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Some applications of substructuring and domain decomposition techniques to radiation and scattering of time-harmonic electromagnetic waves

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Abstract

After a quick review of the domain decomposition methods, and, more particularly on their application to large size problems relative to radiation and scattering of time-harmonic waves, we describe two contributions of the authors in this context. The two contributions are related to the scattering of a time-harmonic electromagnetic wave by a large perfectly conducting structure including a deep cavity. The first contribution is a substructuring technique. It is used to increase the speed of the convergence of the iterative algorithm in a Multi-Level Fast Multipol Method (MLFMM) solution. Numerical experiments illustrate the effectiveness of the approach since the number of iterations of the underlying Krylov iterative method remains almost constant while increasing a characteristic length in the problem. The second contribution proposes an adaptation of the overlapping domain decomposition techniques for a boundary integral formulation. It is used here to perform a hybridization of an exact formulation, used at the opening of the cavity, with an asymptotic high-frequency method employed for the rest of the exterior boundary of the structure. Numerical results demonstrate the reliability and the efficiency of the method. ***To cite this article: N. Balin et al., C. R. Physique 7 (2006).***

Résumé

Quelques applications de techniques de sous-structuration et de décomposition de domaine au rayonnement et à la diffraction des ondes électromagnétiques en régime harmonique. Après une revue rapide des méthodes de décomposition de domaine, et, plus particulièment de leur application aux problèmes relatifs au rayonnement et à la diffraction des ondes électromagnétiques en régime harmonique, nous décrivons deux contributions des auteurs dans ce contexte. Les deux contributions sont liées à la diffraction d'une onde électromagnétique en régime harmonique par une structure parfaitement conductrice de grande taille comportant une cavité profonde. La première contribution est une technique de sous-structuration. Elle est utilisée pour accroître la vitesse de convergence de l'algorithme itératif dans le cadre d'une résolution par la méthode multipôle rapide multi-niveaux. Des expérimentations numériques illustrent l'efficacité de l'approche en ce que la méthode itérative de Krylov afférente converge en un nombre d'itérations qui reste quasiment constant lorsqu'on augmente une longueur caractéristique dans le problème. La seconde

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contribution propose une adaptation de la méthode de décomposition de domaine par recouvrement pour une formulation par équations intégrales de frontière. Elle est utilisée ici pour effectuer une hybridation d'une formulation exacte, utilisée à l'ouverture de la cavitée, avec une méthode asymptotique haute fréquence, employée pour le reste de la frontière extérieure de la structure. Des résultats numériques prouvent la fiabilité et l'efficacité de la méthode. ***Pour citer cet article : N. Balin et al., C. R. Physique 7 (2006).***

Keywords: Domain decomposition methods; Time-harmonic waves

Mots-clés : Méthodes de décomposition de domaine ; Régime harmonique

1. Domain Decomposition Methods and their use in radiation and scattering of electromagnetic waves

Domain Decomposition Methods (DDMs) constitute flexible and powerful techniques to numerically solve Partial Differential Equations (PDEs). As clearly described in [1], they can be used as a "*process of distributing data from a computational model among the processors in a distributed memory computer*", or for "*the separation of the physical domain into regions*" which occurs when using different and adapted solving methods, and finally as "*a process of subdividing the solution of a large linear system* [resulting from the discretization of the PDE on the entire domain] *into smaller problems whose solutions can be used as a preconditioner or a solver*".

One generally can distinguish three kinds of DDMs:

- *Substructuring methods*: The computational domain is first divided into several pieces without overlap. The degrees of freedom (dof) internal to each piece are next removed through a Schur complement technique. As a result, the solution of the entire problem is reduced to that of a problem of smaller size usually called the coarse problem (e.g., [2,3], etc.).
- *Non-overlapping methods*: Roughly speaking, substructuring and non-overlapping DDMs are closely related. When the coarse problem is still too large to be handled by a direct method (e.g., [3]), one has to resort to an iterative method. The derivation of efficient iterative procedures, which is precisely the task of non-overlapping DDMs, can generally be obtained only through a thorough inspection of the physical phenomenon and the properties of its mathematical model.
- *Overlapping methods*: The overlapping DDMs are solution techniques similar to the non-overlapping ones, but based instead on a decomposition of the computational domain into overlapping pieces. According to the procedure that is considered, either of Jacobi or Gauss–Seidel type, the iterative process is called a Schwarz additive or multiplicative alternating method. Because of their greater flexibility for writing the matching conditions linking the functions obtained by solving local problems, the overlapping methods are generally easier to use. This point will be illustrated below when such techniques will be adapted in order to hybridize an exact with a high-frequency asymptotic method.

