SL_2(\mathbb{R}),$$

which satisfies  $L_N \equiv NL_N = (P_{\frac{1}{2}}H_N)^2 \equiv 1$ . We may now rewrite the table above.

$N$	Generators for $\Gamma_0(N)$
18	$P_1, Q, A, B, BH_{18}J_{18}W_{18}J_{18}H_{18}^{-1}W_{18}^{-1}A^{-1},$ $QBJ_{18}^{-2}W_{18}^{-1}A^{-1}, Q(W_{18}J_{18})^{-2}A^{-1}, QP_1H_{18}J_{18}^{-1}W_{18}^{-1}J_{18}^{-1}H_{18}^{-1}B^{-1}.$
20	$P, Q, C, CL_{20}^{-2}W_{20}^{-1}L_{20}^{-1}CQ, CL_{20}^{-2}CL_{20}^{-1},$ $CL_{20}^{-1}W_{20}^{-1}L_{20}^{-1}CL_{20}^{-1}Q, L_{20}^2C^{-1}W_{20}^{-1}L_{20}^{-1}CL_{20}^{-1}, CL_{20}^{-3}.$
24	$P_1, Q, D, L_{24}^2, L_{24}W_{24}P_1^{-1}DL_{24}, QP_{1/2}L_{24}^{-1}D^{-1}P_{1/2}L_{24},$ $H_{24}P_1^{-1}DP_{1/2}^{-1}DP_{1/2}H_{24}^{-1}, QP_{1/2}D^{-1}P_{1/2},$ $L_{24}P_1^{-1}L_{24}^{-1}, L_{24}W_{24}L_{24}^{-1}, (W_{24}^{-1}L_{24}^{-1}P_1)^2.$

Note that these generating sets are far from minimal, for example, the list in the case  $N = 20$  may be simplified to:

$$P_1, Q, C, L_{20}W_{20}L_{20}^{-1}, L_{20}^{-1}W_{20}L_{20}, L_{20}CL_{20}^{-1}, L_{20}^{-1}CL_{20}, L_{20}^3.$$

*Proof in case  $N = 18$ .* Since  $4 \nmid 18$ , we have that  $J_{18} \equiv 1$ . By the list of generators for  $\Gamma_0(18)$  given in the table and equations (3.6), (3.4), and (3.34), it suffices to prove that  $\gamma \equiv 1$  for  $\gamma \in \{A, B\}$ . Noting that

$$B = PH_{18}AH_{18}^{-1}P^{-1}, \quad A \equiv \begin{pmatrix} -11 & -1 \\ -54 & -5 \end{pmatrix} W_{18}^{-1},$$

it is enough to show that  $\begin{pmatrix} -11 & -1 \\ -54 & -5 \end{pmatrix} \equiv 1$ . Because  $3^2 | N$ , we have  $f|_k U_3 = 0$  which implies  $P_{1/3} + P_{-1/3} \equiv -1$ . Therefore:

$$(3.35) \quad (P_{1/3} + P_{-1/3})J_{18} \equiv 1 \equiv J_{18}(P_{1/3} + P_{-1/3}).$$

Multiplying equation (3.35) by  $P_{1/3}J_{18}$ , we deduce:

$$\begin{pmatrix} -11 & -1 \\ -54 & -5 \end{pmatrix} + \begin{pmatrix} 25 & -7/3 \\ -54 & -5 \end{pmatrix} \equiv \begin{pmatrix} 13 & 4/3 \\ -108 & -11 \end{pmatrix} + 1,$$

and on the other we have

$$\begin{pmatrix} 13 & 4/3 \\ -108 & -11 \end{pmatrix} \equiv \begin{pmatrix} 25 & -7/3 \\ -54 & -5 \end{pmatrix}.$$

Combining the previous two equations, we are done.  $\square$

*Proof in case  $k = 1$  and  $N \in \{14, 15, 17, 20, 23, 24\}$ .* For  $N \in \{14, 15, 17, 23\}$ , we may mimic the argument given in [CF95] using Lemma 3.2 in place of [CF95, Lemma 5]. When  $N = 20$ , it suffices to prove that  $C \equiv 1$ . We have

$$(3.36) \quad \left(1 - \begin{pmatrix} 3 & -1 \\ 40 & -13 \end{pmatrix} P_{1/3}\right) + \left(1 - \begin{pmatrix} 3 & -2 \\ 20 & -13 \end{pmatrix} P_{2/3}\right) \equiv 0.$$

Noting that

$$\begin{pmatrix} 3 & -1 \\ 40 & -13 \end{pmatrix} \equiv CL_{20}^{-2}, \quad \begin{pmatrix} 3 & -2 \\ 20 & -13 \end{pmatrix} \equiv C^{-1},$$

equation (3.36) becomes

$$1 - C \equiv (1 - C)C^{-1}P_{1/3}L_{20}^2.$$



The matrix  $C^{-1}P_{1/3}L_{20}^2$  is elliptic of infinite order and so the result follows from Lemma (3.2). When  $N = 24$ , it suffices to prove that  $D \equiv 1$ . We have

$$(3.37) \quad \left(1 - \begin{pmatrix} 5 & -4 \\ 24 & -19 \end{pmatrix} P_{4/5}\right) + \left(1 - \begin{pmatrix} 5 & 4 \\ -24 & -19 \end{pmatrix} P_{-4/5}\right) \\ + \left(1 - \begin{pmatrix} 5 & -2 \\ 48 & -19 \end{pmatrix} P_{2/5}\right) + \left(1 - \begin{pmatrix} 5 & 2 \\ -48 & -19 \end{pmatrix} P_{-2/5}\right) \equiv 0.$$

