Topological insulators/Isolants topologiques

Electromagnetic and thermal responses in topological matter: Topological terms, quantum anomalies and D-branes

Réponses électromagnétiques et thermiques dans la matière topologique : Termes topologiques, anomalies quantiques et D-branes

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A R T I C L E   I N F O

A B S T R A C T

We discuss the thermal (or gravitational) responses in topological superconductors and in topological phases in general. Such thermal responses (as well as electromagnetic responses for conserved charge) provide a definition of topological insulators and superconductors beyond the single-particle picture. In two-dimensional topological phases, the Strěda formula for the electric Hall conductivity is generalized to the thermal Hall conductivity. Applying this formula to the Majorana surface states of three-dimensional topological superconductors predicts cross-correlated responses between the angular momentum and thermal polarization (entropy polarization). We also discuss a use of D-branes in string theory as a systematic tool to derive all such topological terms and topological responses. In particular, we relate the $\mathbb{Z}_2$ index of topological insulators introduced by Kane and Mele (and its generalization to other symmetry classes and to arbitrary dimensions) to the K-theory charge of non-BPS D-branes, and vice versa. We thus establish a link between the stability of non-BPS D-branes and the topological stability of topological insulators.

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R É S U M É

Nous traitons des réponses thermiques (ou gravitationnelles) des supraconducteurs topologiques et, plus généralement, des phases topologiques. Ces réponses thermiques (tout comme les réponses électromagnétiques pour la charge électrique conservée) fournissent une définition des isolants et supraconducteurs topologiques, qui reste valable au-delà du modèle à une particule. Pour les phases bidimensionnelles, la formule de Streda de la conductivité de Hall de charge est généralisée à la conductivité de Hall thermique. Appliquée aux états de surface de Majorana des supraconducteurs topologiques, cette formule conduit à des fonctions de réponse croisées entre le moment angulaire et la polarisation thermique. Nous discutons également de la théorie des cordes (*D-branes*) comme outil

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systématique pour obtenir ces réponses topologiques. En particulier, nous relions l'invariant topologique $\mathbb{Z}_2$ des isolants topologiques introduit par Kane et Mele (ainsi que ses généralisations à d'autres classes de symétrie en dimensions arbitraires) à la charge de certaines *D-branes* (celles qui ne saturent pas la limite de Bogomol'nyi–Prasad–Sommerfield), et vice versa. Nous établissons ainsi un lien entre la stabilité de ces *D-branes* et celle des isolants topologiques.

1. Introduction

A vital role of topology in quantum transport phenomena in solids has been recognized as a driving force of a *dissipationless* current. The Berry curvature [1] in momentum and real spaces induces the velocity of Bloch electrons [2], resulting in various effects, such as the anomalous Hall effect [3] and the spin Hall effect [4]. Even the heat current can be induced by the Berry curvature, as observed in the anomalous Nernst effect [5] and the thermal Hall effect [6]. These quantum transport phenomena of topological origin (or *topological currents*) are regarded as a promising candidate for spintronics with low energy cost.

In metallic systems, the usual transport currents with dissipation are dominant over these topological currents, and hence it is rather difficult to identify the latter. A topological current, however, can be non-vanishing even in insulating systems – such systems on general ground are called topological insulators. This is so since it is the topological properties of electronic wavefunctions, rather than a band structure of energy levels in solids, that are responsible for generation of a topological current.

The quantum Hall effect (QHE) is a canonical example of topological insulators characterized by a topological (i.e., *quantized*) transport law [7–9]. Recently topological insulators realized by strong spin–orbit interactions in two and three dimensions have been discovered [10–17]. In particular, the three-dimensional (3d) topological insulator is characterized by the topological magnetoelectric effect [18,19] (the axion electrodynamics [20]). (See below for more details.) As a consequence of non-trivial electrical wavefunctions in the bulk, these topological insulators support anomalous boundary (edge or surface) modes, whose gapless nature is topologically protected.

Analogously, for superconductors (SCs) and superfluids, one can consider topological properties associated with the wavefunctions of fermionic quasi-particles; Within the BCS mean-field theory, the Bogoliubov–de Gennes (BdG) Hamiltonian is a fermion bilinear in the Nambu spinor. The case where we have a quasi-particle gap everywhere in the momentum space is an analogue of the band insulator. A topological SC is a SC with a full gap and topologically non-trivial quasi-particle wavefunctions. Canonical examples of topological SCs include, e.g., the 2d chiral *p*-wave SC [21,22].

In three dimensions, the B phase of superfluid ^{3}He was recently identified as a new topological SC (superfluid) [23–27]. A recent surface transverse acoustic impedance measurement reported in Refs. [28,29] revealed a signature of the surface Majorana fermion mode on the surface of ^{3}He–B. A copper-doped topological insulator ($Cu_x Bi_2 Se_3$) has been discussed as a candidate of a 3d topological SC [30–32].

The purpose of this article is to describe the response theory of topological phases with a special focus on their thermal, rather than electrical, transport. A motivation for the thermal response is that it is well-defined even for phases in which the electrical charge is not conserved, such as topological SCs, or topological phases in spin systems. While topological SCs (as well as topological insulators) can be defined in terms of a topological invariant built out of, within the BCS mean-field theory, quasi-particle wavefunctions, the response theory gives a physically measurable definition of topological phases, which can be largely insensitive to microscopic details – a lesson we have learned from the physics of the QHE, or the topological magnetoelectric effect in the 3d topological insulator. In particular, formulating the thermal response as a response to an external gravitational field, we will derive a thermal analogue of the topological magnetoelectric effect, which uncovers an interesting cross-correlation between thermal and mechanical responses, in terms of the temperature gradient, and an applied angular velocity, respectively (Section 3).

For the bulk of this article, we will focus on topological SCs in two and three dimensions, such as the 2d chiral *p*-wave SC or ^{3}He–B. However, we will also briefly describe how such response theory can be systematically constructed for a wider class of topological insulators and SCs in the "periodic table" [23,33,34] (Section 4). In fact, we will illustrate that responses in topological phases are closely related to quantum anomalies in field theories. Such characterization of topological phases in terms of anomalies is expected to incorporate arbitrary strong interactions as far as a bulk gap is not destroyed.

Interestingly, such considerations lead to a natural link to topological objects in string theory – D-branes. We will demonstrate, by considering a particular configuration of D-branes, that we can realize a field theory model of topological insulators and superconductors with desired discrete symmetries (Section 4). The stability criterion of D-branes against Tachyon condensation is in one-to-one correspondence with the classification of topological insulators and superconductors (i.e., the periodic table [23,33,34]). From condensed matter point of view, the D-brane construction can be thought of as a convenient tool that bridges K-theory classification of fermionic Hamiltonians, and linear response theory.

2. Electromagnetic response in the QHE and 3d topological insulator

2.1. The QHE in two dimensions

Let us start this review by first illustrating the usefulness of the effective field theory of linear response, by taking the QHE as an example. The electromagnetic response of the quantum Hall fluid is described by the effective Chern–Simons action:

$$I_{\text{eff}} = \frac{e^2 k}{4\pi \hbar} \int \mathrm{d}t \, \mathrm{d}^2 x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad k \in \mathbb{Z} \tag{1}$$

where A_μ is an external electromagnetic gauge field. The Chern–Simons action can be derived by considering a coupling of the (topological) insulator in question to the external (background) electromagnetic field A_μ, and then by integrating out fermions to derive the effective action for A_μ.