More or less, the underlying principle of a domain decomposition method relies on a '*static condensation*' in an extended sense. This explains why these methods have been developed mostly in the framework of strongly coercive PDEs like those involved in structural mechanics problems: to limit the references to some representative or recent developments, we quote the overlapping alternating Schwarz methods [4,5,1], the large class of non-overlapping methods based on a Neumann–Neumann scheme like Finite Element Tearing and Interconnecting Dual Primal (FETI-DP) [6] and Balancing Domain Decomposition on Constraints (BDDC) [3].

Taking their roots in Thévenin's theorem (e.g., [7]), techniques similar to substructuring methods have been also considered for wave propagation problems in acoustics and electromagnetism. In the engineering terminology generally used in this framework, the Neumann–Dirichlet operator or its inverse, basically involved in substructuring methods, are often called impedance or admittance operators (e.g., [8] or [9] limiting the references to some techniques that have been applied to the scattering problem considered below). However, the weak coerciveness of the PDEs, governing the phenomena relative to time-harmonic waves, prevent such operators from being well-defined at some exceptional frequencies linked to local internal resonances, that is, resonances peculiar to waves in any region involved in the decomposition of the computational domain. This perhaps explains why neither DDMs were widely

used nor was a general theory available for their convergence in this context until the break-through achieved by Després [10,11]. Roughly speaking, one of Després's main ideas was to express the matching conditions equivalently in terms of an absorbing boundary condition instead of the Dirichlet and Neumann conditions usual in the standard DDMs. In fact, this way of writing the matching conditions as equivalent to construct a non-overlapping DDM was earlier introduced by P.-L. Lions for the Laplace equation as a limiting case of an overlapping DDM [12]. In his thesis, Després has also shown that the same kind of boundary conditions, when used in the context of an overlapping DDM, greatly improves the convergence of the iterative process comparatively to the usual Dirichlet condition. Several studies have been then devoted to the improvement of Després's approach (e.g., [13–15], etc.) and its extension to various problems and discretization schemes (e.g., [16–19], etc.). Actually, Després's method can be restated as a procedure based on the expression of the matching conditions in terms of the scattering operator instead of the standard impedance or admittance ones with the advantage to be then always well-defined. In this respect, Barka, Soudais and Volpert [8] and Barka and Soudais [20] have used this way to express the matching conditions to develop a general substructuring procedure. However, it seems that it only applies if the equivalent currents can be expanded in waveguide modes.

However, in spite of all these considerable advances, non-overlapping DDMs have faced a difficulty which was overcome only recently. The matching conditions cannot be set in a weak variational form unless resorting to special tools and formulations. For instance, in the approach, adopted early by Després [10], the discretization procedure is based on a mixed Finite Element Method (FEM). The counterpart for usual FEMs of this difficulty is that the equations for the coarse problem cannot be set at the so-called cross-points, that is, points supporting a dof shared by more than two subdomains. For mixed finite element discretizations, any dof can be shared by at most two subdomains thus explaining why they have been used in the above mentioned work by Després [10]. In the case of a standard nodal FEM, a way to overcome this difficulty is developed in [6] by means of a FETI-DP method, in [3] through a BDDC method used in the context of a PDE with strong coerciveness properties, and in [21,22] for the Helmholtz equation. It consists of keeping the strong continuity requirements at cross-points for both the trial and the test functions and using a suitable Schur complement technique to define the coarse problem. The stability and the convergence properties of this approach, in the context of wave propagation problems, have been established for the scalar case only [21,22], mainly because of the high level of technicalities related to the coerciveness properties of Maxwell's equations. Its extension to time-harmonic electromagnetic waves in the framework of a discretization by Nédélec's edge elements is straightforward, at least for an algorithmic purpose.

