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Electromagnetic modelling/Modélisation électromagnétique

Probabilistic modelling of stochastic interactions between electromagnetic fields and systems

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Abstract

Rigorous deterministic models of physical phenomena are important for understanding the fundamental characteristics of interactions, but if we neglect the uncertainties in the various components of these models, the correspondence between predictions based on such models and real-life situations can be very weak. This article addresses the important question of how to account, rigorously, for uncertainties in classical macroscopic electromagnetic interactions between fields and systems of linear material (considering uncertainties in the field incident on the system as well as in the geometrical realisation of the system itself). In the first part, we develop the general ideas which lead to the expression of the variances of the observables of the problem in terms of field- and system covariance operators. In the second part, we study the covariance operator which characterises a stochastic system. In the third part, we investigate stochastic electromagnetic fields and present the definition of a canonical stochastic field of which the spatial covariance operator has the real part of the Green tensor function as kernel function. In the final part, we present an illustration by studying the interaction of a stochastic plane wave with a stochastic twisted pair of wires. ***To cite this article: B.L. Michielsen, C. R. Physique 7 (2006).***
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Résumé

Modélisation probabiliste des interactions stochastiques entre champs électromagnétiques et système. Pour comprendre la nature des interactions physiques, il est important d'avoir des modèles rigoureux et déterministes, par contre, si l'on néglige les incertitudes des différentes paramètres de ces modèles la correspondance entre prédictions basées sur ces modèles et les situations pratiques peut être très faible. Cet article traite de la question importante à savoir comment prendre en compte, rigoureusement, les incertitudes dans l'interaction macroscopique entre champs électromagnétiques et systèmes de matériaux linéaires (en considérant à la fois les incertitudes dans le champ et celles de la réalisation du système). Dans une première partie, les idées générales sont développées conduisant aux expressions des variances d'observables en termes des opérateurs de covariance du champ et du système. Dans la deuxième partie, l'opérateur de covariance qui caractérise le système est étudié en plus de détail. La troisième partie présente l'étude du champ stochastique canonique, qui est telle que sa covariance spatiale est proportionnelle à la partie réelle de la fonction de Green tensorielle du problème électromagnétique de l'environnement. Dans la dernière partie de cet article, un exemple de l'interaction entre une onde plane stochastique et une paire de fils torsadée à géométrie stochastique est élaboré en tout détail. ***Pour citer cet article : B.L. Michielsen, C. R. Physique 7 (2006).***
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1. Introduction

The theory we develop in this article applies to 'observables' in macroscopic, linear time-harmonic electromagnetic interaction theory. Examples of such observables are, in antenna theory, the voltage on the port of a receiving antenna placed in some incident electromagnetic field, in ElectroMagnetic Compatibility (EMC), an interference source induced in an interconnect system due to the presence of some unforeseen electromagnetic source in the system's environment or again, in scattering theory, a scattering coefficient corresponding to the amplitude of a wave in a scattered field expansion due to some field incident on a linear scatterer. In all these cases, we can express the observable as a linear form, determined by the antenna, the interconnect system or the scatterer etc., on an 'incident' field as it would exist in absence of the antenna, the interconnect system, the scatterer etc. In order to fix ideas, we shall present our development in the context of electromagnetic compatibility theory, where the need to account for uncertainties in both the system and the incident field is most obvious. This means that we shall concentrate on the characterisation of interference sources induced in multi-port systems built of linear conductors and dielectrics. In particular, we want to show how uncertainties in the system geometry and in the incident field influence the strength of these interferences. (Of course, this also serves to characterise noise sources in antennas, etc.).

The interaction between electromagnetic fields and electronic systems plays an important role in every-day technological applications. Ever since the Maxwell equations have been established, computational models have been elaborated to describe the behaviour of real-life phenomena and to predict the essential characteristics of them. The usefulness of such models is two-fold: in the first place, the model improves the understanding of the physical phenomena and, in the second place, the model makes possible the study of the behaviour of systems, relying on these phenomena, without having to realise them. That means, if we are able to make reliable predictions on the behaviour of a physical system by computation, we can make faster and safer technological progress.

The truth of the story is that actual technological progress is built on incomplete models. Maxwellian models have been simplified into Low-Frequency circuit-theory models and transmission-line models, for example, and all other sorts of pragmatical hypotheses have been introduced to cope with the computational limits of man and computing machines, however powerful they may be these days. In many cases, these approximate models are sufficiently precise for the purpose they were derived for, i.e., to describe the system's behaviour under certain idealised working conditions complying with the hypotheses underlying the approximation. The typical situation, however, is that the actual operational context of the system is much richer than these idealised conditions allow for. It is therefore often a matter of good-luck that the system works correctly in many different practical circumstances. This may be due to a certain fault tolerance built into the system, implicitly or explicitly.

The heart of research in ElectroMagnetic Compatibility (EMC) consists of handling situations where systems have to work in operational contexts their design did *not* sufficiently account for. The immediate reaction to such situations is to change the system by adapting it to the new situation. A more thoughtful approach would be to change (more precisely to *extend*) the models designers use when conceiving their systems. This model extension should have two stages: a deterministic one and a probabilistic one. In the deterministic extension, one includes representations of the system's true operational conditions into the approximations to the (supposedly) complete Maxwellian models. In the probabilistic extension, one goes one step further and one also models the uncertainties which exist in both the environment the system has to operate in, and the physical-geometrical realisation of the system itself. (Of course, similar extensions are useful in other domains than EMC, any model wanting to be realistic should be extended in some way or another to account for uncertainties in the realisation.)

There exists a vast literature on the modelling of the interaction of electromagnetic fields and systems, in which induced perturbations are described through a great number of different approaches, theoretical models and numerical computation techniques (see [1] for further references). The majority of these studies, though, is limited to the application of classical, deterministic, electromagnetic theory. Models of stochastic fields have been proposed in the past and the statistical point of view has been elaborated in much detail (see [2]). In particular, the statistical description of fields in so-called Mode Stirred Chambers (MSC) has been the subject of many investigations (see [3] and, for a probabilistic approach, [4,5]). Much less work has been done, though, on the interaction with stochastic *systems*. The case of stochastic cable bundles seems to be the most accessible and has been investigated from the statistician's point of view in [6–8]. There is an early sketch of ideas for the analysis of a probabilistic interaction theory with stochastic cable bundles (see [9]), but the proposed way to use reciprocity does not seem to be the most convenient one. Alternatively, rigorous, but non-probabilistic methods have been developed such as those related to sensitivity analysis (see [10]).

A complete probabilistic formulation of the theory of interaction between stochastic fields and stochastic systems seems to be lacking today. It is the purpose of this paper to establish a fundamental approach to this important problem area.

1.1. Outline of the article

As announced above, this article presents an analysis of a rigorous way to make the deterministic and probabilistic extensions to macroscopic electromagnetic interaction models as they are used in ElectroMagnetic Compatibility theory. For that purpose, we shall concentrate on the time-harmonic electromagnetic interaction with linear multi-port systems, which is the most frequently used interaction model for ElectroMagnetic Compatibility problems.

In Section 2, we recall the basic multi-port models used in electronic system design. Then we single out the Thévenin model and concentrate on the Thévenin source vector for illustrating the interaction theory. The dominant statistics, being the mean and variance (or standard deviation), can be readily expressed in terms of mathematical objects associated to the stochastic system, i.e., the system with a probabilistic model for its geometry and physical constitution, and the stochastic fields incident on the system.