The Chern–Simons action encodes all types of topological responses in the QHE. (i) The current $\delta j^k(t, x)$ induced by A_μ is computed, for a time-independent vector potential, as

$$\delta j^k(t, x) = \frac{\delta I_{\text{eff}}}{\delta A_k(t, x)} = \frac{e^2 k}{2\pi \hbar} \epsilon^{ki0} \partial_i A_0 \tag{2}$$

This is the QHE, $J^x = \sigma_H E^y$, with the Hall conductivity $\sigma_H = ke^2/(2\pi\hbar)$. (ii) Similarly, one can compute the charge $\delta\rho(t, x)$ induced by A_μ as

$$c\delta\rho(t, x) = \delta j^0(t, x) = \frac{\delta I_{\text{eff}}}{\delta A_0(t, x)} = \frac{e^2 k}{2\pi \hbar} \epsilon^{0ij} \partial_i A_j \tag{3}$$

where c is the speed of light. This means that a (solitonic) flux tube binds k units of charge, which is, in cylinder geometry, nothing but charge-pumping in Laughlin's thought experiment. For uniform magnetic field B^z, the change in the total electron charge is

$$c\delta Q_e = \sigma_H \delta B^z \quad \Rightarrow \quad c\frac{\delta Q_e}{\delta B^z} = ce\frac{\delta N_e}{\delta B^z} = \sigma_H \tag{4}$$

where N_e is the total electron number. This is nothing but the Středa formula [35,36]. (iii) The Chern–Simons term can be written, for static configurations of A_μ, as $I_{\text{eff}} = \sigma_H \int \mathrm{d}t \, \mathrm{d}^2 x \, \epsilon^{0ij} A_0 \partial_i A_j = (\sigma_H/c) \int \mathrm{d}t \, \mathrm{d}^2 x \, \phi B^z$, where $A_0 =: \phi/c$. If we define (the change in) magnetization by $\delta M^z = \frac{\delta I_{\text{eff}}}{\delta B^z} = (\sigma_H/c)\delta\phi$, then

$$ec\frac{\delta M^z}{\delta\mu} = \sigma_H \tag{5}$$

where $\mu = e\phi$ is the chemical potential $[\mu N_e = (\mu/e)eN_e = \phi Q_e]$.

2.2. Topological insulator in three dimensions

Similarly, for the 3d topological insulator, the electromagnetic response is encoded in the effective action [18–20]:

$$I_{\text{eff}} = \frac{\theta e^2}{32\pi^2 \hbar c} \int \mathrm{d}t \, \mathrm{d}^3 x \, \epsilon^{\mu\nu\kappa\lambda} F_{\mu\nu} F_{\kappa\lambda} = \frac{\theta e^2}{4\pi^2 \hbar c} \int \mathrm{d}t \, \mathrm{d}^3 x \, \boldsymbol{E} \cdot \boldsymbol{B} \tag{6}$$

This "axion" term can be derived, similarly to the Chern–Simons action, by integrating out fermions in the presence of the background electromagnetic field. The Dirac quantization condition and time-reversal symmetry (TRS) restrict θ to be quantized, $\theta = 0, \pi \pmod{2\pi}$; $\theta = 0$ for trivial insulators or vacuum, whereas $\theta = \pi$ inside topological insulators.

As inferred from the axion term, the topological insulator features the topological magnetoelectric (ME) effect:

$$\boldsymbol{M} = \frac{\delta I_{\text{eff}}}{\delta \boldsymbol{B}} = \frac{\theta}{\pi} \frac{e^2}{2hc} \boldsymbol{E} = \frac{\theta}{\pi} \frac{\alpha}{4\pi} \boldsymbol{E} \tag{7}$$

$$\boldsymbol{P} = \frac{\delta I_{\text{eff}}}{\delta \boldsymbol{E}} = \frac{\theta}{\pi} \frac{e^2}{2hc} \boldsymbol{B} = \frac{\theta}{\pi} \frac{\alpha}{4\pi} \boldsymbol{B} \tag{8}$$

or, quantized electromagnetic polarizability,

$$\frac{\delta P_i}{\delta B_j} = \frac{\delta M_i}{\delta E_j} = \delta_{ij} \frac{\theta}{\pi} \frac{\alpha}{4\pi} \tag{9}$$

where $\alpha = e^2/(\hbar c)$ is the fine structure constant. Eqs. (8) and (7) are the 3d analogues of Eqs. (4) and (5), respectively.
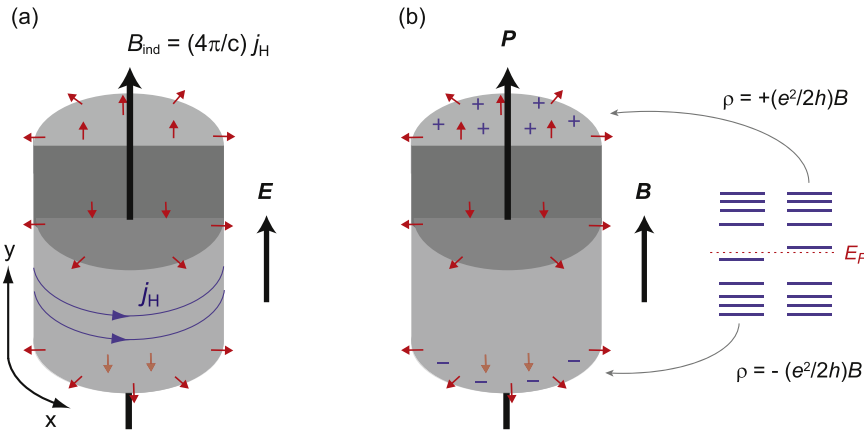
Fig. 1. The three-dimensional topological insulator in cylindrical geometry; (a) the induced magnetization follows the direction of the applied electric field; (b) the induced polarization follows the direction of the applied magnetic field. In both cases, time-reversal symmetry is broken on the surface, by putting, say, magnetic impurities, as indicated by red arrows. In (a), the magnetization is caused by the surface Hall current j_H. In (b), the polarization is caused by an excess or a deficit charge on the top or bottom surface [see Eq. (4)]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The magnetization M in Eq. (7), which follows the direction of the external electric field E, is generated by the surface QHE (Fig. 1(a)); when a topological insulator is in contact with a topologically trivial insulator (or simply vacuum), the θ-angle in the axion term jumps by π at the interface. As the axion term is the total derivative of the Chern–Simons term, such interface (where TRS is weakly broken) is accompanied by the half quantized surface Hall current with $\sigma_H = \pm e^2/(2h)$, which generates a bulk magnetization M. Similarly, when an external magnetic field B is applied, according to Eq. (4), it induces an excess or a deficit of charge on the surfaces that are orthogonal to B – this is the source of a bulk electric polarization P in Eq. (8) (Fig. 1(b)).

3. Gravitational response in topological superconductors

3.1. Gravitoelectromagnetism

Energy–gravity coupling   We now develop the thermal response theory of topological phases, with an eye, in particular, on applications to topological SCs. Our strategy here is, based on Luttinger's idea [37], to follow as closely as possible the linear response theory for the electric charge: when we study the charge response, we consider a small external probe field for the charge density, $H_I \sim \int d^d x \phi(\mathbf{x}) \rho(\mathbf{x})$. The same strategy can be adopted since energy is conserved; as in the charge response, we can consider an external (fictitious) source term that couples to the energy (Hamiltonian) density $\varepsilon(\mathbf{x})$, $H_I \sim \int d^d x \phi_g(\mathbf{x}) \varepsilon(\mathbf{x})/v^2$, where we have introduced v, which has the dimension of velocity to assign a proper dimension to ϕ_g (see below).

