C. R. Physique 16 (2015) 723–728

Contents lists available at ScienceDirect

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Prix Langevin de l'Académie des sciences

Landau damping and inhomogeneous reference states

Amortissement Landau et états de référence non homogènes

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ARTICLE INFO

Article history:
Available online 18 April 2015

Keywords:
Landau damping
Plasma physics
Fluid dynamics
Astrophysics
Vlasov equation
Inhomogeneous stationary state

Mots-clés :
Amortissement de Landau
Physique des plasmas
Dynamique des fluides
Astrophysique
Équation de Vlasov
État stationnaire non homogène

ABSTRACT

Landau damping is a fundamental phenomenon in plasma physics, which also plays an important role in astrophysics, and sometimes under different names, in fluid dynamics, and other fields. Its theoretical discussion in the framework of the Vlasov equation often assumes that the reference stationary state is homogeneous in space. However, Landau damping around an inhomogeneous reference stationary state, a natural setting in astrophysics for instance, induces new mathematical difficulties and physical phenomena. The goal of this article is to provide an introduction to these problems and the questions they raise.

RÉSUMÉ

L'amortissement Landau est fondamental en physique des plasmas, et joue, parfois sous un nom différent, un rôle important en astrophysique, en dynamique des fluides et dans d'autres domaines. Son traitement théorique à partir de l'équation de Vlasov suppose souvent que l'état stationnaire de référence est homogène en espace. Néanmoins, un état stationnaire non homogène en espace, un cadre naturel en astrophysique par exemple, induit des difficultés mathématiques et des phénomènes physiques nouveaux. Le but de cet article est de fournir une introduction à ces problèmes et aux questions qu'ils soulèvent.

1. Introduction

Landau damping was first introduced almost 70 years ago [1], and it has since then been an active subject of research, whose importance goes far beyond its original realm of the Vlasov equation in plasma physics. The purpose of this article is to provide an introduction to the specificities of Landau damping close to a stationary state of the Vlasov equation which is not homogeneous in space: we describe in particular the various singularities governing the behavior of a perturbation (Table 1), and how they yield a two-step relaxation, see Fig. 3. This situation is in particular relevant for astrophysical applications of the Vlasov equation, but it is striking that a phenomenology similar to the one we will describe here is

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http://dx.doi.org/10.1016/j.crhy.2015.03.013

encountered in very different physical systems: fluids described by Euler equation [2], Kuramoto model for oscillators synchronization [3], sound propagation in bubbly fluids [4]... For self-consistency, we will first recall some basic facts about the Vlasov equation and standard Landau damping as it was first introduced in plasma physics, close to a homogeneous-in-space stationary state.

2. The Vlasov equation

When one tries to describe the positions and velocities of a large number of interacting particles by a phase space density, different situations may occur: if the interactions between particles are strong and rare, so that they involve only two particles each time, one expects a Boltzmann-like equation; if on the contrary one particle feels the effect of many others, the individual effect of each one being weak, one enters the realm of the Vlasov equation. This scaling limit, sometimes called mean-field limit, appears in many areas of physics, the two main examples being the particles in Coulombian interaction, and the particles in Newtonian interaction. In the plasma physics context, it was indeed introduced by Vlasov [5]; in the astrophysical context, the Vlasov equation is usually called "collisionless Boltzmann equation". We write down the equation in its Hamiltonian form:

$$\partial_t f + \{f, h\} = 0 \tag{1}$$

where f is the phase space density, normalized to 1, $\{f, g\}$ is the Poisson bracket

$$\{f, g\} = \frac{1}{M}\left(\nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{v}} g - \nabla_{\mathbf{v}} f \cdot \nabla_{\mathbf{x}} g\right) \tag{2}$$

M is the particles' mass and $h(\mathbf{x}, \mathbf{v}, t)$ the Hamiltonian

$$h(\mathbf{x}, \mathbf{v}, t) = \frac{M\mathbf{v}^2}{2} + \phi(\mathbf{x}, t) \tag{3}$$

The potential $\phi(\mathbf{x}, t)$ is created by f. For Coulomb or Newton interaction, ϕ is related to the spatial density $\rho = \int f \mathrm{d}\mathbf{v}$ through the Poisson equation:

$$\Delta\phi = C\rho \tag{4}$$

where $C = 4\pi G$ for the Newton interaction, with G the gravitational constant, and $-4\pi e^2/M$ for the Coulomb one. More generally, ϕ may be given by the convolution of the density ρ and a kernel K:

$$\phi(\mathbf{x}) = \int K(\mathbf{x} - \mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{5}$$

For instance, K is proportional to $1/|\mathbf{x}|$ for Coulomb (resp. $-1/|\mathbf{x}|$ for Newton) interaction in 3D. Physically, the Vlasov equation is simply a transport equation, meaning that each particle moves in the field that all particles create. Hereafter, we set $M = 1$ for simplicity.