This theoretical development identifies the kind of objects we have to compute for the extended models. On the system side, the probabilistic model of the system should result in an average current distribution and a covariance operator of a stochastic current distribution with zero mean. On the incident field side, we need a probabilistic model for the ambient field, from which the average field and the field's covariance operator can be derived.

In Section 3, we introduce the basic formulation of a system with a stochastic geometry. There remains much work to be done on the computation of stochastic distributions, but a non-trivial case can readily be studied using a zeroth order approximation for the fluctuating values of the distribution and a first-order approximation in its stochastic support combined with a Taylor expansion of the test functions it evaluates on. This leads to the simplest representation of a non-trivial stochastic distribution representing the response of a stochastic system to an incident field. In this case, we can derive explicit expressions for the system's covariance operator in terms of the probabilistic model for its geometry.

In Section 4, we present a definition of canonical stochastic electromagnetic fields in reciprocal lossless configurations. This is a complement to earlier results in [11], where we presented a constructive proof for the existence of stochastic electric fields of which the spatial covariance function is proportional to the real part of the 'electric current to electric field' part of the Green tensor function of the Maxwell equations. This section, presents the complete electromagnetic covariance operator, including cross-covariances between the electric and magnetic fields, of such canonical fields.

In Section 5, we combine the system covariance operator of Section 3 and the field covariance operator of Section 4, in the expressions derived in Section 2, to compute the variance of the voltage induced on the terminal ports of a stochastic twisted pair of wires in a stochastic isotropic plane wave, which is the canonical field in free space. The results show some typical aspects of how probabilistic modelling can fill-in the gap between conventional deterministic modelling and practical situations.

2. Basic interaction theory

In the linear theory of electromagnetic interactions the electromagnetic field itself is not considered an observable. The field is only there to describe the way certain distributions of matter behave under the remote influence of each other. In this paper, we shall be concerned with electronic systems applications and we shall consider the coefficients of the linear multi-port models, relating the voltages and currents on the ports, as observables. In a certain way, these coefficients are the measurable responses of the system when we apply internal or external excitations to the system.

The most general representation of a linear observable in electromagnetic field theory has the form of the evaluation of two 'current' distributions on the *incident* electric and magnetic field, as they would exist in absence of the system,

$$A = \langle j, E^i \rangle + \langle k, H^i \rangle$$

The two current distributions, j (electric current) and k (magnetic current), which represent the system, correspond to solutions of specific boundary value problems (BVP) for the Maxwell equations. The precise definition of these BVPs depends on the exact observable one is interested in. Generally speaking, these are problems where sources are imposed on the responsive system itself (internal excitations).

The above representation is a not-so-direct consequence of the field reciprocity relations. It may be worth noting that there is an essential difference, which is not always well-understood, between, on the one hand, making a direct use of field reciprocity to express the induced voltage as an integral over the source of the incident field and, on the other hand, using reciprocity for deriving the above representations, where the integrals extend over the incident field and system determined current distributions (see [12]). In the latter, the fields scattered by the system have been shown not to contribute to the value of the observable. As will become clear below, this makes the above representations much more useful than those where the total fields in the two states appear.

In the particular case of electronic multi-port systems, the current distributions, j and k, correspond to applying unit sources to the ports of interest. The various representations of multi-port systems in terms of Thévenin, Norton, Hybrid or S-parameter models all lead to similar formulations. For ease of presentation, we shall simply handle the Thévenin case, the reader will understand how to transform the presentation to adapt it to other choices. Our system model is therefore

$$V = \mathscr{E} + ZI$$

where V and I are the vectors of port voltages and currents respectively and Z is the system's impedance, or response, matrix. The inhomogeneity of this system of equations, $\mathscr{E}$, represents the port voltages in absence of port current excitations and, therefore, represents the presence of electromagnetic fields incident on the system.

In what follows, we shall assume that any inhomogeneity in the model is a perturbation with respect to the desired model. We shall concentrate on this model extension exclusively. We can safely assume that the perturbations in the response coefficients are second order effects and because the very design of the system needs to have the coupling to the environment weak we can neglect their effect with respect to the effects of $\mathscr{E}$.

We can obtain integral representations for the components of the Thévenin source vector (see [12]):

$$\mathscr{E}_p = \langle j_p, E^i \rangle + \langle k_p, H^i \rangle$$

where $\{E^i, H^i\}$ is the incident electromagnetic field and j_p and k_p are electric and magnetic contrast current distributions corresponding to a unit current source applied to the pth port of the system.

In order to simplify the presentation a little, we shall write this representation in a more succinct form, which also emphasises the fact that the formulation applies to general linear observables and not only to induced voltages

$$A = \langle \omega, \psi \rangle$$

where ψ represents the pair of regular incident electric and magnetic fields, $E^i \oplus H^i$, and $\omega = j \oplus k$ the pair of distributions with support on the stochastic system corresponding, for the induced voltage observable, to a unit current source at the port considered, or, more generally, to the system in an appropriate (internal excitation) state.

What we have obtained thus far is an explicit expression of a model extension which allows us to account for the presence of externally generated electromagnetic fields. As it appears we have to solve a number of boundary value problems, one for each port we are interested in, and the resulting current distributions on the system allow us then to compute the model modifications for any incident electromagnetic field our system might be placed in.

We now pass on to the second model extension, which should account for uncertainties in the interaction configuration. That means, uncertainties in the externally generated field and uncertainties in the realisation of the system. The uncertainties in the environment lead us to consider stochastic incident fields (precise models for such fields will be presented in one of the subsequent sections). The uncertainties in the realisation of the system make that the distributions, introduced above, must be stochastic distributions resulting from boundary value problems for the Maxwell equations with a stochastic system geometry and/or stochastic physical constitution.

From these basic integral representations, presented above, we derive the following representations for the two essential statistics of the observable $A = \mathscr{E}_p$, assuming that the stochastic distribution ω and the stochastic field ψ are statistically independent. The average is given by,

$$\mathbb{E}(A) = \big\langle \mathbb{E}(\omega), \mathbb{E}(\psi) \big\rangle$$

and the variance is given by,

$$\operatorname{var}(A) = \mathbb{E}\big(\langle \omega, \psi \rangle \overline{\langle \omega, \psi \rangle} \big) - \big| \mathbb{E}(A) \big|^2$$

where the first term can be written as a scalar product of operators

$$\mathbb{E}\big(\langle \omega, \psi \rangle \overline{\langle \omega, \psi \rangle} \big) = \operatorname{Tr}(C_\omega C_\psi) = \langle C_\omega, C_\psi \rangle$$

where C_ψ is the covariance operator[1] of the stochastic incident field and C_ω the covariance operator of the stochastic distribution ω. The scalar product in the last equation is the inner product in the Hilbert space of Hilbert–Schmidt operators, in which we would like to represent the introduced covariance operators. However, the feasibility of this has yet to be shown and necessitates some development, in particular on the side of the distributions ω. As to the field covariance operator there is no problem as soon as we define stochastic fields as finite, or sufficiently rapidly convergent infinite, stochastic linear combinations of solutions of the homogeneous Maxwell equations. Also in the particular case of canonical stochastic fields, according to the definition given in [11], we have sufficiently explicit knowledge concerning the covariance to show that the operator in question is compact (see Section 4). In Section 3, we shall elaborate a particular type of stochastic systems for which the covariance operators are indeed of a suitable type for evaluating the scalar product. We then get the satisfactory situation that the first and second moments of our observables are both given in terms of a scalar product. The average is determined by a scalar product defining a pairing of distributions and functions. The variance is determined by a scalar product defining a pairing between operators.