The external source ϕ_g can be thought of as a fictitious gravitational field in the presence of the Lorentz invariance, as it allows us to identify the energy as the mass, $\varepsilon(\mathbf{x}) = m(\mathbf{x})v^2$, where $m(\mathbf{x})$ is the mass density. For the gravitational theory of nature (not for Luttinger's fictitious gravity, which is a mere device to develop a linear response theory), v should be replaced by the speed of light c, whereas in condensed matter systems with emergent Lorentz symmetry, v is the "Fermi" velocity (see below for more discussion). The coupling of the system to the potential ϕ_g can then be written as $H_I \sim \int d^d x \phi_g(\mathbf{x}) m(\mathbf{x})$, where, from the analogy with electromagnetism, mass can be thought of as a "charge" coupled with a gravitational field.

With the Lorentz invariance, in more covariant language (in the Lagrangian language), the gravitational coupling with the (thermal) energy current is introduced as follows: the energy density ε and the energy current j_E are components of the energy–momentum tensor $T^{\mu\nu}$, $\varepsilon = T^{00}$ and $j_E^i = v T^{0i}$. They are thus coupled with the variation of space–time metric $g_{\mu\nu}$ as $-(1/2) \int dt\, d^d x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}$ in the Lagrangian, where $g = \det g_{\mu\nu}$ – in a way analogous to the way the charge current j^μ couples with the external electromagnetic potential as $- \int dt\, d^d x\, j^\mu A_\mu$.

Gravitoelectric field   A spatial gradient in energy density inevitably causes a temperature gradient, as one can infer from the thermodynamic equality $dU = T\, dS$ as follows. (Here, U is the internal energy, S the entropy, and T the temperature.) Let us divide the total system into two subsystems (subsystems 1 and 2). The equilibrium between the two is achieved when the total entropy is maximized: $dS = dS_1 + dS_2 = 0$. Since energy is conserved, $dE_2 = -dE_1$, and hence $dS_1/dE_1 - dS_2/dE_2 = 0$, i.e., $T_1 = T_2$. Let us now turn on a gradient in the gravitational potential, so that the gravitational potential felt by subsystems 1 and 2 differs by $\delta\phi_g$. In this case, we have $dE_2 = -dE_1(1 + \delta\phi_g/v^2)$. This suggests the generation

of a temperature difference $T_2 = T_1(1 + \delta\phi_g/v^2)$. In other words, we can view the "electric" field $\boldsymbol{E}_g$ associated with the gradient of ϕ_g, which we call "gravitoelectric field", as a temperature gradient:

$$\boldsymbol{E}_g := -\boldsymbol{\nabla}\phi_g = v^2 T^{-1}\boldsymbol{\nabla}T \tag{10}$$

Gravitomagnetic field The analogy with electromagnetism can be further put forward – such formalism is called gravitoelectromagnetism [38]. For our purpose to develop a linear response theory for thermal transport, we put an external gravity field which is infinitesimally small. We can thus write the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the metric of flat space–time. In the presence of matter, in the post Newtonian limit, we keep only h_{00}, h_{ii}, and h_{0i} [which are of order $\mathcal{O}(v^{-2})$], whereas $h_{ij} = \mathcal{O}(v^{-4})$ $(i \neq j)$ are neglected [38]:

$$ds^2 = v^2\big(-1 + 2\phi_g/v^2\big)\,dt^2 - \frac{4}{v}\boldsymbol{A}_g \cdot d\boldsymbol{x}\,dt + \big(1 + 2\phi_g/v^2\big)\,d\boldsymbol{x} \cdot d\boldsymbol{x} \tag{11}$$

where we have introduced the gravitomagnetic potential, $A_{g,i} := -(v^2/2)h_{0i}$. For small $h_{\mu\nu}$, the Einstein equation can be linearized up to these non-zero components of $h_{\mu\nu}$ kept in Eq. (11), and the resulting equation looks structurally identical to the Maxwell equation. (However, note that we are considering an external gravity field as a source that does not have its own dynamics.) As we have seen, the gravitoelectric field $\boldsymbol{E}_g$ corresponds to temperature gradient. What does the gravitational analogue of the magnetic field, the gravitomagnetic field, $\boldsymbol{B}_g$, correspond to? It turns out that $\boldsymbol{B}_g$ can be understood as the angular velocity vector of rotating systems:

$$\boldsymbol{B}_g := \boldsymbol{\nabla} \times \boldsymbol{A}_g = v\boldsymbol{\Omega} \tag{12}$$

For a system rotating with the frequency $\boldsymbol{\Omega} = \Omega^z \boldsymbol{z}$, this can be understood by making a coordinate transformation from the rest frame to the rotating frame in which the metric in the polar coordinates (t, r, φ, z) takes the form $ds^2 \simeq -v^2\,dt^2 + 2\Omega^z r^2\,d\varphi\,dt + r^2 d\varphi^2 + dr^2 + dz^2$. One can then read off, from the definition of the gravitoelectromagnetic field, the non-zero gravito gauge potential $A_g^\varphi = (v/2)\Omega^z r$. In Cartesian coordinates, $\boldsymbol{A}_g = (v/2)\Omega^z \hat{z} \times \boldsymbol{x}$, and $\boldsymbol{B}_g = \boldsymbol{\nabla} \times \boldsymbol{A}_g = v\Omega^z \hat{z}$.

3.2. Thermal Strěda formula for 2d topological SCs

We now use the formalism described above to study the thermal transport of topological SCs in two and three dimensions. The goal here is to establish a gravitational analogue of the Strěda formula (4)–(5) and the topological ME effect (7)–(8) [39].

Let us start with a 2d topological fluid that does not have time-reversal symmetry. To draw a parallelism with the charge response, we introduce the moment of the thermal current $\boldsymbol{M}_T$ as $\boldsymbol{M}_T = \boldsymbol{M}_E - (\mu/e)\boldsymbol{M}$, where $M_E^{\mu\nu} = \langle x^\mu T^{0\nu} - x^\nu T^{0\mu}\rangle$, and $M_E^z = M_E^{12}$. (Similarly, the thermal current can be decomposed into $\boldsymbol{j}_T = \boldsymbol{j}_E - (\mu/e)\boldsymbol{j}$, where $\boldsymbol{j}$ is the electric charge current.) Here the average has to be taken at finite temperature: $\langle \cdots \rangle \equiv \sum_n f(\varepsilon_n)\langle n|\cdots|n\rangle$, where ε_n and $|n\rangle$ are the eigenvalue and the eigenstate of the Hamiltonian $\mathcal{H}$, and $f(\varepsilon_n)$ is the Fermi distribution function.[1,2] We can then obtain

$$\kappa_H = \frac{v}{2}\frac{\partial M_T^z}{\partial T} \tag{13}$$

from the Kubo formula for the thermal Hall conductivity[3] [39].