The convergence of the discrete dynamics of N particles towards the continuous Vlasov equation has been rigorously proved [6,7], but under the hypothesis that the kernel K is regular enough. In particular, this excludes the Coulomb and Newton cases!

3. Stationary states and their stability

The Vlasov equation (1) admits an infinite number of stationary states. Indeed, any function $f = \varphi(h(\mathbf{x}, \mathbf{v}))$ that depends on the phase space coordinates only through the Hamiltonian h is a stationary solution. An easy special case is provided by a function f that depends only on $\mathbf{v}^2$ (this implies that f is homogeneous in space), with a vanishing self-consistent potential. Space inhomogeneous stationary solutions, needed for instance to describe self-gravitating systems, are more difficult to construct, since one needs to solve the self-consistent equation (4) for the potential.

Studying the Vlasov dynamics close to a stationary solution proved a difficult endeavor, with surprises even at the linear level. When the linearized Vlasov operator has an eigenvalue with positive real part, the corresponding stationary state is clearly unstable. Furthermore, since the Vlasov equation has a Hamiltonian structure, it is easy to realize that the spectrum of the linearized operator is symmetric with respect to the imaginary axis; hence, if there is an eigenvalue with a negative real part, there is also one with a positive real part, and the corresponding stationary state is also unstable. Thus, the stable stationary states are only marginally stable, the spectrum being included in the imaginary axis. Yet, a perturbation around a stable stationary state may decay in some sense exponentially, with a well-defined rate: this surprising exponential decay for a time-reversible equation is the well-known Landau damping phenomenon, first described in 1946 [1]. In the following, we consider such damping of a perturbation δf around a stable stationary state f_0.

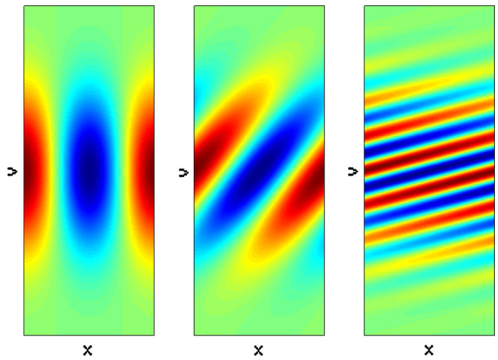

Fig. 1. (Color online.) Damping of an initial spatial structure by free transport (periodic spatial boundary condition are used). Left = initial condition; center and right: the shear in phase space creates finer and finer structures. After integration over the v variable, the spatial structure damps.

4. Landau damping, homogeneous case

We start with a simple setting, 1D and periodic in space, and consider a homogeneous-in-space stationary solution $f_0(v)$. The linearized Vlasov equation around f_0 for a small perturbation δf reads:

$$\partial_t \delta f + v\partial_x \delta f - f_0' \partial_x \delta\phi = 0 \tag{6}$$

$$\delta\phi(x) = \int K(x-y)\delta\rho(y)\mathrm{d}y \tag{7}$$

where $\delta\rho$ and $\delta\phi$ are the density and potential perturbations respectively.

To understand the basic mechanism of Landau damping, we start by forgetting the interaction, keeping only the free transport: each particle moves with constant velocity v. Velocities of initially close particles are different; this shear develops finer and finer structures in the (x,v) phase space; any initial spatial organization is thus damped by this phase mixing process: see an illustration in Fig. 1. Introducing the interaction term does not modify much this basic physical mechanism, but has an important effect on the type of damping. See [8] for details on these differences between free transport and Landau damping.