Noting that

$$\begin{pmatrix} 5 & -4 \\ 24 & -19 \end{pmatrix} \equiv QD^{-1} \begin{pmatrix} 5 & 4 \\ -24 & -19 \end{pmatrix} \equiv DP, \\ \begin{pmatrix} 5 & -2 \\ 48 & -19 \end{pmatrix} \equiv QL_{24}^{-2} \equiv 1, \quad \begin{pmatrix} 5 & 2 \\ -48 & -19 \end{pmatrix} \equiv Q(W_{24}^{-1}L_{24}^{-1}P)^2 \equiv 1,$$

equation (3.37) becomes

$$1 - D \equiv (1 - D)D^{-1}P_{3/5}.$$

The matrix  $D^{-1}P_{3/5}$  is elliptic of infinite order. The result now follows from Lemma 3.2.  $\square$

**3.5. Proof of Theorem 2.4.** Given a function  $f$  on  $\mathcal{H}$  and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ , define the function  $f|{}^k\gamma$  by

$$(f|{}^k\gamma)(z) = \det(\gamma)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Let  $f$  be defined as in (2.17). By equation (2.17) (resp. (2.16)), we know that

$$f = f|{}^kP, \quad f = f|{}^kH_N.$$

We may now argue as in Section 3.4, replacing Lemmas 3.1 and 3.2 by:

**Lemma 3.3** (Lemma 5 in [CF95]). *Suppose  $F$  is holomorphic on  $\mathcal{H}$  and  $E \in SL_2(\mathbb{R})$  is elliptic. If  $f|{}^kE = f$ , then either  $E$  has finite order or  $f$  is constant.*

Holomorphy of  $f$  at the cusps follows from convergence of  $L_f(s)$  in some right half-plane, as in [Ogg69, V-14].

## REFERENCES

- [BBB<sup>+</sup>18] S. Bettin, J. Bober, A. Booker, B. Conrey, M. Lee, G. Molteni, T. Oliver, D. Platt, and R. Steiner. A conjectural extension of Hecke’s converse theorem. *Ramanujan J.*, 47(3):659–684, 2018.
- [BCK19] A. Booker, P. Cho, and M. Kim. Simple zeros of automorphic L-functions. *Compos. Math.*, 155:1224–1243, 2019.
- [BK13] A. Booker and M. Krishnamurthy. Weil’s converse theorem with poles. *International Mathematics Research Notices*, 127:1–12, 2013.
- [Bum98] D. Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics, 1998.
- [CF95] J.B Conrey and D. Farmer. An extension of Hecke’s converse theorem. *International Mathematics Research Notices*, 9:445 – 463, 1995.
- [CFOS07] J.B Conrey, D. Farmer, B.E. Odgers, and N.C. Snaith. *Journal of Number Theory*, 122(2):314–323, 2007.
- [DPZ02] A. Diaconu, A. Perelli, and A. Zaharescu. A note on the GL(2) converse theorem. *C. R. Math. Acad. Sci. Paris*, 334:621–624, 2002.

- [GM04] S. Gelbart and S. Miller. Riemann’s zeta function and beyond. *Bull. Amer. Math. Soc.*, 41:59–112, 2004.
- [GR15] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products (eighth edition)*. Translation edited and with a preface by Daniel Zwillinger and Victor Moll. Elsevier/Academic Press, 2015.
- [Hec36] E. Hecke. Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung. *Math. Ann.*, 112(1):664–699, 1936.
- [HO] L. Hochfilzer and T. Oliver. Ratios of Artin L-functions. arXiv:1910.02821.
- [JL70] H. Jacquet and R. P. Langlands. *Automorphic Forms on  $GL(2)$* . Lecture Notes in Mathematics, 1970.
- [Li81] W-C. W. Li. On converse theorems for  $GL(2)$  and  $GL(1)$ . *Amer. J. Math.*, 103(5):851–885, 1981.
- [Maa49] Hans Maass. Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Math. Ann.*, 121:141–183, 1949.
- [MSSU] T. Miyazaki, F. Sato, K. Sugiyama, and T. Ueno. Converse theorems for automorphic distributions and Maass forms of level  $N$ . arXiv:1905.049909.
- [NO20] M. Neururer and T. Oliver. Weil’s converse theorem for Maass forms and cancellation of zeros. accepted in *Acta Arithmetica*, 2020.
- [Ogg69] A. Ogg. *Modular forms and Dirichlet series*. W. A. Benjamin, 1969.
- [PS71] I. I. Piatetski-Shapiro. On the Weil–Jacquet–Langlands theorem. *Lie groups and their representations (Proc. Summer School, Bolyai, János Math. Soc., Budapest)*, pages 583 – 393, 1971.
- [Raz77] M. Razar. Modular forms for  $\Gamma_0(N)$  and Dirichlet series. *Trans. Amer. Math. Soc.*, 231:489 – 495, 1977.
- [Wei67] A. Weil. Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Mathematische Annalen*, 168:149–156, 1967.

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