3. Stochastic systems

In the present section, we shall elaborate on the covariance operator of a stochastic system. In order to keep the presentation as simple as possible, we shall only consider systems made of perfectly conducting material. Then, in order to obtain the stochastic distributions which characterise the system's response to externally generated electromagnetic fields, we have to solve boundary value problems for the Maxwell equations on domains with stochastic boundaries. These distributions can also be found from boundary integral equations over stochastic surfaces.[2] However, the complete mathematical treatment of this problem seems not yet accessible and we will have to content ourselves with a few approximations.

In the first place we assume that the fluctuations of the boundary surfaces can be parameterised as deviations from an average surface, which are small relative to the variations of the functions the distributions are to be evaluated on. Let X denote the stochastic surface and let M be the, smooth, average surface. We assume that X is parameterised by M through,

$$\mu : M \to X$$

a stochastic diffeomorphism. This diffeomorphism induces a transformation,

$$\mu^* : F(X) \ni u \mapsto [\mu']^{-1} u \circ \mu \in F(M)$$

where $[\mu']$ is the Jacobian matrix and $F(X)$ and $F(M)$ are spaces of vector fields over X and M respectively. This provides a parameterisation of fields over X by fields over M. The inverse image, $(\mu^*)^{-1} u$, for some vector field u

[1] The covariance operator of a zero-average, stochastic field, ψ, is formally defined as $\mathbb{E}(\psi\psi^\dagger)$, i.e., the average of the rank 1 operator, $\psi\psi^\dagger$ ($\psi^\dagger$ is the linear form associated to the field, this determines the domain of the operator). The kernel distribution of this operator is given by the spatial covariance function $\mathbb{E}(\psi(x)\overline{\psi}(y))$. Of course, care has to be taken as to the nature of the stochastic field, but in our analysis we shall only work with smooth stochastic fields and stochastic distributions which are finite stochastic linear combinations of compactly supported normal distributions.

[2] At the time of publication, the analysis of stochastic interaction problems through stochastic integral equations is the subject of a joint research project at ONERA and the TU/e in Eindhoven, The Netherlands.

over M will be called the μ-deformation of u. This corresponds to the vector field on X obtained by following, 'in a natural way', the deformation of M into X through μ^{-1}.

In order to keep the equations relatively simple without having to introduce too much machinery from differential geometry, we shall consider electric and magnetic current distributions as a special kind of vector fields (vector-valued densities, in fact) which transform under induced transformations getting the Jacobian, $|\mu'|$, as a multiplier, e.g., $\mu^* j = |\mu'|[\mu']^{-1} j \circ \mu$. This implies that the evaluation of a distribution with support on X can be replaced by one on M in the following way:

$$\langle j, E\rangle_X = \langle \mu^* j, \mu^* E\rangle_M$$

We shall use the stochastic μ-deformation of the deterministic current distribution, j_0, representing the average system with surface M, as a zeroth order approximation of the true stochastic distribution j. This means $j = (\mu^*)^{-1} j_0$ or,

$$\mu^* j = j_0$$

In this way, we get the simplest, non-trivial, representation of a stochastic distribution representing an observable defined by a stochastic system. In this 'zeroth order' approximation, we only have to solve for the average system distribution, j_0.

In order to get convenient expressions for the corresponding covariance operator, we shall assume that the deformation μ is a deviation in a direction along the unit normal to M,

$$\mu : M \ni m \mapsto m + h(m)\nu(m)$$

with ν the unit normal on M and h a stochastic function over M, given as a stochastic linear combination

$$h(m) = \sum_q A_q h_q(m)$$

where each h_q is a smooth scalar function on M. It can be shown that, if M is a smooth surface, the set $\{(m + h\nu(m)) \in \mathbb{R}^3\colon m \in M \wedge h \in \mathbb{R}\}$, equipped with a suitable topology, is locally (i.e., for sufficiently small h) diffeomorphic to an open neighbourhood of M (see [13]). Therefore, we can indeed represent sufficiently small deformations in this way.

We shall now transform the stochastic distributions with stochastic support into stochastic distributions with support on the fixed average manifold. The first step is to use the parameterisations discussed above. In the case of electric surface current distributions, we have

$$\langle j, E\rangle_X = \langle \mu^* j, \mu^* E\rangle_M$$

and, using the 0th order approximation introduced above,

$$= \langle j_0, \mu^* E\rangle_M$$

Now j_0, the current distribution characterising the average system, has fixed support, but $\mu^* E$ is a stochastic parameterisation which makes the evaluation non-linear in the field E. Because of the supposed smallness of the fluctuations (with respect to the variations in the test functions they have to be evaluated on) we can use a Taylor expansion of the test function on a neighbourhood of the average manifold.

The second step, then, is to approximate the values of the test functions using

$$E\big(m + n(m)\big) = E(m) + \big[\big(n(m)\cdot\nabla\big)E\big](m) + \mathscr{O}\big(\big|n(m)\big|^2\big)$$

where n is some normal vector field over M. The first-order approximation gives

$$E\big(m + n(m)\big) \approx E(m) + \big[\nabla\big(n(m)\cdot E\big)\big](m) - \big[n(m)\times(\nabla\times E)\big](m)$$

using the fact that the test function corresponds to an incident electromagnetic field satisfying vacuum Maxwell equations on a neighbourhood of the surface, we get

$$E\big(m + n(m)\big) \approx E(m) + \big[\nabla\big(n(m)\cdot E\big)\big](m) + j\omega\mu_0 n(m)\times H(m)$$

This is a point-wise equation where the scalar product with the fixed vector $n(m)$ precedes the gradient operator. This is inconvenient for functional manipulations. Therefore, we rewrite this to make the underlying differential operator explicit,

$$= E(m) + D_n(m)E(m) + j\omega\mu_0 n(m)\times H(m)$$

where $D_n(m) = \sum_{k=1}^{3} n_k(m)\nabla i_k^t$, with i_k the unit vector along the kth coordinate on $\mathbb{R}^3$ and $n_k(m) = i_k \cdot n(m)$. According to the definitions above, $(\mu^* E)(m) = (\mu'(m))^{-1} E(m + n(m))$. Substitution into the representation gives

$$\begin{aligned}\langle j, E\rangle_X = \langle j_0, \mu^* E\rangle_M &= \langle j_0, (\mu')^{-1}[E + D_n E + j\omega\mu_0 n \times H]\rangle_M \\ &= \langle \left(\mu'^t\right)^{-1} j_0, [E + D_n E + j\omega\mu_0 n \times H]\rangle_M\end{aligned}$$

This representation can be put in the form of an evaluation of electric and magnetic current distributions on an electromagnetic field using the transposition $D_n^t u = \sum_{k=1}^{3} i_k \nabla \cdot (n_k u)$,

$$\langle j, E\rangle_M = \langle p, E\rangle_M + \langle m, H\rangle_M$$

where,

$$p = \left(\mathbb{I} + D_n^t\right)\left(\mu'^t\right)^{-1} j_0, \qquad m = -j\omega\mu_0 n \times \left(\mu'^t\right)^{-1} j_0$$

Given the special form of the parameterisation, we can derive explicit expressions for the derived map.