The standard mathematical treatment of (6)–(7) is as follows. After a Fourier transform in space and a Laplace transform in time, which is defined for the function $u(x,t)$ as

$$\hat{u}(k,\omega) = \int \mathrm{e}^{-\mathrm{i}kx}\left(\int_0^\infty \mathrm{e}^{\mathrm{i}\omega t} u(x,t)\mathrm{d}t\right)\mathrm{d}x \tag{8}$$

the linearized Vlasov equation (6)–(7) is reduced to a diagonalized algebraic equation and is solved formally. Simple manipulations give the Fourier–Laplace component as

$$\widehat{\delta\phi}(k,\omega) = \frac{G(k,\omega)}{D(k,\omega)} \tag{9}$$

where

$$D(k,\omega) = 1 - K_k \int \frac{k f_0'(v)}{kv-\omega}\mathrm{d}v, \quad G(k,\omega) = -\mathrm{i}K_k \int \frac{\tilde{f}_{\mathrm{ini}}(k,v)}{kv-\omega}\mathrm{d}v \tag{10}$$

and K_k and $\tilde{f}_{\mathrm{ini}}$ are the Fourier transforms of $K(x)$ and the initial distribution, respectively. The time evolution of the kth Fourier component of the potential $\widehat{\delta\phi}(k,\omega)$ is then given by an inverse Laplace transform.

The asymptotic behavior of a time-dependent function is intimately related to the regularity of its Laplace transform: the more singular is the Laplace transform, the slower is the decay at large times. Thus to understand the large-time behavior of $\delta\phi$, we have to study the singularities of the D and G functions (10), or rather the singularities of their analytic continuation. Although they are not eigenvalues of the linearized operator (6)–(7), roots of the continued D function yield an exponential damping of $\delta\phi$: this is Landau damping.

We remark that an algebraic damping may happen if the stable stationary state f_0 or perturbation δf is not analytic [9]. In the next section, we show that the algebraic damping is actually unavoidable in the inhomogeneous case even if f_0 and δf are analytic.

5. Landau damping, inhomogeneous case

In astrophysics, the analog of Landau damping takes place, and is considered a standard mechanism in the relaxation of stellar structures, see [8] for a textbook reference. However, it is remarkable that actual computations of Landau damping rates for self-gravitating structures are very rare; indeed, we are only aware of one instance of such a computation [10,11].[1] Clearly, the reason lies in the technical difficulty, as will become clear later. Inhomogeneous stationary states are also a common feature in plasma physics; in this context, Landau damping has been studied in the limit of vanishing velocity dispersion [12]. We feel that this situation calls for a more precise study of Landau damping close to inhomogeneous stationary states.

For simplicity, let us stick to a 1D setting with periodic boundary conditions, and consider an inhomogeneous stationary state of the Vlasov equation, of the form $f_0(x,v) = \varphi(h(x,v))$.

In the inhomogeneous case, the derivative of δf with respect to v survives in the linearized Vlasov equation, and this prevents us from obtaining an algebraic equation. A solution to this problem is to introduce the angle-action variables (θ, J) associated with the Hamiltonian h, which is always integrable, since we stick to 1D and f_0 (and thus h) is stationary. The angle-action variables eliminate the derivative with respect to J, and the Fourier transform with respect to θ yields an algebraic equation just as in the homogeneous case. However, in the inhomogeneous case, the Fourier bases for x, $\mathrm{e}^{\mathrm{i}kx}$, and for θ, $\mathrm{e}^{\mathrm{i}m\theta}$, are not orthogonal, and the algebraic equation is not diagonalized: all modes are mixed. Thus, in general, the algebraic equation is described by using an infinite-size matrix D whose (l,m)-element is denoted by $D(l,m,\omega)$ and an infinite size vector G. The Fourier–Laplace component of potential is expressed in the form

$$\widehat{\delta\phi(l,\omega)} = \sum_m D^{-1}(l,m,\omega) G(m,\omega) \tag{11}$$

where $D^{-1}(l,m,\omega)$ is the (l,m)-element of the inverse matrix D^{-1}. Each element of D and G has a similar form as in the homogeneous case, it reads as an integral:

$$F(\omega) = \int \frac{\psi(J)}{\Omega(J) - \omega/m} \mathrm{d}J \tag{12}$$

where $\Omega(J) = (\mathrm{d}h/\mathrm{d}J)(J)$. This solution "in matrix form" was introduced in the 1970s [13,14]. Let us discuss the singularities of the function F, since we know they control the asymptotic behavior of the potential.

Formally, the roots of the determinant of D correspond to poles of $\widehat{\delta\phi}$, and these poles yield exponential dampings just as in the homogeneous case [15].