This article reports some contributions of the authors in the development of DDMs for solving problems related to radiation and scattering of time-harmonic electromagnetic waves. To illustrate the application of these techniques, we consider the scattering of an electromagnetic wave by a large perfectly conducting structure with a deep cavity. The objective is mainly to speed up the numerical solution of this problem in order to be able to consider higher frequencies than those that are characteristic of the resonance regime. In the first section, we consider a Boundary Element Method (BEM) for the solution of the scattering problem. Beyond the resonance regime, the effective solution of the associated linear system, at least at the exterior of the cavity, can only be undertaken by an iterative Krylov method (e.g., [23]) with a fast computation of the matrix–vector product by means of the MLFMM. It is well known that the cavity then slows down the convergence of the iterative solver making much less efficient the use of the MLFMM. We will show how the introduction of a special substructuring technique makes it possible to restore the efficiency of such a solution procedure. The second section is devoted to the description of an adaptation of the additive Schwarz alternating method to the so-called Electric Field Integral Equation (EFIE). This adaptation can be used either as a fast solver or as a way to couple an exact method with an asymptotic high-frequency method. We will show below that the hybridization technique reduces the computation time considerably as compared to a full-wave solution of the above scattering problem.

2. Application of a substructuring method to the electromagnetic scattering by an electrically deep cavity

After giving the statement of the scattering problem and making some remarks on the difficulties arising when its solution is undertaken by means of standard methods, we describe the substructuring approach and present some numerical experiments to illustrate its efficiency.

2.1. Statement of the problem

Consider a large structure with an arbitrarily-shaped electrically deep cavity as schematically depicted in Fig. 1.

Let be given an incident plane wave $(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}})$ such that $|\mathbf{E}^{\text{inc}}| = 1$. Recall that $|\mathbf{E}^{\text{inc}}|$ indicates the Euclidian norm of a vector with three Cartesian complex components. The usual notation will be used without further comments.

The main issue is to compute the bistatic Radar Cross-Section (RCS) of this obstacle which can be expressed in the direction $(\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ from the following relationships

$$s(\varphi, \theta) := 10 \log_{10}\left(4\pi \left|\mathbf{A}(\theta, \varphi)\right|^2\right)$$

where $\mathbf{A}(\theta, \varphi)$ is the radiation function of the scattered electric field. This function can be obtained by solving the following boundary-value problem

$$\begin{cases} \nabla \times \mathbf{E} - \mathrm{i}kZ\mathbf{H} = 0 \\ \nabla \times \mathbf{H} + \mathrm{i}kZ^{-1}\mathbf{E} = 0 \end{cases} \text{in } \Omega \\ \begin{cases} \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = 0 \quad \text{on } \Gamma \\ \mathbf{E}(x) = \mathbf{E}^{\text{inc}}(x) + (\mathrm{e}^{\mathrm{i}kr}/r)\mathbf{A}(\varphi, \theta) + \mathcal{O}(1/r^2) \end{cases} \tag{1}$$

in which k is the wavenumber and $Z \approx 120\pi$ is the intrinsic impedance of vacuum in SI units.

It is well known that this kind of cavity, like for instance an inlet in an aircraft, can significantly contributes to the overall RCS. Standard full-waves methods, like BEMs or FEMs, can loose their efficiency, specially for higher frequencies, because of several difficulties:

- Direct solvers rapidly become unusable because of the large increase in memory storage and CPU time;
- Internal resonances inside the cavity severely slow down the convergence of an iterative solver despite the use of highly performing preconditioners. In this way, the advantage gained from a fast evaluation of the matrix–vector product by means of the MLFMM could be completely lost;
- The geometry of the cavity is well adapted to the use of a FEM and of a frontal forward substitution to remove all the internal dofs [24]. However, because of the weak coerciveness of the PDEs related to time harmonic waves, the algorithm can break down. Even worse, its stability is not guaranteed. Moreover, the depth and the narrowness of the cavity prevent the use of low order FEMs otherwise the large amount of dispersion errors then makes any computation of the RCS meaningless [25].