$$\mu' = \mathbb{I} + \nu dh + h\omega_n$$

where ω_n is the connection form on the surface M corresponding to the differential of the normal, $d\nu = \omega_{nq}^k \tau_k \theta^q$, and (τ_1, τ_2, ν) is an orthonormal moving frame over M and $\theta^q = \tau_q^t$. Using the smallness of h again, we get an approximation for the inverse,

$$\mu'^{-1} = \mathbb{I} - \nu dh - h\omega_n = \mathbb{I} - d_n h$$

where $d_n h = \nu dh + h\omega_n$. This means that the distributions characterising the stochastic system, are given by

$$p = \left(\mathbb{I} + D_n^t\right)\left(\mathbb{I} - (d_n h)^t\right) j_0, \qquad m = -j\omega\mu_0 h\nu \times \left(\mathbb{I} - (d_n h)^t\right) j_0$$

and, to first order in h, by

$$p = j_0 + \left(D_n^t - (d_n h)^t\right) j_0, \qquad m = -j\omega\mu_0 h\nu \times j_0$$

where $(d_n h)^t = \nabla h \nu^t + h\omega_n$. (Observe that, in the case of perfect conductors, the magnetic current distribution has no zeroth order term, it arises only for non-vanishing deviations from the average M.) The stochastic perturbation with respect to the average is written as a stochastic linear combination

$$\pi^e = p - p_0 = \sum_q A_q \pi_q^e, \qquad \pi^h = m - m_0 = \sum_q A_q \pi_q^h$$

with

$$\pi_q^e = \left(D_{n;q}^t - (d_n h_q)^t\right) j_0, \qquad \pi_q^h = -j\omega\mu_0 \nu \times h_q j_0$$

Supposing that the amplitudes $\{A_p\}$ in the representation of the stochastic deformations are statistically independent, we can readily write the explicit expression for the covariance operator of the chosen port of the stochastic system

$$C_\omega : \mathscr{E}^6 \ni \psi \mapsto \sum_q \operatorname{var}(A_q)\omega_q \langle \omega_q, \psi\rangle \in \mathscr{E}'^6$$

where

$$\omega_q = \left(\pi_q^e \oplus \pi_q^h\right)$$

This completely defines the covariance operator for a stochastic system in terms of objects associated to the average surface and the (small) fluctuations of that surface. The operator can be seen to be of the type we have introduced in our presentation of the general interaction theory. With that result we have the operator which characterises the response of the system to a field. If the incident field would be a deterministic field, we can readily evaluate the variance of the system's response using a degenerate form of the formula, i.e. by replacing the field covariance operator by a simple tensor product. In the next section, we study the operator which characterises a stochastic incident field such that the complete stochastic interaction can be evaluated.

4. Canonical stochastic fields

The general problem we have to face in the field-to-system interaction modelling is the choice of a model for the stochastic field. It appears that in the free space situation a natural choice of stochastic field is an isotropic plane wave. This corresponds to assuming that there is a very remote source of electromagnetic energy but there is no knowledge about its position nor about its orientation and only the mean and standard deviation of its strength are given. Such ideas lead to the isotropic plane wave model of which it is known that the spatial covariance coincides with the real part of the Green function in free space. As this isotropic plane wave seems to be the most natural choice in a free space context, we may call it the canonical stochastic field in free space.

Inspired by these ideas, we can turn the situation inside out and take the Green function as the starting point. This is also because the tensorial Green function encompasses, in principle, all information concerning the geometry and physical constitution of the configuration. This opens the way to defining generalisations of the free space situation. The natural generalisation of the isotropic plane wave is then defined as the stochastic electric field of which the spatial covariance is proportional to the real part of the corresponding electric field Green function in the configuration. A constructive proof of the existence of such canonical fields in arbitrary configurations with reciprocal, linear and lossless material, has been given in [11].

According to the general interaction theory, we need the following operator,

$$C_\psi = \mathbb{E}\big(\psi\psi^\dagger\big) = \mathbb{E}\big((E \oplus H)(\overline{E} \oplus \overline{H})^t\big) = \begin{bmatrix} \mathbb{E}(EE^\dagger) & \mathbb{E}(EH^\dagger) \\ \mathbb{E}(HE^\dagger) & \mathbb{E}(HH^\dagger) \end{bmatrix}$$

The construction of the canonical fields in [11], was entirely dedicated to the electric field covariance, $\mathbb{E}(EE^\dagger)$. Therefore the cross-covariance between electric and magnetic fields was not presented there. In this section, we shall give the complete covariance operator in terms of the Green tensor functions.

For the definition of the Green tensor functions, we shall use the following system of equations,

$$\begin{bmatrix} -j\omega\varepsilon & +\nabla\times \\ -\nabla\times & -j\omega\mu \end{bmatrix} \begin{bmatrix} G^{ee} & G^{eh} \\ G^{he} & G^{hh} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}$$

where $\mathbb{I} = \sum_{k=1}^{3} e_k e_k^t \delta$ is the tensor valued Dirac measure, defined by $\forall\varphi \in \mathscr{E}^3\ \mathbb{I}(\varphi) = \varphi(0)$, and (e_1, e_2, e_3) a global orthonormal frame. The Maxwell equations, supplemented with boundary conditions on the boundary of the domain of the fields, are supposed to represent lossless and reciprocal material, so ε and μ are real and symmetric tensor functions.

From this definition, it follows

$$G^{he}(x, y) = \frac{1}{-j\omega\mu}\nabla_x \times G^{ee}(x, y), \qquad G^{eh}(x, y) = \frac{1}{j\omega\varepsilon}\nabla_x \times G^{hh}(x, y)$$

and from the field reciprocity relation, we derive

$$G^{ee}(x, y) = G^{ee}(y, x)^t, \qquad G^{hh}(x, y) = G^{hh}(y, x)^t, \qquad G^{eh}(x, y) = -G^{he}(y, x)^t$$

For a canonical stochastic field, the spatial covariance of E and H can both be found using the method developed in [11]. This gives,

$$c_{EE}(x, y) = \mathbb{E}\big(E(x)\overline{E^t(y)}\big) = 2\,\mathrm{Re}\big(G^{ee}(x, y)\big)$$

and

$$c_{HH}(x, y) = \mathbb{E}\big(H(x)\overline{H^t(y)}\big) = 2\,\mathrm{Re}\big(G^{hh}(x, y)\big)$$

We can now derive the cross-covariance operators for a canonical stochastic electromagnetic field. Note that the stochastic magnetic field follows from the stochastic electric field through the local relation

$$H = \frac{1}{-j\omega\mu}\nabla \times E$$

As a consequence, the cross-covariance of E and H is

$$c_{HE}(x, y) = \mathbb{E}\big(H(x)\overline{E^t(y)}\big) = \frac{1}{-j\omega\mu}\nabla_x \times \mathbb{E}\big(E(x)\overline{E^t(y)}\big) = \frac{1}{-j\omega\mu}\nabla_x \times 2\,\mathrm{Re}\big(G^{ee}(x, y)\big)$$

and, according to the relations above,

$$= j2\,\mathrm{Im}\big(G^{he}(x,y)\big)$$

and similarly for $c_{EH}(x,y)$

$$c_{EH}(x,y) = \mathbb{E}\big(E(x)\overline{H^t(y)}\big) = \frac{1}{j\omega\varepsilon}\nabla_x \times \mathbb{E}\big(H(x)\overline{H^t(y)}\big) = \frac{1}{j\omega\varepsilon}\nabla_x \times 2\,\mathrm{Re}\big(G^{hh}(x,y)\big) = j2\,\mathrm{Im}\big(G^{eh}(x,y)^t\big)$$

We observe that the real part of the functions G^{ee} and G^{hh} satisfy a source-free Helmholtz equation with real C^{∞} coefficients (because we included losslessness in the definition of the canonical field). Therefore, this real part of the Green state is infinitely differentiable. From the above development, it follows that, in fact, all the components of the spatial covariance tensor function of a canonical stochastic field are infinitely differentiable.