To see the physical meaning of $\boldsymbol{M}_E$, note that the definition of $M_E^{\mu\nu}$ is similar to the orbital angular momentum: $L^{\mu\nu} = (1/c)\langle x^\mu T^{\nu 0} - x^\nu T^{\mu 0}\rangle$. Indeed, when there is a relativistic invariance, the energy–momentum tensor can be symmetrized so that $T^{\mu\nu} = T^{\nu\mu}$, and thus $M_E^{\mu\nu} = cL^{\mu\nu}$. While this is not the case in condensed matter systems in general, the (pseudo) Lorentz invariance can emerge in solids at low energies, such as in graphene or in topological superconductors where electrons or quasi-particles obey the Dirac or Majorana equation. In these systems, the Fermi velocity v plays a role of c, and the $M_E^{\mu\nu}$ tensor is related to the orbital angular momentum as $\boldsymbol{M}_E = v\boldsymbol{L}$. Thus, $\kappa_H = (v^2/2)(\partial L^z/\partial T)$. This is an analogue of the electromagnetic Strěda formula (5).

To derive "the other half" of the Strěda formula [a thermal analogue of Eq. (4)], we note that the variation of the free energy is given by $dF = -S\,dT - \boldsymbol{L} \cdot d\boldsymbol{\Omega}$, where S is the entropy. In terms of the gravitoelectric and gravitomagnetic fields,

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[1] Below, we will consider the part of the thermal current carried by $\boldsymbol{j}_E$ (and hence $\boldsymbol{M}_E$) specializing to the case of $\mu = 0$. This is valid in discussing topological SCs.

[2] Our convention for ϕ_g, $\boldsymbol{A}_g$, and $\boldsymbol{M}_E$ in this review differs from the one employed in Ref. [39]; there is a factor-of-two difference in defining ϕ_g and $\boldsymbol{A}_g$ [see Eq. (11)]. Accordingly, there is a similar factor of $1/2$ difference in defining $\boldsymbol{M}_E$. These factors are due to the spin 2 nature of gravitons, and are convenient, since the resulting linearized Einstein equation does not have such factors of two, and the similarity with the Maxwell equation is clearer. Also, in the definition of ϕ_g, we have put an extra factor of v^2.

[3] While in the bulk, in the absence of interactions, the Kubo formula gives us one of the most direct ways to get the Strěda formula for σ_H and κ_H, the applicability of the Strěda formulas is not limited to non-interacting systems, as they can be also derived assuming the presence of chiral edge states. See Ref. [39] for details.

this can be written as $dF = -(TS/v^2)(v^2T^{-1}\,dT) - (\boldsymbol{L}/v) \cdot d(v\boldsymbol{\Omega}) = -(Q_T/v^2)(d\phi_g) - (\boldsymbol{M}_E/v^2) \cdot d\boldsymbol{B}_g$, where $Q_T = TS$ is the thermal energy density, which couples with ϕ_g/v^2. From the Maxwell relation, we thus obtain

$$\kappa_H = \frac{v^3}{2T}\left(\frac{\partial M_E^z}{\partial \phi_g}\right)_{B_g^z} = \frac{v^3}{2T}\left(\frac{\partial Q_T}{\partial B_g^z}\right)_{\phi_g} \tag{14}$$

This is the thermal analogue of the Strĕda formula for the charge Hall conductivity, in that Q_T is the zeroth component of the energy current as eN_e is in the charge current. With the pseudo-relativistic invariance, this can be written as

$$\kappa_H = \frac{v^2}{2}\left(\frac{\partial L^z}{\partial T}\right)_{\Omega^z} = \frac{v^2}{2}\left(\frac{\partial S}{\partial \Omega^z}\right)_T \tag{15}$$

3.3. Cross-correlated response of 3d topological SC

The thermal Strĕda formula derived above for 2d topological fluid can be used to study the response of 3d topological SCs to the temperature gradient and the rotation; given that the fermionic quasi-particles are fully gapped in the bulk, all (topological) transport phenomena, in the presence of a boundary (surface), can essentially be discussed by looking at the surface transport. For the sake of simplicity, one can consider a cylindrical geometry (as in Fig. 1), apply the temperature gradient $\partial_z T$ or external rotation Ω^z, and discuss the responses, which are mediated by the surface. (As in the electromagnetic responses, we weakly break TRS at the surface and hence the surface is gapped.) Such thought experiments lead to, for the (induced) thermal polarization $\boldsymbol{P}_E$ defined by $\delta Q_T = -\nabla \cdot \boldsymbol{P}_E$, and for the moment of the energy current $\boldsymbol{M}_E$,

$$\boldsymbol{M}_E = \left(T\kappa_H/v^3\right)\boldsymbol{E}_g \tag{16}$$

$$\boldsymbol{P}_E = \left(T\kappa_H/v^3\right)\boldsymbol{B}_g \tag{17}$$

$$\frac{\delta M_{E,i}}{\delta E_{g,j}} = \frac{\delta P_{E,i}}{\delta B_{g,j}} = \delta_{ij}\frac{\theta}{\pi}\frac{T\kappa_H}{v^3} \tag{18}$$

The parallel between the electromagnetic and gravitational cases are obvious. Observe, however, that the gravitational response is not quantized as in the charge response, because of the presence of the velocity v. The situation is somewhat similar to the detection of the conformal anomaly (central charge) from specific heat in 1d quantum systems at criticality; while the central charge C for a given quantum critical system in 1d is a dimensionless universal parameter, it shows up in specific heat C_V together with the velocity v as $C_V = C\pi T/(3v)$.

Possible experiments The thermal (gravitational) analogue of the Strĕda formula [Eq. (15)] and of the topological ME effect [Eq. (18)] can be tested experimentally. An external angular velocity Ω^z results in the change in temperature (in 2d topological SCs) and thermal polarization (in 3d topological SCs). If the heat capacity of the system is sufficiently small, these may not be difficult to measure.

In three dimensions, the "dual" response, i.e., the response to the applied temperature gradient [Eq. (17)], can be detected by making use of the Einstein–de Hass effect. Let us assume a cylindrical 3d topological SC is suspended by a thin string and apply thermal gradient (as in Fig. 1(a)). This induces surface energy current with angular momentum L^z, according to Eq. (17). By the conservation law of the total angular momentum, it must be compensated by a mechanical angular momentum of the material, which can be directly measured in principle.

Note that in both responses to Ω^z and to temperature gradients, the part of the responses of our interest are contributions from the fermionic quasi-particles. They should be distinguished from the contributions from bosonic excitations such as vortices. If Ω^z is larger than the critical angular velocity Ω_{c1} above which vortices are introduced in the bulk of the sample, an extra contribution to thermal polarization would be generated.

4. Anomaly ladder and D-branes

4.1. Integrating out fermions and chiral anomaly

We have constructed the response theory of 3d topological SCs starting from the thermal Strĕda formula of 2d topological SCs. We now discuss the response of topological SCs (and insulators) to an external gravitational field from a field theoretical point of view [40,41]. Our goal here is to derive a gravitational analogue of the axion term in the electromagnetic response, and show that it is related to quantum anomaly (chiral anomaly). In this section, we will use natural units $e = c = \hbar = 1$, and set the Fermi velocity v to be unity for simplicity.