2.2. The substructuring process

We start from a special non-overlapping decomposition $\{\Omega_p\}_{p=1}^{p=N+1}$ of the computational domain Ω. As indicated in Fig. 2, Ω_p $(p = 1, \ldots, N)$ is a sectional domain of the cavity, and Ω_{N+1} is the domain exterior to both the cavity and the structure. The sectional surfaces respectively separating Ω_p from Ω_{p+1} $(p = 1, \ldots, N)$ are denoted by Σ_p.

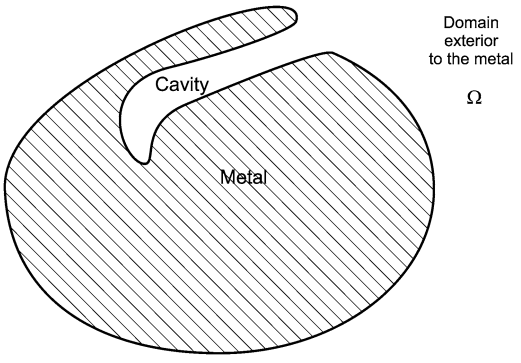


Fig. 1. Schematic view of a deep cavity residing in a large structure.

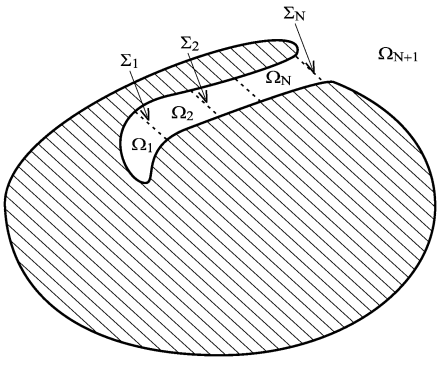


Fig. 2. Decomposition of the cavity in nonoverlapping sectional domains.

Let $\mathbf{n}_p$ ($p = 1, \ldots, N+1$) stands for the unit normal to the boundary $\partial\Omega_p$ of Ω_p, directed outward from Ω_p. We first state an equivalent formulation of problem (1) as a transmission problem

$$\begin{cases} \nabla \times \mathbf{E}_p - \mathrm{i}kZ\mathbf{H}_p = 0 \quad \text{in } \Omega_p \\ \nabla \times \mathbf{H}_p + \mathrm{i}kZ^{-1}\mathbf{E}_p = 0 \quad \text{in } \Omega_p \\ \mathbf{n}_p \times (\mathbf{E}_p \times \mathbf{n}_p) = 0 \quad \text{on } \Gamma \\ \mathbf{E}_{N+1}(x) = \mathbf{E}^{\mathrm{inc}}(x) + (\mathrm{e}^{\mathrm{i}kr}/r)\mathbf{A}(\varphi, \theta) + \mathcal{O}(1/r^2) \end{cases} \tag{2}$$

We express the matching conditions in a way more adapted to the substructuring method, we have in view,

$$\mathbf{J}_p + \mathbf{J}_{p+1} = 0 \text{ and } \mathbf{M}_p + \mathbf{M}_{p+1} = 0 \quad \text{on } \Sigma_p (p = 1, \ldots, N) \tag{3}$$

where $\mathbf{J}_p = \mathbf{n}_p \times \mathbf{H}_p|_{\partial\Omega_p}$ and $\mathbf{M}_p = \mathbf{E}_p \times \mathbf{n}_p|_{\partial\Omega_p}$ are the equivalent currents relative to Ω_p according to the engineering terminology that is usual in this context. This task can be considered as a first step common to every substructuring or non-overlapping DDM. It is worth noting that the boundary condition implies that $\mathbf{M}_p = 0$ on Γ.