This property has the immediate consequence that the four covariance operators of a canonical stochastic field in Ω, are continuous mappings

$$C : \mathscr{E}'(\Omega)^3 \to \mathscr{E}(\Omega)^3$$

This determines the complete covariance operator of a canonical electromagnetic field. The idea is that, in absence of any knowledge concerning the sources of electromagnetic energy in a certain configuration, one resorts to a canonical stochastic field and if the Green function is accessible one does not have to construct the stochastic field to compute the variance of the response of the system to it.

Of course, Green functions are explicitly known in only relatively few configurations with many symmetries. However, the identification of the covariance operator and the Green function can also serve in a perturbation analysis where one estimates the modification of the Green functions in a certain region due to geometrical or constitutive modifications in another. In this way, conventional scattering theory can provide information on the relevance of such modifications for the fluctuations of the observables under consideration. It has been observed, for example, that the free space covariance operator can even provide useful results in Mode Tuned Reverberation Chambers, which are very far from free space but which tend to present a kind of isotropic plane wave ambient field for sufficiently high frequencies (see the example in [5]).

The most important general conclusion of this section is that we have covariance operators applicable to any *distributions* of electric and magnetic contrast sources representing observables. That means that we can compute the variances of observables characterised not only by regular functions on a volume but also by point-, line- or surface-distributions. In the following section, we shall see an example of a stochastic distribution with support on a line.

5. Example: a stochastic pair of wires in a stochastic isotropic plane wave

The twisted pair of wires is a typical example where conventional deterministic modelling fails to give directly usable information. If the twisted pair is analysed with deterministic modelling, the (more or less perfect) symmetry of the twisted pair and the smallness of the cross-section can give very low coupling to certain incident fields. However, an additional quarter of a twist, which is a negligible geometrical perturbation in practical situations, or a slightly different incident field can completely change the response characteristics. In order to get information concerning realistic behaviour, one therefore has to either consider large numbers of different interaction geometries or, as we propose in this paper, turn to probabilistic modelling.

The interaction of a deterministic plane wave with a stochastic twisted pair of wires has been presented before (see [14]), but here we extend the interaction computation to the case where the incident field is also a stochastic wave. The model for the stochastic twisted pair is recalled in Section 5.1. As to the stochastic incident field, we shall consider the most simple model, the isotropic plane wave, detailed in Section 5.2. The results of the interaction computations presented in Section 5.3 illustrate the essential characteristics of a complete probabilistic model for the interaction between stochastic fields and stochastic systems.

5.1. A stochastic pair of wires

In this section, we present the model of the stochastic twisted pair. The average geometry will be taken to be the perfectly regular twisted pair. The geometry fluctuations are modelled as a stochastic linear combination of spatial harmonics and in accordance with the perturbation theory presented in Section 3, the stochastic distributions defining the observable are obtained from the stochastic deformation of the deterministic current distribution on the average geometry. In addition, this current distribution is assumed to be essentially the twisted deformation of a distribution known from the transmission-line theory for parallel pairs of wires. All these approximations are quite well-established in the engineering literature and it is not our purpose here to be overly pedantic as to the accuracy of the numerical results. We want to present an illustration of the general ideas and the reader will know what additional effort has to be done to apply stochastic interaction theory in more complicated situations where the various distributions are to be obtained from numerical solutions of boundary value problems.

5.1.1. The perfect twisted pair of wires

The stochastic twisted pair of wires we want to consider is defined by geometrical fluctuations around an average geometry for which we take a perfectly regular twisted pair (see Fig. 1). The geometry of a perfect twisted pair cable of length L can be defined by the parameterisations of the two wire axes, $\mathscr{W}^+ \cup \mathscr{W}^-$, in terms of the axial coordinate, $z \in \mathscr{Z} = (0, L) \subset \mathbb{R}$ (the cable is assumed to be lying on the third coordinate axis in $\mathbb{R}^3$):

$$\mu_\pm : \mathscr{Z} \ni z \mapsto \mu_\pm(z) = \bigl(\pm r_T(z), z\bigr) \in \mathbb{R}^3$$

where $r_T : \mathscr{Z} \ni z \mapsto r_T(z) = (a\cos(\xi z), a\sin(\xi z)) \in \mathbb{R}^2$ is the transversal position of the axis of $\mathscr{W}^+$, a is the cable radius and $\xi = 2\pi/T$ the twist number for a given twist period T. Note the following relations: the Jacobian of the parameterisation is $|\mu'_\pm(z)|$, which, in the perfect twist case, is a constant given by $|\mu'_\pm(z)| = \sqrt{1 + (\xi a)^2}$, $\tau^\pm(z) = \mu'_\pm(z)/\sqrt{|\mu'_\pm(z)|}$ is the unit tangent vector on the wire axis, the z-component of this vector is given by $\tau^\pm_z = 1/\sqrt{|\mu'_\pm(z)|}$ and the transversal position vector is related to the transversal part of this tangent vector by $\tau^\pm_T(z) = \pm i_z \times r_T(z)/\sqrt{|\mu'_\pm(z)|}$. Also, in the general case, we have $|\mu'_\pm(z)|\tau^\pm_T(z) = \pm\partial_z r_T(z)$.

5.1.2. Model for the geometry fluctuations

The stochastic twisted pair of wires is defined through a perturbation, ρ, of the transversal position function r_T. This ρ is chosen to be a stochastic vector function with vanishing average, such that the average of the stochastic twisted pair is the above described perfectly regular twisted pair.

We further suppose that these fluctuations vanish at the terminals because the wire positions on the ends are usually fixed by the connectors, so

$$\rho(0) = 0 = \rho(L)$$

In our computations, we shall use a finite harmonic expansion consistent with the above considerations,

$$\rho(z) = \sum_{p=1}^{N} A_p \sin(p\pi z/L) \tag{1}$$

where the amplitudes A_p are, statistically independent, stochastic vectors in two dimensions, the components of which are given by two uniform probability distributions on an interval $(-\Delta/p, \Delta/p)$, for some $\Delta < a$. Although, any finite linear combination of the harmonics gives a smooth perturbation we make the intervals decrease as $1/p$ such that even in the limit $N \to \infty$ the perturbation remains regular.

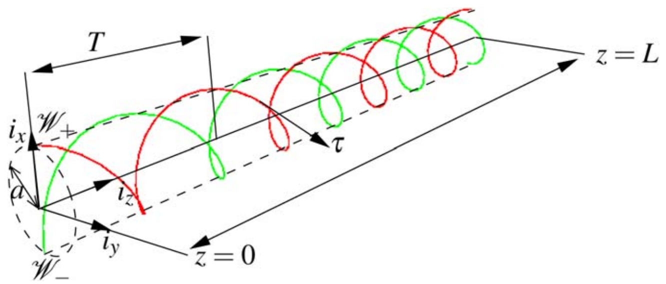

Fig. 1. Geometry of a perfect twisted pair.

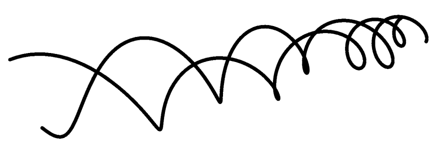

Fig. 2. Sample realisation of a stochastic twisted pair according to the model of Eq. (1).

Fig. 2 shows an example of one realisation of a twisted pair of wires issued from this model to give an idea of the kind of cables we are considering here.