Let us work with an example; A canonical example of the 3d topological SC is the B phase of ^{3}He, which is described, in the momentum space, by the following BdG Hamiltonian: $H = (1/2)\int d^3k\,\Psi^\dagger(\boldsymbol{k})\mathcal{H}(\boldsymbol{k})\Psi(\boldsymbol{k})$, where $\Psi^\dagger(\boldsymbol{k}) = (c_{\uparrow,\boldsymbol{k}}^\dagger, c_{\downarrow,\boldsymbol{k}}^\dagger, c_{\uparrow,-\boldsymbol{k}}, c_{\downarrow,-\boldsymbol{k}})$ is the Nambu spinor composed of fermionic creation/annihilation operators ($c_{s,\boldsymbol{k}}^\dagger/c_{s,\boldsymbol{k}}$) of a ^{3}He atom with spin s and momentum $\boldsymbol{k}$, and the kernel $\mathcal{H}(\boldsymbol{k})$ takes the following form:

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} \xi(\boldsymbol{k}) & \Delta(\boldsymbol{k}) \\ \Delta^\dagger(\boldsymbol{k}) & -\xi(-\boldsymbol{k}) \end{pmatrix}, \quad \xi(\boldsymbol{k}) = \frac{k^2}{2M} - \mu, \ \Delta(\boldsymbol{k}) = |\Delta| \boldsymbol{k} \cdot \boldsymbol{s}(is_y) \tag{19}$$

where M is the mass of a ^{3}He atom, μ is the chemical potential, and $|\Delta|$ is the amplitude of the pair potential. With the $\boldsymbol{d}$-vector pointing parallel to momentum, $\boldsymbol{d}(\boldsymbol{k}) = |\Delta|\boldsymbol{k}$, there is an isotropic gap everywhere on the 3d Fermi surface. The critical point at $\mu = 0$ separates topologically trivial ($\mu < 0$) and non-trivial ($\mu > 0$) phases, which are characterized by an integral topological invariant of symmetry class DIII (the winding number) $\nu = 0$ and $\nu = 1$, respectively [23]. We will henceforth set $|\Delta| = 1$ and drop the $\mathcal{O}(k^2)$ term in $\xi(\boldsymbol{k})$, $\xi(\boldsymbol{k}) \to -\mu \equiv m$. With a suitable unitary transformation, the BdG Hamiltonian is written in terms of the 4×4 Dirac matrices $\{\alpha_{i=1,2,3}, \beta\}$ as $\mathcal{H} = -i\sum_{i=1}^3 \alpha_i \partial_i + m\beta$. Upon this linearization, μ can be thought of as a "Dirac mass term".

We introduce an external gravity field which couples to the Dirac Hamiltonian (Lagrangian) as, $S[\bar{\psi}, \psi, e] = \int dt \, d^3x \sqrt{-g}\mathcal{L}$,

$$\mathcal{L} = \bar{\psi} e_a{}^\mu i \gamma^a \left(\partial_\mu - \frac{i}{2} \omega_\mu{}^{cd} \Sigma_{cd} \right) \psi - m\bar{\psi}\psi \tag{20}$$

where $\mu, \nu, \ldots = 0, 1, 2, 3$ is the space–time index, and $a, b, \ldots = 0, 1, 2, 3$ is the flat index; $e_a{}^\mu$ is vielbein, and $\omega_\mu{}^{ab}$ is a spin connection; $\Sigma_{ab} = [\gamma_a, \gamma_b]/(4i)$. The effective action $I_{\text{eff}}[m, e]$ for the gravitational field is then obtained from the fermionic path integral $I_{\text{eff}}[m, e] = -i \ln \int \mathcal{D}[\bar{\psi}, \psi] \exp(iS[\bar{\psi}, \psi, e])$.

Below, we are interested in the topological term of the effective action, which is, in the imaginary time path integral, an imaginary part of the $S_{\text{eff}}[m, e]$. This part can be computed by making use of the chiral anomaly as follows [42]. We first observe that the continuum Hamiltonian $\mathcal{H}$ enjoys a continuous chiral symmetry: we can flip the sign of mass, in a continuous fashion, by the following chiral rotation $\psi \to \psi = e^{i\phi\gamma_5/2}\psi'$, $\bar{\psi} \to \bar{\psi} = \bar{\psi}'e^{i\phi\gamma_5/2}$, under which the mass is rotated as

$$m'(\phi) = me^{i\phi\gamma_5} = m(\cos\phi + i\gamma_5 \sin\phi) \tag{21}$$

so that $m'(\phi = 0) = m$ and $m'(\phi = \pi) = -m$. Since m can continuously be rotated into $-m$, one would think, naively, $I_{\text{eff}}[m] = I_{\text{eff}}[-m]$. This naive expectation is, however, not true, because of the chiral anomaly. The chiral transformation that rotates m continuously costs the Jacobian $\mathcal{J}$ of the path integral measure,

$$\mathcal{D}[\bar{\psi}, \psi] = \mathcal{J}\mathcal{D}[\bar{\psi}', \psi'] \tag{22}$$

This chiral anomaly (the chiral Jacobian $\mathcal{J}$) is responsible for the gravitational analogue of the axion term (the θ-term). The Jacobian $\mathcal{J}$ can be computed explicitly by the Fujikawa method [43] as

$$I_{\text{eff}}^\theta = -\ln\mathcal{J} = \frac{1}{2}\frac{1}{2}\frac{\theta}{384\pi^2} \int d^4x \sqrt{-g}\epsilon^{cdef} R^a{}_{bcd}R^b{}_{aef} \tag{23}$$

where $R = d\omega + \omega \wedge \omega$ ($R^a{}_{b\mu\nu} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + [\omega_\mu, \omega_\nu]^a{}_b$) is the Riemann curvature tensor; as in the electromagnetic response, the θ-angle is fixed to either $\theta = 0$ or $\theta = \pi$ by time-reversal symmetry. The former corresponds to a topologically trivial state, and $\theta = \pi$ to a topologically non-trivial state. The theta term in the gravitational effective action (23) ("the gravitational instanton term") is an analogue of the axion term $\propto \theta \boldsymbol{E} \cdot \boldsymbol{B}$ in the electromagnetic effective action; there is an obvious structural parallelism between the electromagnetic and gravitational cases [44].

To make the connection with the existence of topologically protected surface modes, we note, when there are boundaries (say) in the x^3-direction at $x^3 = L_+$ and at $x^3 = L_-$, the gravitational instanton term I_{eff}^θ, at the non-trivial time-reversal invariant value $\theta = \pi$ of the angle θ, can be written in terms of the gravitational Chern–Simons terms at the boundaries, $I_{\text{eff}}^\theta = I_{\text{CS}}|_{x^3=L_+} - I_{\text{CS}}|_{x^3=L_-}$, where $(i, j, k = 0, 1, 2)$

$$I_{\text{CS}} = \frac{1}{2}\frac{1}{4\pi}\frac{c'}{24} \int d^3x \, \epsilon^{ijk} \text{tr}\left(\omega_i \partial_j \omega_k + \frac{2}{3}\omega_i\omega_j\omega_k \right) \tag{24}$$

with $c' = 1/2$. This value of the coefficient of the gravitational Chern–Simons term is one-half of the canonical value $(1/4\pi) \times (c'/24)$ with $c' = 1/2$. As discussed by Volovik [45] and by Read and Green [21] in the context of the 2d chiral p-wave SC, the coefficient of the gravitational Chern–Simons term is directly related to the thermal Hall conductivity, which in our case is carried by the topologically protected surface modes.

4.2. Anomaly ladder and periodic table

All types of responses we have discussed so far (the QHE, the topological ME, and their gravitational (thermal) analogues) are related to the quantum anomaly; the Chern–Simons term in the QHE (both electromagnetic and gravitational) is a manifestation of the parity anomaly; the axion term can be derived, as we have seen above, from the chiral anomaly. In fact, a wider class of topological insulators and SCs in the periodic table (Table 1) can be related to quantum anomalies.