Usual substructuring techniques, including a finite element modelling for the interior of the cavity, determine (an approximation of) the impedance operator, defined through the following relationship $\mathbf{n}_p \times \mathbf{M}_p = Z_p\mathbf{J}_p$ to carry out the process of removing the internal dofs (e.g., [9,8], etc.). This operator plays the role of the Neumann to Dirichlet operator for the equations posed in Ω_p in problem (2). However, observe that writing this relation already amounts to assuming that the following boundary-value problem is well-posed

$$\begin{cases} \nabla \times \mathbf{E}_p - \mathrm{i}kZ\mathbf{H}_p = 0 \quad \text{in } \Omega_p \\ \nabla \times \mathbf{H}_p + \mathrm{i}kZ^{-1}\mathbf{E}_p = 0 \quad \text{in } \Omega_p \\ \mathbf{n}_p \times (\mathbf{E}_p \times \mathbf{n}_p) = 0 \quad \text{on } \Gamma \\ \mathbf{n}_p \times \mathbf{H}_p = \mathbf{J}_p \quad \text{on } \Sigma_{p-1} \cup \Sigma_p \end{cases} \tag{4}$$

with an obvious adaptation for $p = 1$. Elementary cases, as, for instance, a domain that is a section of a waveguide, make it possible to check that this property is not true in general. Barka and Soudais [20] propose to express the matching conditions in terms of the scattering matrix associated to an expansion of the electromagnetic field in waveguide modes to avoid these resonances. Another way to proceed is adopted below.

Two of the authors [26,27] have designed a procedure that makes it possible to remove the internal dofs in a stable way and brings back the solution of the scattering problem to a well-posed EFIE-like integral equation set on $\partial\Omega_{N+1}$. However, this equation is suitable for a direct solution only. It is inefficient in the context of an iterative solution which is the only way to solve the resulting linear system when using the MLFMM. We now give some indications on how this approach has been adapted in [28]. The reduction of the cavity is based on the determination of the impedance operator of the cavity at Σ_N. Of course, the procedure can then break down. But this failure cannot occur during the elimination process of the dofs internal to the cavity. Moreover, it is possible to check whether the numerical procedure fails, or even cannot be performed in a stable way, before starting the solving process on $\partial\Omega_{N+1}$. One can then safely reduce the equations related to the dofs internal to the cavity and next use the so-called Combined Field Integral Equation (CFIE) whose suitability to iterative methods is well-established. Let us now describe the solving process in a more precise manner. The main trick is to use the Rumsey reaction concept [29] to equivalently write the matching conditions at the sectional surfaces Σ_p ($p = 1, \ldots, N-1$) at the interior of the cavity and the link between $\mathbf{J}_N$ and $\mathbf{H}$ at its opening Σ_N in the following form:

$$\sum_{p=1}^{N} \int_{\partial\Omega_p} (\mathbf{E}_p \cdot \mathbf{J}'_p - \mathbf{H}_p \cdot \mathbf{M}'_p)\,\mathrm{d}S = \int_{\Sigma_N} \mathbf{n}_N \times \mathbf{J}_N \cdot \mathbf{M}'_N\,\mathrm{d}S \tag{5}$$

for all test currents $\{(\mathbf{J}'_p, \mathbf{M}'_p)\}_{p=1}^{p=N}$ satisfying the same relations (3) than the unknown currents, and furthermore such that $\mathbf{J}'_N|_{\Sigma_N} = 0$. The Maxwell equations in the interior of the cavity are enforced by means of the Stratton–Chu (e.g., [30]) integral representations of $\mathbf{E}_p|_{\partial\Omega_p}$ and $\mathbf{H}_p|_{\partial\Omega_p}$

$$\mathbf{E}_p = \frac{1}{2}\mathbf{n}_p \times \mathbf{M}_p + \mathrm{i}kZ\,T_p\mathbf{J}_p + K_p\mathbf{M}_p, \qquad \mathbf{H}_p = -\frac{1}{2}\mathbf{n}_p \times \mathbf{J}_p + \mathrm{i}kZ^{-1}\,T_p\mathbf{M}_p - K_p\mathbf{M}_p$$