5.1.3. Model for the current distribution

In the integral representation of the induced voltage, $\mathscr{E}$, a normalised current distribution appears, which corresponds to a source applied to the port under consideration. On the twisted pair of wires, this current distribution can be approximated in transmission-line theory. We shall work with the following approximation:

$$I|_{\mathscr{W}^{\pm}}(z) = \pm \sin\bigl(\beta(L-z)\bigr)/\sin(\beta L)$$

applying to the voltage on the terminal port in $z = 0$. The propagation coefficient β will be chosen to account for phase speed reduction due to the average twist speed, i.e., we shall use

$$\beta = k_0\sqrt{1+(\xi a)^2}$$

with $k_0 = \omega/c_0$ the wave number of free space and ξ the twist number, but we suppose that the geometrical fluctuations do not influence the phase velocity along the transmission-line axis (see the approximation hypotheses introduced in Section 3).

5.1.4. The covariance operator of the stochastic twisted pair of wires

The model for the stochastic geometry and the approximation of the fluctuating current distribution through deformation of the deterministic one allows us to make a representation for the covariance operator, along the lines of Section 3.

As the evaluation of wire currents on fields can be advantageously handled in the so-called thin-wire approximation, the perturbation theory, presented above, should be adapted to the case of a stochastic curve in three-dimensional space. Now, we could choose the perfect wire geometry as the parameterising manifold, however, as these wires themselves are very close to the cable axis we might as well use the simple straight line segment as the parameterising manifold which serves already as such for the perfect wires. The first step then is to substitute a Taylor expansion of the electric incident field, to first order around the cable axis. This gives, with the above introduced notations,

$$E\bigl(r_T(z), z\bigr) = E(0,z) + \bigl[\bigl(r_T(z)\cdot\nabla\bigr)E\bigr](0,z)$$

where $r_T \in \mathbb{R}^3$ is the (stochastic) transversal coordinate of the centre of the wire we consider. Using basic relations from vector analysis, and the Maxwell equation $\nabla \times E = -j\omega\mu_0 H$, we can show that this is equivalent to,

$$E\bigl(r_T(z), z\bigr) = E(0,z) + \sum_{k=1}^{3}\bigl(r_T(z)\cdot i_k\bigr)\bigl[\nabla(i_k\cdot E)\bigr](0,z) + j\omega\mu_0\bigl[r_T(z)\times H\bigr](0,z)$$

where i_k is the unit vector along the kth coordinate axis in $\mathbb{R}^3$. The integral over a wire, $\mathscr{W}$, appearing in the general integral representation of induced voltages, can now be rewritten as an integral over functions evaluated on the (fixed) cable axis

$$\int_{\mathscr{W}} j\cdot E = \int_{z=0}^{L} I_W(z)\Biggl[\tau(z)\cdot E(0,z) + \sum_{k=1}^{3}\bigl(r_T(z)\cdot i_k\bigr)\tau(z)\cdot\bigl(\nabla(i_k\cdot E)\bigr)(0,z) + j\omega\mu_0 H(0,z)\cdot\bigl(\tau(z)\times r_T(z)\bigr)\Biggr]\bigl|\mu'(z)\bigr|\,\mathrm{d}z$$

where $\tau(z)$ is the unit vector tangent to the wire at z. Using the symmetry properties of the twisted pair, which are preserved in the stochastic perturbations, we can derive the following expression for the induced voltage source

$$\mathscr{E} = 2\Biggl\{a i_x\cdot E^i(0,0) + \int_{z=0}^{L} I_W(z)\Biggl[\bigl|\mu'(z)\bigr|\tau_T(z)\cdot E(0,z) + \sum_{k=1}^{3}\bigl(r_T(z)\cdot i_k\bigr)(\partial_z i_k\cdot E)(0,z) + j\omega\mu_0 H\cdot\bigl(i_z\times r_T(z)\bigr)\Biggr]\mathrm{d}z\Biggr\}$$

where $2ai_x \cdot E^i(0,0)$ is the contribution due to the unit current distribution on the port in $z=0$ and we used $|\mu'(z)|\tau_z = 1$. In this representation we can do a partial integration to transpose the z-derivative and obtain a linear form on the electric and magnetic field on the cable axis. This integral representation now has the general form

$$\mathscr{E} = \int_{\mathfrak{X}} (p \cdot E^{\mathrm{inc}} + m \cdot H^{\mathrm{inc}}) \tag{2}$$

where

$$p(z) = -2(\partial_z I_W)(z) r_T(z), \qquad m(z) = 2 I_W(z) j\omega\mu_0 \big(i_z \times r_T(z)\big)$$

The p and m are distributions on the cable axis depending linearly on the transversal position function r_T. In this way we have obtained a representation in terms of evaluations on a fixed domain, $\mathfrak{X}$, and we can represent the covariance operator as an integral operator on functions over this domain.

In order to find the covariance operator of the twisted pair, we study the variance of the observable on a deterministic field. First note that

$$\mathscr{E} - \mathbb{E}(\mathscr{E}) = \int_{z\in\mathfrak{X}} \big[\pi^e(z) \cdot E^i(0,z) + \pi^h(z) \cdot H^i(0,z)\big]\,\mathrm{d}z$$

where

$$\pi^e(z) = p - \mathbb{E}(p) = \sum_k A_k \pi_k^e, \qquad \pi^h(z) = m - \mathbb{E}(m) = \sum_k A_k \pi_k^h$$

and, owing to the linear dependence of p and m on r_T, the expressions for $\pi^{e,h}$ are obtained from those of p and m, respectively, by simply replacing r_T with ρ. The harmonic constituents $\pi_k^{e;h}$ are obtained by substitution of the kth harmonic of the expansion of the fluctuation into the expressions. Now, we get

$$\mathrm{var}(\mathscr{E}) = \int_{z\in\mathfrak{X}} \begin{bmatrix} E^i \\ H^i \end{bmatrix}(0,z) \cdot \left[\int_{\zeta\in\mathfrak{X}} C(z,\zeta) \cdot \begin{bmatrix} \overline{E^i} \\ \overline{H^i} \end{bmatrix}(0,\zeta)\,\mathrm{d}\zeta\right]\mathrm{d}z$$

and the covariance operator of the twisted pair, $C(z,\zeta)$, is seen to be given by

$$C(z,\zeta) = \sum_{kl} \mathbb{E}(A_k \overline{A_l}) \begin{bmatrix} \pi_k^e(z) \otimes \overline{\pi_l^e}(\zeta) & \pi_k^e(z) \otimes \overline{\pi_l^h}(\zeta) \\ \pi_k^h(z) \otimes \overline{\pi_l^e}(\zeta) & \pi_k^h(z) \otimes \overline{\pi_l^h}(\zeta) \end{bmatrix}$$

and due to the statistical independence of the amplitudes A_k

$$C(z,\zeta) = \sum_{k} \mathrm{var}(A_k) \begin{bmatrix} \pi_k^e(z) \otimes \overline{\pi_k^e}(\zeta) & \pi_k^e(z) \otimes \overline{\pi_k^h}(\zeta) \\ \pi_k^h(z) \otimes \overline{\pi_k^e}(\zeta) & \pi_k^h(z) \otimes \overline{\pi_k^h}(\zeta) \end{bmatrix}$$

This concludes the analysis of the stochastic twisted pair of wires.

5.2. *Stochastic isotropic plane wave*

The isotropic plane wave is a canonical stochastic field in free-space according to the definitions given in [11], that means that the real part of the Green function for Maxwell's equations in free space provides the field's spatial covariance. We can therefore write down the explicit expressions for the field covariance operators

$$(C_{EE}J)(x) = -\omega\mu_0 \int C_x J - \frac{1}{\omega\varepsilon_0}\nabla\nabla \cdot C_x J, \qquad (C_{EH}K)(x) = j\nabla \times \int C_x K$$

$$(C_{HE}J)(x) = -j\nabla \times \int C_x J, \qquad (C_{HH}K)(x) = -\omega\varepsilon_0 \int C_x K - \frac{1}{\omega\mu_0}\nabla\nabla \cdot C_x K$$

where

$$C_x(y) = 2\frac{\sin(k_0\|x-y\|)}{4\pi\|x-y\|}$$

is twice the imaginary part of the free-space Green function of the Helmholtz equation.