Poles are not the only singularities in the inhomogeneous case. Indeed the integration range for the action has a minimum value J_v, whereas the integration range for the velocity in (10) was $\mathbb{R}$. As a result, a logarithmic singularity appears on the real axis of ω. A simple example helps to understand the creation mechanism of the new singularity. Suppose $m = 1$ and $\psi(J) = 1$, and $\Omega(J)$ is expanded around J_v as

$$\Omega(J) = \Omega(J_v) + a(J - J_v) + \cdots \tag{13}$$

The F function is then expressed as

$$F(\omega) = \int_0^\infty \frac{\mathrm{d}J}{aJ - (\omega - \Omega(J_v))} + \cdots \tag{14}$$

and it is clear that a logarithmic singularity appears at $\omega = \omega_v \equiv \Omega(J_v)$. This singularity lies on the real axis, since the function Ω is real, and gives rise to a *branch cut* when viewed in the complex plane; hence the analytic continuation of the function F is multi-valued. In the homogeneous case, the integral runs over the whole real line, see (10), hence the singularities on the real axis cancel: integrals on the left and the right of the singularity diverge in the same way, with opposite sign. Formally, the mechanism is that analytic continuation turns the integrals in the D and the G functions into principal values. The existence of the minimum point J_v prevents us from applying the cancellation mechanism to the inhomogeneous case.

We stress that this logarithmic singularity comes from the inhomogeneous nature of the stable stationary state f_0. Thus, the singularity is unavoidable, and shows up even when the reference state f_0 and the initial perturbation are perfectly smooth.

Another type of singularity arises when $\Omega(J)$ has a local extremum. Generically, detailed computations reveal that the singular part of F is [16,17]

$$F^{\mathrm{sing}}(\omega) = C_1(\omega - \omega)^\nu \ln|\omega - \omega_v| + C_2(\omega - \omega)^\mu H(\omega - \omega_v) \tag{15}$$

where $H(x)$ is the Heaviside step function, and a non-negative integer ν and a positive real value $\mu + 1$ come from the leading powers in the expansion of ψ at $J = J_v$. Known singularities of the Laplace transform $\widehat{\delta\phi}$ are summarized in Table 1.

[1] Computing *instability rates* is easier, since it does not require any analytic continuation.

Table 1
Singularities of the Laplace transform of the potential for an analytic stable stationary state f_0 and perturbation δf with induced asymptotic dampings. The checkmarks signal the presence of the singularities.

Singularity	Asymptotic damping	Homogeneous	Inhomogeneous
Pole	Exponential	✓	✓
Log. branch point	Algebraic		✓
Heaviside step	Algebraic		✓

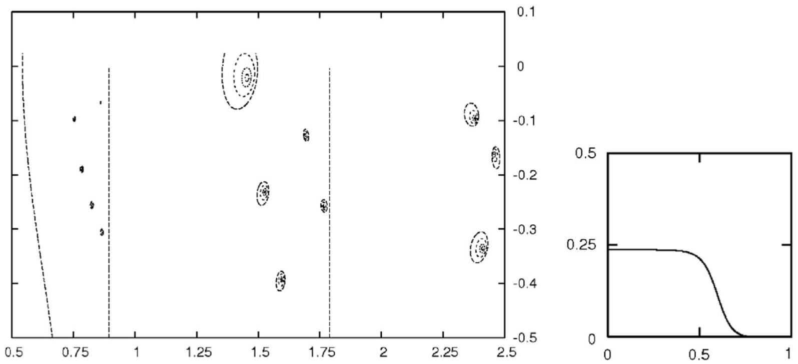

Fig. 2. Left panel: Contour plot of the modulus of the dispersion function Det(D) in the ω complex plane; the horizontal (vertical) axis is the real (imaginary) part of ω. The vertical dashed lines are branch cuts, separating different Riemann sheets; the contour levels are chosen so that the roots are clearly visible. Right panel: the reference stationary state φ around which the dispersion function shown on the left panel is computed. It is represented here as a function of the variable $\sqrt{2h}$, so that the statistical equilibrium distribution would be a Gaussian.

The inverse Laplace transform of the logarithmic and the Heaviside step singularities in (15) both yield an oscillatory algebraic damping, with exponent $1+\nu$ and $1+\mu$ and frequency ω_ν respectively. The analysis shown in this section can be extended to higher dimensional systems [17]. The value of the exponent ν or μ giving the slowest damping can be obtained by a detailed analysis of the leading powers around the singular points on J in the reference state f_0 and perturbation δf.