T_p and K_p being the surface integral operators

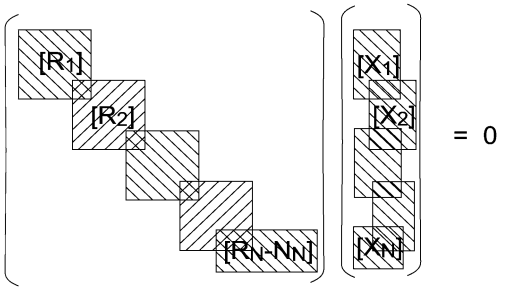


Fig. 3. Structure of the linear system relative to the Rumsey reactions.

$$T_p\mathbf{J}_p(x) = \frac{1}{k^2}\nabla_S \int\limits_{\partial\Omega_p} G(x,y)\nabla_S\cdot\mathbf{J}_p(y)\,\mathrm{d}S(y) + \Pi_S \int\limits_{\partial\Omega_p} G(x,y)\mathbf{J}_p(y)\,\mathrm{d}S(y)$$

$$K_p\mathbf{M}_p(x) = \Pi_S \int\limits_{\partial\Omega_p} \nabla_y G(x,y)\times\mathbf{J}_p(y)\,\mathrm{d}S(y)$$

where ∇_S, $\nabla_S\cdot$ and Π_S are respectively the surface gradient and divergence and the projection onto the tangent plane and $G(x,y) = \exp(\mathrm{i}k|x-y|)/4\pi|x-y|$ is the Green kernel giving the outgoing solutions to the Helmholtz equation. These integral equations can be viewed also as an extension of the PMCHW formulation (e.g., [31]) which seems to have be used only when the interfaces separating two dielectrics and the metallic surfaces do not cut each other. Meshing the surfaces in triangles and approximating the currents by the usual divergence-conforming edge elements, we are led to the system, schematically depicted in Fig. 3, where each block corresponds to the matrix associated with the bilinear form which expresses the Rumsey reaction in a sectional domain

$$\int\limits_{\partial\Omega_p} (\mathbf{E}_p\cdot\mathbf{J}'_p - \mathbf{H}_p\cdot\mathbf{M}'_p)\,\mathrm{d}S = [X'_p]^{\mathrm{T}}[R_p][X_p]$$

The symbols $[X_p]$ and $[X'_p]$ represent column-wise vectors of complex numbers collecting the dofs of the unknown and the test equivalent currents related to Ω_p. We have also denoted by $[N_N]$ the matrix associated with the bilinear form

$$[X'_N]^{\mathrm{T}}[N_N][X_N] = \int\limits_{\Sigma_N} \mathbf{n}_N\times\mathbf{J}_N\cdot\mathbf{M}'_N\,\mathrm{d}S$$

Introducing successively the sectional domains $\Omega_1, \Omega_2, \ldots, \Omega_p$, it is possible to remove at each step all the dofs that are not on Σ_p, with the guarantee that the elimination process cannot breakdown and that it can be done in a stable way. (See [26,27] for a proof in the 2D case. The same proof applies for the 3D case also, at least in a formal way.) As a result, the system (5) can be put in the form

$$[A][J_N] + [B][M_N] = 0 \tag{6}$$

where $[J_N]$ and $[M_N]$ are respectively the dofs of $\mathbf{J}_N|_{\Sigma_N}$ and $\mathbf{M}_N|_{\Sigma_N}$ and $[B]$ is a square matrix.

The algorithm can break down only after this step: the impedance matrix of the cavity can be safely determined if and only if matrix $[B]$ in (6) can be inverted in a stable way. Since this matrix is moderately sized, this property can be controlled by making use of a direct solver for dense matrices, such as for instance the public domain library `Lapack`, which, while computing the matrix $[Z_c] = -[B]^{-1}[A]$, also yields an approximate value for the condition number of $[B]$.