5.3. Computing the variance of the induced voltage

In this section, we detail the computation of the variance of the observable voltage appearing on a terminal port of the twisted pair of wires. According to the formulation presented in Section 2, the variance is given by the following expression:

$$\operatorname{var}(\mathscr{E}) = \operatorname{Tr}(C_\omega C_\psi) = \sum_p \operatorname{var}(A_p)\langle \omega_p, C_\psi \overline{\omega_p}\rangle \tag{3}$$

where we have used the covariance operator of the incident field, C_ψ, and the spectral expansion of the covariance operator of the twisted pair, $C_\omega = \sum_p \operatorname{var}(A_p)\omega_p \otimes \omega_p^\dagger$ with $\omega_p = \pi_p^e \oplus \pi_p^h$.

To illustrate the theory we shall present the results of some computations on a twisted pair with the following characteristics:

- Cable length, $L = 0.5$ m,
- Cable radius, $a = 2$ mm,
- Twist period, $T = 0.01$ m,
- Number of harmonics in the fluctuation expansion, $N = 5$,
- Fluctuation extent, $\Delta = 1$ mm.

The incident field is an isotropic stochastic plane wave with an electric field polarisation with unit variance, i.e., zero mean value and a standard deviation of 1 V/m.

Before presenting the illustrations of the stochastic interaction theory, we show some results on deterministic cases which illustrate a typical situation where probabilistic modelling is almost unavoidable. In Fig. 3, we show the induced voltage due to a plane wave with propagation direction along the y-axis and electric field of strength 1 V/m parallel to the z-axis, i.e., the cable axis. It can be seen that the induced voltage is very low and increases monotonously with frequency in the frequency range considered. If we change the propagation direction by turning it over 45 degrees towards the cable axis and keep the electric field of equal strength in the y–z plane orthogonal to the propagation direction, we observe that the behaviour of the induced voltage as function of the frequency changes radically. There are apparent resonance peaks in this case, which make the induced voltage orders of magnitude greater than in the case of transversal incidence. What happens physically here is that, in the case of transverse incidence, the integer number of complete twists makes the induced voltage a superposition of contributions with alternating signs. Hence,

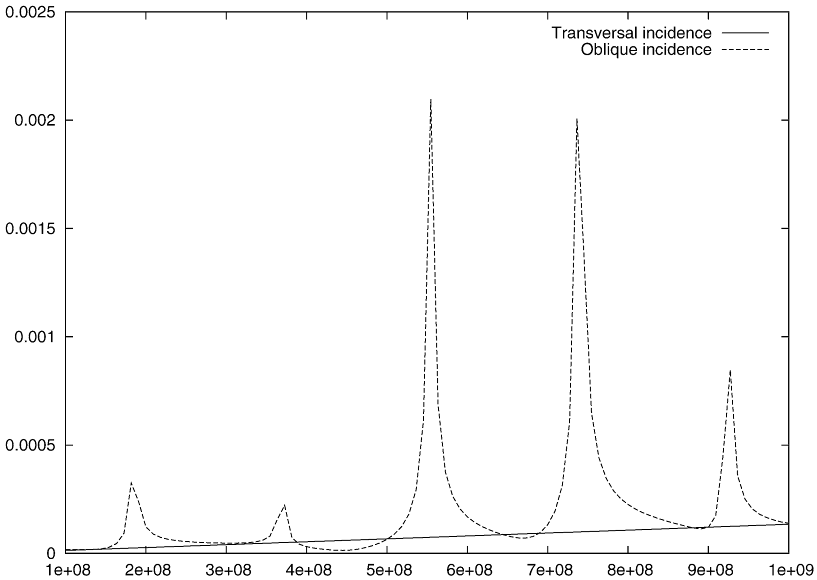

Fig. 3. Comparison of two deterministic interaction situations leading to completely different responses.

the perfect symmetry together with the negligible phase variations over the cable cross-section leads to very weak interaction. In many other situations with the same cable this accidental cancelling does not occur.

The problem with this kind of modelling is that one does not know whether one has chosen the most favourable case, a typical case or the worst case. For non-trivial interaction problems, this information can only be obtained from large series of different case studies or by specific mathematical elaborations dedicated to the extraction of these global characteristics. Probabilistic modelling is, in fact, one way to study the problem of characterising the nature of the global behaviour we are interested in. As we shall see, the variance or standard deviation is capable to give much of the information we are after.

We shall now test the representation of the stochastic twisted pair through the line distributions presented above, by comparing the induced voltage computed from the representation formula (2), with the value obtained from a numerical average and variance computation using an increasing number of realisations drawn from the ensemble. The fixed incident plane wave, oscillating at 200 MHz, propagates parallel to the y-axis and has electric field polarisation parallel to the principal axis of the twisted pair.

The results are shown in Fig. 4. In fact the induced voltages are imaginary for all sample configurations. This is due to the special choice of a wave with transversal incidence. It also appears that the induced voltages are quite uniformly distributed on an interval. This is related to the fact that the probabilistic model for the stochastic twisted pair leads to a stochastic induced voltage given by a weighted superposition of only a few statistically independent stochastic variables, each of them uniformly distributed but on different intervals. For the present model, it seems evident that twice the standard deviation corresponds to a hard upper limit. In this paper, we shall not try to precisely characterise the distribution of the voltages.[3] The results show that a brute force computation of the average and standard deviation converges to the values obtained from probabilistic modelling. The practical importance of this model is also clear because the perfect situation, giving an induced voltage of the order of 10^{-5} V, is shown to be quite exceptional as the standard deviation due to relatively weak geometry fluctuations is of the order of 10^{-3} (both counting on a typical electric field strength of 1 V/m).

The standard deviation around one of the resonance frequencies is shown in Fig. 5, together with the sample values at 200 MHz.

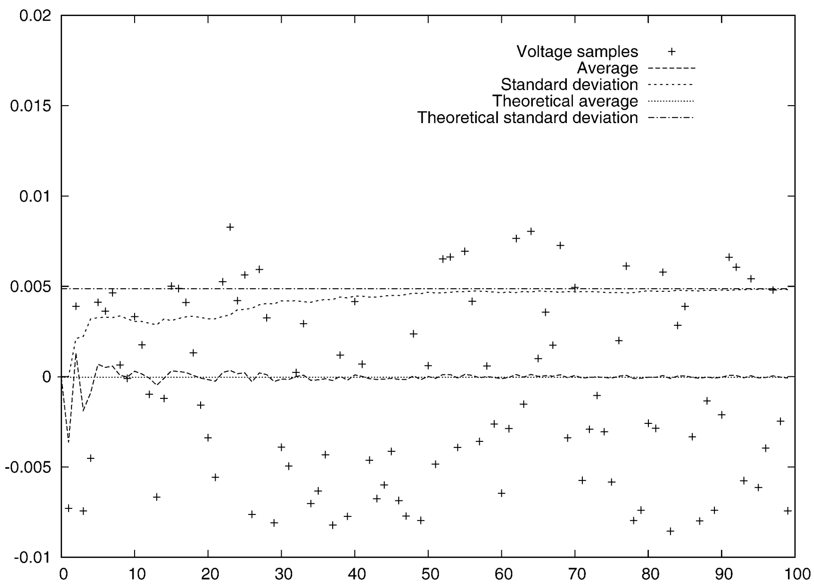

Fig. 4. Average and variance of a port-voltage induced by a plane wave in a stochastic twisted pair of wires. Convergence of a 'Monte Carlo' computation to the values computed from the probabilistic model.