Table 1

Periodic table of topological insulators and superconductors for the 10 symmetry classes in various spatial dimensions. Topological insulators (superconductors) in the complex symmetry classes (A and AIII) are related to the chiral U(1) anomaly. The primary series of the topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification in the eight real symmetry classes are located on the diagonal in the table. In even space–time dimensions (odd space dimensions) they are predicted from the chiral anomaly in the presence of background gravity ($\mathbb{Z}^\diamond$), and from the chiral anomaly in the presence of both background gravity and U(1) gauge field ($\mathbb{Z}^\clubsuit$). The topological response of topological phases in odd space–time dimensions ($\mathbb{Z}^\heartsuit$ and $\mathbb{Z}^\spadesuit$) follows from their higher-dimensional ancestor ($\mathbb{Z}^\diamond$ and $\mathbb{Z}^\clubsuit$), respectively. (For interpretation of color symbols in this table, the reader is referred to the web version of this article.)

Symmetry class\d	0	1	2	3	4	5	6	7	⋯
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	⋯
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	⋯
AI	$\mathbb{Z}^\spadesuit$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	⋯
BDI	$\mathbb{Z}_2$	$\mathbb{Z}^\clubsuit$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	⋯
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^\heartsuit$	0	0	0	$2\mathbb{Z}$	0	⋯
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^\diamond$	0	0	0	$2\mathbb{Z}$	⋯
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^\spadesuit$	0	0	0	⋯
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^\clubsuit$	0	0	⋯
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^\heartsuit$	0	⋯
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^\diamond$	⋯

Let us consider topological insulators and SCs with an integer topological invariant located on the diagonal of the periodic table (Table 1), which we call the "primary series". (The $\mathbb{Z}_2$ topological phases which "descend" from the primary series can be called 1st and 2nd descendants [34].) For the primary series, the effective action for the electromagnetic and gravitational response can be derived essentially by repeating the procedure discussed above; taking a Dirac representative in $D = d + 1 =$ even, coupling it to the background electromagnetic and gravitational fields, and then integrate out fermions; the chiral anomaly in $D = d + 1 =$ even dimensions allows us to calculate the imaginary part of the action (i.e., the part that encodes topological part of the response). The result is summarized as follows:

$$\delta \ln Z = 2\pi i \int_{M_D} \delta\theta \, \mathrm{ch}(E) \hat{A}(R) \Big|_D \tag{25}$$

$\delta \ln Z$ represents the change in the effective action under the infinitesimal chiral transformation $\delta\theta$; by integrating it one gets the topological action for the linear response. $\mathrm{ch}(E)$ denotes the Chern character of the vector bundle E, which is explicitly given by $\mathrm{ch}(E) = \mathrm{tr}[e^{F/(2\pi)}]$ in terms of its field strength F. $\hat{A}(R)$ is the A-roof genus and takes the form:

$$\hat{A}(R) = 1 + \frac{1}{192\pi^2} \mathrm{tr}[R^2] + \cdots \tag{26}$$

where R is the curvature two-form on the manifold M_D. Since (25) measures the number of chiral fermion zero modes minus that of anti-chiral ones as follows from the chiral rotation, (25) is equivalent to the index theorem in mathematics.

Topological terms for $d =$ even can be derived from the topological terms in $d + 1$ dimensions considered above: they are all Chern–Simons type and obtained as a boundary contribution from a $(d + 1)$-dimensional topological term.

One could check that the topological response derived from the chiral anomaly is fully consistent with the periodic table; (i) the topological terms derived in this way preserve correct discrete symmetries for primary series; (ii) for symmetry classes that are realized as topological SC (i.e., no charge conservation), the anomaly polynomial predicts that there is no topological response for the EM field; only a gravitational (thermal) response exists.

One could think of this as an alternative "derivation" of the periodic table (for the primary series).

4.3. Topological phases and D-branes

Introduction We would now like to point out an interesting connection between topological phases in condensed matter and D-branes. D-branes are topologically stable objects in string theory; at the level of classical (super)gravity, which is a low-energy effective field theory of string theory, D-brane can be visualized as a $(p + 1)$-dimensional solitonic solution to the 10d Einstein equation (such D-brane is called Dp-brane). Besides such geometrical attribute in gravity theory, an important property of D-branes for our purposes is the fact that open string excitations on D-branes give rise to a gauge field theory. This dual nature of D-branes has been proved to be a useful tool to understand non-perturbative phenomena in gauge field theories, such as monopoles, Seiberg–Witten theory [46], etc. (Such dual nature of D-branes also plays a key role in AdS/CFT or holography. For our discussions in this article, however, we will not use AdS/CFT.)

We will describe below what D-branes are, how we can construct topological phases from D-branes, and why they give us an insight on the periodic table. In fact, the opposite is also true; topological phases deepen our understanding of the

Table 2

Dp-brane charges from K-theory, classified by $K(\mathbb{S}^{9-p})$, $K^{-1}(\mathbb{S}^{9-p})$ and $KO(\mathbb{S}^{9-p})$. A $\mathbb{Z}_2$ charged Dp-brane with p even or p odd represents a non-BPS Dp-brane or a bound state of a Dp and an anti-Dp-brane, respectively.

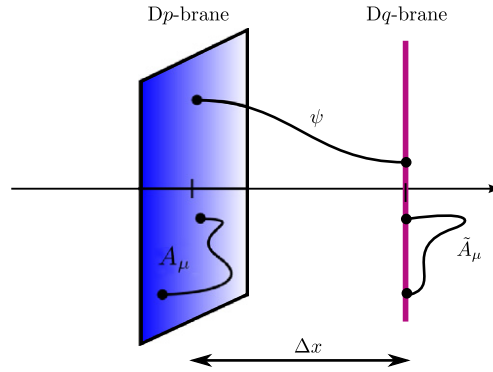|	D(-1)	D0	D1	D2	D3	D4	D5	D6	D7	D8	D9
Type IIB	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
Type IIA	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
O9$^-$ (type I)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Fig. 2. A typical D-brane configurations that realize a topological insulator (superconductor).

stability of D-branes. As we will argue below, the stability of the so-called "non-BPS" D-branes is directly related to the $\mathbb{Z}_2$ topological index of topological insulators.

K-theory charge of D-branes As mentioned above, D-branes are a topologically stable object in sting theory; Dp-brane is a $(p+1)$d solitonic object in 10d space–time of classical (super)gravity theory. What is behind the stability of D-branes is the fact that there is a "charge" associated with them. These charges are quantized, and hence they cannot change for a smooth deformation of field configurations; D-branes are thus stable.

The charge of a stable D-brane can be either integer- or $\mathbb{Z}_2$-valued, depending on the types of D-branes and sting theory. This is summarized in Table 2. Observe that for type-IIA and -IIB string theory, "0" and "$\mathbb{Z}$" appear in an alternating fashion. On the other hand, for type-I sting theory, the way K-theory charges "0", "$\mathbb{Z}$" and "$\mathbb{Z}_2$" appear closely follows the Bott periodicity. Compare entries "type IIB" and "type IIA" in Table 2 with "complex" symmetry classes (A and AIII) in Table 1, and entries "type I" in Table 2 with eight "real" symmetry classes in Table 1. In fact, it was argued that D-brane charges are classified by K-theory [47–49]; one then cannot help speculating on a possible connection between D-branes and the periodic table of topological phases.