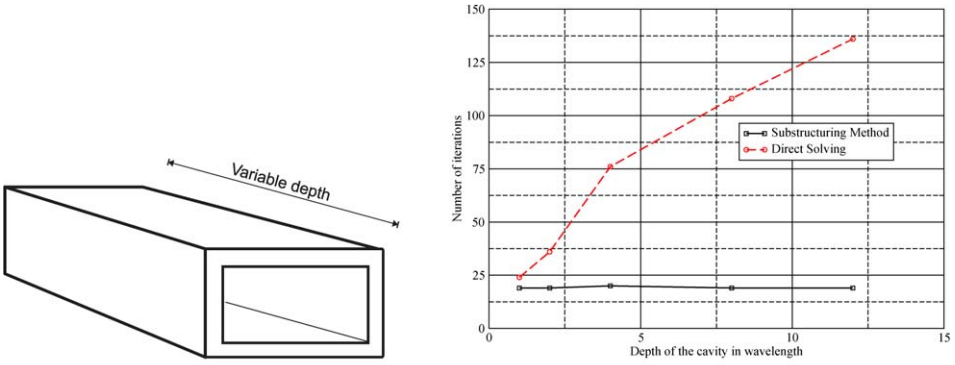
After the determination of $[Z_c]$, we can express $\mathbf{M}_{N+1}$ using $\mathbf{J}_{N+1}$. We can then reduce the scattering problem to a CFIE posed on $\partial\Omega_{N+1}$. More precisely, we obtain a boundary integral equation which can be considered as the usual CFIE for the case where $\partial\Omega_{N+1}$ is the boundary of a perfectly conducting obstacle in which the electric and magnetic fields are perturbed by the contribution of a distribution of magnetic currents on the opening Σ_N of the cavity. This makes it possible to solve this equation in a very efficient way by using a SPAI preconditioner associated with the usual CFIE on $\partial\Omega_{N+1}$ (see [32] for details on the definition and the construction of such a preconditioner).

2.3. Some numerical experiments

The first case is destined to compare the performances of this method with a broad approach based on the use of a CFIE without the substructuring technique. To bring out the effect of the cavity, we have considered a box with thick walls and opened at one side as depicted by left figure in Fig. 4. The plot on the right figure in Fig. 4 clearly shows that the number of iterations grows almost linearly with the depth of the cavity in the case of a direct application of the CFIE formulation while it remains constant for the method considered here.

The second example is related to a COBRA cavity within a fuselage, depicted in Fig. 5, which was the test-case n° 1 of the workshop EM-JINA 2004.[1]

Because of the thin surface at the opening of the cavity, only the EFIE can be used in the context of a direct approach. In Table 1, we compares the CPU time necessary to get the solution to the direct EFIE, first using a usual Krylov iterative solver, next a Krylov solver specially customized to handle such kinds of problems which can be considered as extreme cases for an iterative method, with the one used by the present procedure.



(a) The box cavity with thick walls

(b) Number of iterations versus the depth of the cavity

Fig. 4. Behavior of the GMRES iterative solver with and without using the substructuring process.

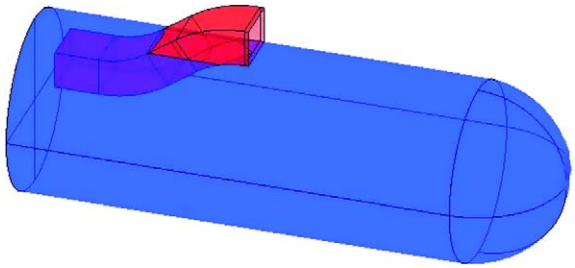


Fig. 5. COBRA cavity within a fuselage.

Table 1
Behavior of various solving procedures for the RCS computation of a COBRA cavity within a fuselage

Solver	Number of dofs	CPU time (cavity)	CPU time (exterior)	Total CPU time
GMRES	216 605	–	–	309 h
FGMRES	216 605	–	–	89 h
This method	223 907	1 h	17 h	18 h

[1] See the Web page http://www.elec.unice.fr/pages/congres/jina2004/workshop/test1.pdf.

This comparison clearly indicates that this method is more than 4 times faster than the most powerful purely algebraic process.

3. Hybridization of a BEM with a high-frequency asymptotic method

In this section, we show how the Schwarz additive domain decomposition techniques can be adapted to develop an efficient hybridization procedure for coupling a BEM with a high-frequency method.