[3] However, if one is interested in computing the probability that a given level will be superseded, the nature of the precise probability distribution must be known.

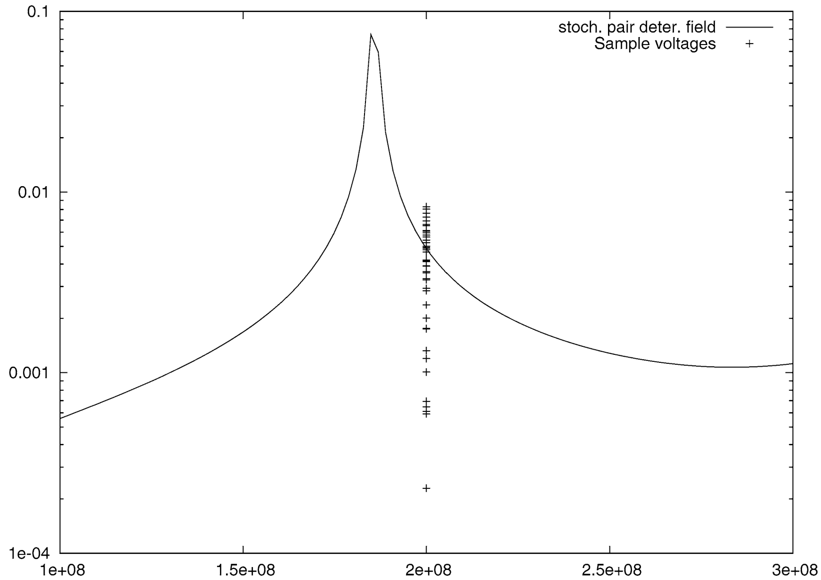

Fig. 5. Standard deviation of the induced voltage as function of frequency (Hz) and the sample voltages at 200 MHz of a stochastic twisted pair in a deterministic plane wave.

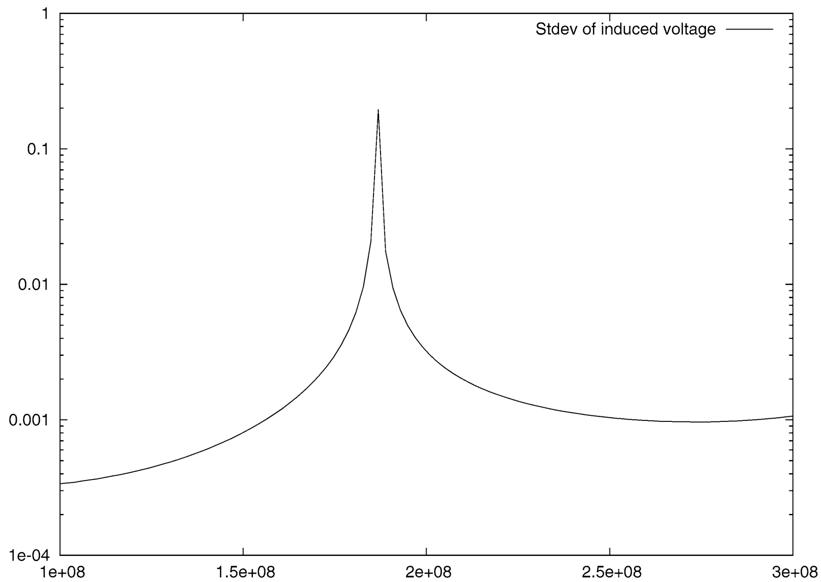

Fig. 6. Standard deviation of the induced voltage as function of the frequency (Hz) in a perfect twisted pair in an isotropic plane wave.

Now we characterise the induced voltage when the twisted pair is perfect and the incident field is an isotropic stochastic plane wave. In this case the covariance operator of the twisted pair reduces to the simple tensor product of the distributions appearing in the representation formula (2). The results are shown in Fig. 6.

Finally, we compute the induced voltage in a stochastic twisted pair in an isotropic stochastic plane wave using the complete interaction theory. The results in Fig. 7, show that the standard deviation captures both the differences due to the fluctuations in the cable cross-section as those due to fluctuations in the incident field. However, the nature of the two constituents is different and the composition of the two is not a simple addition.

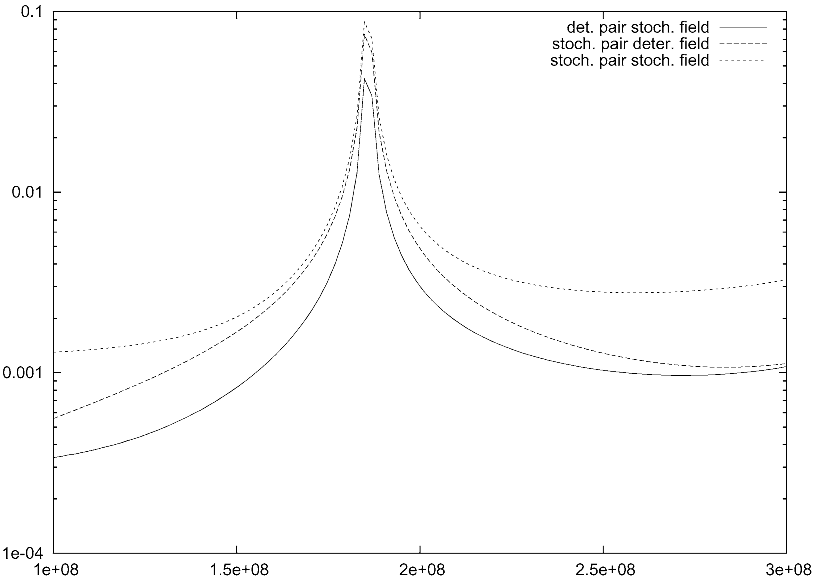

Fig. 7. Computed standard deviations of the induced voltage as function of the frequency (Hz) for various combinations of stochastic and deterministic components.

6. Conclusion

In this article, we have developed an analysis of the probabilistic modelling of the interaction of electromagnetic fields and systems, where both of them have uncertainties. The uncertainties in the incident fields represent uncertainties on the operational environment of the system and the uncertainties of the system represent uncertainties in the manufacturing of the system and, possibly, incomplete specification of the positioning of various of its parts. In particular, the routing of cables through conduits or even through only vaguely specified zones is a true source of uncertainty when the ambient fields are rapidly oscillating.

As to the stochastic fields, we have presented a straightforward extension of a previously developed analysis of canonical stochastic fields, with a given spatial covariance in the electric field component. Here, we have presented the complete field covariances, including the cross-covariance terms between the electric and magnetic fields, in terms of the Green tensor functions of the configuration.

For the case of relatively small fluctuations in systems of perfectly conducting surfaces, we have been able to derive a first, non-trivial, approximation to the stochastic distributions which characterise the response of the system. In this approximation, only the geometrical fluctuations have to be modelled and the values of the stochastic distributions are induced from those on the deterministic average system.

The resulting probabilistic model gives an estimate of the average and variance statistics of observables defined by stochastic systems in stochastic fields. The theory has been illustrated through a complete, but simplified, example concerning a stochastic twisted pair of wires in an isotropic stochastic plane wave. This example shows some interesting features. Most notably, the fact that isolated deterministic computations can possibly miss important resonances in an interaction which do indeed show up in the computed standard deviation. This corroborates the conjecture that deterministic computations alone cannot be used to make reliable estimates about practical interaction situations.

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