For some cases where a D-brane has an integral charge, D-branes are an electric or a magnetic source of an Abelian p-form gauge field $C_{(p)} = (1/p!)C_{(p)}^{\mu_1\mu_2\cdots\mu_p}\,dx_{\mu_1}\,dx_{\mu_2}\cdots dx_{\mu_p}$, the so-called "Ramond–Ramond" (RR) gauge field. An integral of the RR-gauge flux generated by a magnetic D-brane, $\int_S dC_{(p)}$, measures the charge of the D-brane, where S is a hypersurface that encloses the D-brane. For D-branes with a $\mathbb{Z}_2$ charge, while we do see K-theory charge exist, it is not possible to write it down as a quantized integral of the RR flux.

D-brane configurations for topological phases For our purposes, we consider D-brane configurations that consist of two types of D-branes, a Dp-brane and a Dq-brane. They are located in parallel and do not intersect (Fig. 2) [50–52]. One can then ask what kind of (gauge) field theory is realized in such D-brane configuration. In string theory, quantum fields are realized as a vibration of a fundamental string.

In our configuration, an open string can have its end points on D-branes. Let us first consider a string which has one end on a Dp-brane and the other on a Dq-brane. Analyzing the vibrations of such string, one finds a massive fermion in the string spectrum. The mass of the fermion is proportional to the distance between the D-branes. For a relativistic fermion which is fully gapped by a mass term, following the periodic table of topological phases, one can discuss (under a suitable set of discrete symmetries) its topological stability against adiabatic deformation; one can "compute" its topological invariant or topological charge. On the other hand, for the D-brane configuration at hand, if it is stable, one can assign a topological (K-theory) charge to it, following the K-theory charge of D-branes. One can check that these two types of topological charges agree precisely. This is the first implication of the correspondence between the periodic table and D-branes.

The Dirac fermion is not the only quantum field realized in the D-brane configuration. Let us now turn our attention to a string that has its both ends on the Dp-brane. Such string vibration gives rise to a gauge field A_μ living on the D-brane. (Its gauge group depends on the type of sting theory and D-branes.) What is the dynamics of such gauge field on the

D-brane? This is answered by the effective action of D-branes. The topological part (the so-called "Wess–Zumino term") of the effective action for a Dp-brane in a flat space is[4]

$$S_{WZ} = \sum_q \int_X C_{(q+1)} \wedge \mathrm{ch}(E) \hat{A}(TX) \tag{27}$$

where integration ($\int_X$) is over the $(p+1)$d world-volume X of the Dp-brane.[5] In this action, $C_{(q+1)}$ is the background $(q+1)$-form RR-gauge field (which is, in our situation, sourced by the Dq-brane). On the other hand, $\mathrm{ch}(E)$ and $\hat{A}(TX)$ depends on the field configuration on the Dp-brane; the U(1) gauge bundle on the brane is denoted by E; TX and NX are the tangent bundle and normal bundle. Plugging the RR-field generated by the Dq-brane, whose integral $\int dC_{(q+1)}$ is quantized as it measures the topological charge of the Dq-brane, the Wess–Zumino term of the Dp-brane recovers precisely the response theory that we discussed in terms of anomaly.

As an example, let us consider the case with $p = q = 5$[6,7]; this D-brane configuration realizes the 2d QHE. The vibration of an open sting stretching between the D-branes gives rise to a $(2+1)$d massive Dirac fermion, $\mathcal{H} = \psi^\dagger(-i\sum_{i=x,y}\sigma_i\partial_i + m\sigma_z)\psi$ where ψ is the two-component complex Dirac fermion field. On the other hand, the WZ action of Dp-brane is given by the Chern–Simons term:

$$\begin{aligned} S_{WZ} &= \frac{1}{2(2\pi)^2} \int_X C_{(2)} \wedge F \wedge F = \frac{1}{2(2\pi)^2} \int_{Y\times Z} C_{(2)} \wedge d(A \wedge F) \\ &= \frac{-1}{8\pi^2} \int_Z dC_{(2)} \int_Y A \wedge F = \frac{\pm m}{8\pi|m|} \int_Y A \wedge dA \end{aligned} \tag{28}$$

Here, the integral is over the 6d world volume of the Dp-brane, which we split into the common directions of the Dp- and Dq-branes (Y) and the compliment thereof (Z), and we noted that the Dq-brane couples magnetically to the RR two-form $C_{(2)}$, and hence the integral $(2\pi)^{-1}\int dC_{(2)} = \pm 1/2$ measures the RR-charge of the Dq-brane. (We have done a partial integration.) The sign $\pm$ in front of the Chern–Simons term corresponds to $\mathrm{sgn}\,\Delta x = \pm 1$ (see Fig. 2). Similarly, from the WZ coupling $S_{WZ} = \int_X C_{(2)} \wedge \hat{A}(R) = \frac{1}{12}\int_X C_{(2)} \wedge \frac{iR}{4\pi} \wedge \frac{iR}{4\pi}$, we obtain the gravitational Chern–Simons term $\sim \omega \wedge d\omega + \frac{2}{3}\omega^3$, as expected from the responses.

Observe the structural parallelism between the topological terms in the response theory of topological phases, and the Wess–Zumino action of D-branes. In the former, the effective action looks, typically, as $I_{eff} \propto$ (topological invariant) $\times \int d^{d+1}x$ (topological term in gauge theory), where the "(topological invariant)" is the topological invariant of the topological phase in question, and the "(topological term in gauge theory)" is the term of topological origin in gauge theories such as the Chern–Simons term, or the axion term. For example, $I_{eff} \propto \mathrm{Ch} \times \int d^3x \, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ in the QHE, where Ch is the TKNN integer. In the D-brane construction of topological phases, the coefficient in front of the topological term is given by the integral of the RR-field, and measures the K-theory charge of the Dq-brane [54,55].

5. Conclusion

We have described, in the order of increasing spatial dimensions, from two, three, and to arbitrary dimensions, the theory of thermal response in topological phases. Emphasized is a close analogy to electromagnetic responses of topological insulators by adopting the language of gravitomagnetism, which revealed a cross-correlation between thermal and mechanical (rotational) responses.

Let us close with a discussion on the effects of interactions. The fact that topological currents (either in electromagnetic or thermal response) are related to anomalies in field theories suggests that topological phases with topological currents should be stable against interactions. This can be seen simply by observing that a term of topological origin in the effective action, once it exists, has its coefficient which is quantized (in some case in the presence of some discrete symmetry). Thus, small interactions should not destroy topological properties of a given topological phase.[8]

We have computed the theta term (both electromagnetic and gravitational ones) by making use of chiral anomaly (which arises as a Fujikawa Jacobian). Let us imagine repeating the same calculations in the presence of interactions such as $\varphi\bar{\psi}\psi$ (a coupling to a bosonic scalar field φ), or electromagnetic interactions. The Adler–Bardeen theorem says that the anomaly (the Fujikawa Jacobian) will not be altered, even in the presence of such interactions. (Historically, this non-renormalization

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[4] For the discussion below, we will focus on the situation where the topological charge of D-branes is an integer, for presentational simplicity.

[5] This is obtained from a more general expression (29) by noting that in our situation $TX \otimes NX$ is a trivial bundle, and hence $\hat{A}(TX \otimes NX) = \hat{A}(TX) \wedge \hat{A}(NX) = 1$.