From the discussion of this section, it is clear that the slowest decay is algebraic, and that it should rule the asymptotic behavior of a perturbation (assuming of course that the linearization remains valid at large times). Does it mean that exponential "Landau damping", coming from the roots of the determinant of the matrix D, is irrelevant? Of course not: it turns out that it often plays an important role at intermediate time scales, before algebraic damping dominates.

6. Some explicit computations on a simple model

The very small number of explicit computations of Landau damping rates for self-gravitating structures is due to the inherent analytical and numerical difficulties of the needed analytic continuation; in [10,11] this part is performed entirely numerically using Padé approximants, a procedure that is probably stable only close to the real axis and far from the branch cut singularities discussed above. If one accepts to give up the realism of 3D $1/|x|$ potential and turn to a toy model for gravitation, it becomes possible to perform explicit computations of the Landau poles. For this purpose, we will use in this section the Hamiltonian mean-field model [18]. The spatial domain is $[0, 2\pi)$, and the interaction kernel is a cosine:

$$\phi(x) = \int_0^{2\pi} [1 - \cos(x-y)]\rho(y)\mathrm{d}y \tag{16}$$

Hence the self-consistent potential itself is of the form $-M\cos(x-\vartheta)$. In a stationary state, M and ϑ are constant, thus the motion of each particle is that of a simple pendulum. It is well known that all trajectories can then be expressed in terms of elliptic functions and integrals [15], which can be readily analytically continued. Analytic continuation of the coefficients appearing in the matrix method is then relatively easy. Furthermore, thanks to the simple structure of the interaction kernel (only one Fourier mode is present), the determinant one has to compute is actually 2×2 and diagonal, rather than infinite in the general case.

Thanks to these simplifications, it becomes possible to explore all the "Landau poles" and follow their bifurcations when varying the stationary state. It is still a delicate task, as the landscape for $\widehat{\delta\phi}$ presents the full complexity of inhomogeneous stationary states: one should in principle explore an infinity of Riemann sheets. An example of the poles and branch cut singularities is given in Fig. 2.

Direct numerical simulations confirm that the dominant pole of the dispersion function (closest to the real axis) indeed dominate the dynamics over an intermediate time scale [15], and the asymptotic algebraic damping as shown on Fig. 3.

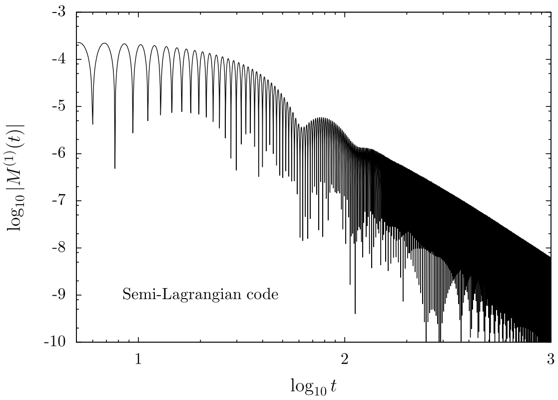

Fig. 3. Evolution of the perturbed field amplitude $M^{(1)}$ close to an inhomogeneous stationary state (see [16] for a precise definition of $M^{(1)}$). At small times, the amplitude oscillates and decreases exponentially (not easily seen in log-log scale): the dynamics is governed by the dominant Landau pole. At large times ($t \gtrsim 100$), the algebraic asymptotic damping becomes clear. The simulations use a semi-Lagrangian method (see [19] for details on the algorithm); the results are consistent with the N-body simulations of [16].

7. Conclusion

Although Landau damping around inhomogeneous stationary solutions is often considered as a direct analog of the much easier homogeneous case, we hope to have illustrated that spatial inhomogeneity requires a somewhat different mathematical structure, which in turn induces important physical differences. In the present short article, we have restricted to the linearized Vlasov equation, which is already very rich. The weakly nonlinear behavior of the Vlasov equation has also been the subject of many studies, old and new (for a few references, see [20–25]); a similar systematic study close to non-homogeneous stationary state largely remains to be done, and the linear theory will be a necessary building block. In view of the differences showing up already at the linear level, surprises may be expected.

Acknowledgement

YYY acknowledges the support of JSPS KAKENHI Grant No. 23560069.

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