[6] Since we can apply the T-duality equivalence which shifts the value of p and q by one [53], we can fix the values of p, say $p = 5$.

[7] This configuration has 6 ND directions.

[8] An alternative derivation of the electrical/thermal Strěda formula mentioned in Footnote 3 also provides another link between topological currents and anomalies, suggesting the stability of topological currents against interactions.

of the chiral anomaly was important to predict the number of quark colors, even at the time when the details of the strong interaction were not known.)

Insensitivity of anomalies to interactions can further be illustrated by the "anomaly matching condition" proposed by 't Hooft; an anomaly (i.e., topological current) can be computed in terms of either the infra-red (IR) or ultra-violet (UV) degrees of freedom, and the results should be the same. For example, the degrees of freedom in solids at UV are of "free-fermion" type, while deep in the IR region such description can be replaced by quasi-particles (such as excitons) that arise due to interactions.

The discussion above naturally echoes in the D-brane construction of topological phases; the field theories realized by Dp–Dq systems come with, in addition to massive fermions we discussed, other fields and interactions among them. As we can understand more or less geometrically, the topological phases are nevertheless stable.

While we have discussed the stability of non-interacting topological phases against interactions, it remains largely an open problem what is the nature of strongly interacting topological phases that arise solely because of interactions, if they exist at all beyond the fractional quantum Hall effect. The existence of a topological current, however, should be a hallmark that we can use to characterize even for these putative "fractional topological insulators".

Acknowledgements

Appendix A. A short course for D-branes

A.1. What is a D-brane?

In string theory [53], we consider a string as a fundamental object instead of an elementary particle. The vibration of a closed string produces, among others, the gravitational field at low energies. On the other hand, the vibration of an open string, which looks topologically like an interval with two ends, produces, among others, gauge fields in the low-energy limit.

A D-brane is defined as an object where open strings can end on it [53,56]; a D-brane can be considered as an object giving a boundary condition to string vibrations – the name D-brane originates from "Dirichlet" brane.

In terms of the 10d gravity theory, which is the low-energy limit of string theory, D-branes are a stable solitonic object, which is quite massive. A Dp-brane extends in p-spatial directions and in time-direction; i.e., its world volume is $(p+1)$d. For example, a D0-brane and a D1-brane look like a particle (called D-particle) and a string (called D-string), respectively.

For our purposes, we will note the following properties (see Ref. [56]): (i) On a D-brane, an Abelian gauge theory is realized. The fluctuations of the gauge field correspond to the fluctuations of the end points of open string. If we have N coincident Dp-branes, we obtain a non-Abelian U(N) gauge theory. (ii) An intersection of two D-branes realizes massless fermions. (iii) D-branes are characterized by a K-theory charge. The last point will further be discussed below.

A.2. K-theory classification of stable D-branes

Gravitons and gauge fields are not the only fields that arise in superstring theory. In addition to them, the vibration of a closed string generates Abelian p-form gauge fields, the Ramond–Ramond (RR) gauge fields. (Here, we are focusing for a moment on a particular type of superstring theory, type-IIA and -IIB superstring theory (with or without orientifolds).) Supersymmetric (or BPS) Dp-branes are charged under these gauge fields; they have the RR-charges, and hence are stable. A Dp-brane directly (electrically) couples to the $(p+1)$-form RR-field $C_{(p+1)}$. An anti-Dp-brane is defined to be the one with a negative RR-charge. In type IIA superstring, p takes only even integer values i.e., $p = 0, 2, 4, 6, 8$ and in type IIB, p takes only odd ones $p = -1, 1, 3, 5, 7, 9$.

Moreover, a Dp-brane couples to other RR q-forms with $q < p$ in the presence of the gauge flux on the brane. This is clearly described by the following formula of the RR couplings of a Dp-brane [57]:

$$S_{RR} = \sum_q \int_X C_{(q+1)} \wedge \text{ch}(E_D) \sqrt{\frac{\hat{A}(TX)}{\hat{A}(NX)}} \tag{29}$$

where TX and NX are the tangent bundle (i.e., the vector bundle tangent to the D-brane world-volume X) and normal bundle (i.e., the vector bundle which is normal to TX). The U(1) gauge bundle on the brane is denoted by E_D. This formula (29) means that if there is a non-trivial gauge bundle on the Dp-brane, it is possible that there exist charges which correspond to lower-dimensional D-brane charges. Such a configuration can be interpreted as a bound state of a Dp-brane and the lower-dimensional D-branes.

In fact, there are D-branes that do not have any RR-charges and are nevertheless stable. They are not supersymmetric and are called non-BPS D-branes [58]. Also a system of a Dp-brane and an anti-Dp-brane sometimes forms a stable bound state. Such a system is called a brane anti-brane system [59]. They typically exist in the presence of the special projection called the orientifold projection. These brane configurations exhaust all possible D-branes in string theory. The orientifold projection means that we require the invariance of string theory under the action $\tilde{\Omega}_q = I_q \cdot \Omega$, defined by the product of the parity I with respect to q-spatial coordinates and the orientation reverse Ω of the string world-sheet. The set of fixed points of I_q is called the orientifold $(9 - q)$ plane.

Being a stable object in string theory, one could imagine what is protecting them from "decay". It turns out that one can assign a K-theory charge to D-branes, which is the reason for stability. Indeed, the K-theory provides a very systematic classification of D-branes in string theory [48,49]. In mathematics, (topological) K-theory classifies vector bundles. More precisely, we start from a pair of two bundles (E_1, E_2) on a manifold X and consider its difference. In other words, we introduce the identification:

$$(E_1 \oplus H, E_2 \oplus H) \simeq (E_1, E_2) \tag{30}$$

This defines the K-group $K(X)$.

In string theory, this identification is naturally interpreted as follows. We start with a brane anti-brane system. The gauge bundles on the D-brane and the anti-D-brane are regarded as E_1 and E_2. Typically, a brane anti-brane system becomes unstable because the total RR-charge is vanishing and it can pair-annihilate. In other words, there appears a so-called tachyon field in the open string between the brane and the anti-brane [60]. The tachyon field has unstable potential energy and condenses like the Higgs effect, which makes the system decay into a lower-dimensional D-brane. This procedure is mathematically described by (30), which means that the charge is conserved under the tachyon condensation. If the gauge field configurations are the same i.e., $E_1 = E_2$, then the brane and anti-brane are completely annihilated, and nothing remains after the tachyon condensation. If $E_1 \neq E_2$, then eventually the system decays into a D-brane, which corresponds to the difference between E_1 and E_2. We presented the K-theory classification of D-branes in type-IIA, type-IIB and type-I string theory, where we take $X = \mathbb{S}^{9-p}$ for a Dp-brane via a compactification procedure. Notice that type-I string theory is defined to be the projection of type-IIB string theory by an (SO type) orientifold 9-plane. For type IIA and type I, we need to use a different K-theory called $K^{-1}(X)$, which just shifts the dimension by one, and $KO(X)$, which is the real valued version of $K(X)$.

In particular, if we ignore the torsion of the K-group $K(X)$, then it is known that $K(X)$ is reduced to even-dimensional cohomology $\bigoplus_{i \geqslant 0} H^{2i}(X, \mathbb{Q})$. Indeed, this is explicitly given by the Chern character and this nicely matches with the RR coupling formula (29) [47]. On the other hand, the argument based on K-theory with torsion is quite general and includes the case where the D-branes do not have any RR-charges as is so for the non-BPS D-brane.

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