3.1. An overlapping decomposition method for the boundary

For clarity, assume that the boundary integral equation is the EFIE, either in 2D or 3D, which, once suitably discretized (cf., e.g., [33]), amounts to solving the following dense linear system

$$[Z][X] = [U] \tag{7}$$

The method, we develop, can be viewed as an extension of a method introduced by Balabane and Tirel [34] that is first recalled now. Assume that the boundary Γ has two distinct components Γ_1 and Γ_2. In this case, distinguishing the dofs on Γ_1 and on Γ_2 yields a natural partitioning of this system in the form

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

These authors proved then that the bock Jacobi iterative method (e.g., [23]) converges if Γ_1 and Γ_2 are sufficiently far away from each other. When Γ_1 and Γ_2 are not separated and constitute a non-overlapping decomposition of Γ, they resort to a fictitious surface Σ to separate the two parts.

To improve the convergence properties of this method, we extend it by using instead an overlapping decomposition of the boundary Γ. For clarity, we limit ourselves to a decomposition of Γ in two overlapping components Γ_1 and Γ_2.

We first make a geometrical distribution of the dofs by assuming that each of them is attached to a particular node. For the 2D case, this can be obtained directly from the definition of the dofs. For the 3D one, it is sufficient to attach any flux through an edge to the mid-point of this edge. Let $[\mathbb{I}]$ be the identity matrix with the same number of rows and columns as $[Z]$. We define a Boolean matrix $[\mathbb{I}_i]$ $(i = 1, 2)$ by deleting from $[\mathbb{I}]$ every row whose number corresponds to a node that is not in the interior of Γ_i. This matrix makes it possible to extract the dofs at the interior of Γ_i from $[X]$ through the matrix–vector product $[X_i] = [\mathbb{I}_i][X]$.

To carry on with the construction of the decomposition method, we now introduce a partition of unity α_1 and α_2 associated with this decomposition of Γ. Recall that this means that α_1 and α_2 are two non-negative functions defined on Γ which are equal to 0 in the exterior of respectively Γ_1 and Γ_2 and that they satisfy $\alpha_1 + \alpha_2 = 1$ on Γ. We now associate a diagonal matrix $[\alpha_j]$ $(j = 1, 2)$ with respectively α_j $(j = 1, 2)$. This matrix simply describes the linear mapping obtained by multiplying each component $[X_j]_i$ of $[X_j]$ by the respective coefficient $\alpha_j(x_i)$ where x_i is the node in the interior of Γ_j associated with $[X_j]_i$. In this way, the partition of unity yields an additive decomposition of any vector $[X]$ of dofs in the following manner

$$[X] = [\mathbb{I}_1]^{\mathrm{T}}[\alpha_1][\mathbb{I}_1][X] + [\mathbb{I}_2]^{\mathrm{T}}[\alpha_2][\mathbb{I}_2][X] \tag{8}$$

Clearly, using the decomposition (8) several times, we can now equivalently put the system (7) successively in the following forms

$$\begin{cases} [\alpha_1][\mathbb{I}_1][Z][X] = [U_1], & [U_1] := [\alpha_1][\mathbb{I}_1][U] \\ [\alpha_2][\mathbb{I}_2][Z][X] = [U_2], & [U_2] := [\alpha_2][\mathbb{I}_2][U] \end{cases}$$

$$\begin{cases} [\mathbb{I}_1][X] + [Z_{11}]^{-1}[Z_{12}][\mathbb{I}_2][X] = [Z_{11}]^{-1}[U_1] \\ [Z_{22}]^{-1}[Z_{21}][\mathbb{I}_1][X] + [\mathbb{I}_2][X] = [Z_{22}]^{-1}[U_2] \end{cases}$$

$$\begin{aligned} &\big([\mathbb{I}] + [\mathbb{I}_1]^{\mathrm{T}}[\alpha_1][Z_{11}]^{-1}[Z_{12}][\mathbb{I}_2] + [\mathbb{I}_2]^{\mathrm{T}}[\alpha_2][Z_{22}]^{-1}[Z_{21}][\mathbb{I}_1]\big)[X] \\ &\quad = [\mathbb{I}_1]^{\mathrm{T}}[\alpha_1][Z_{11}]^{-1}[U_1] + [\mathbb{I}_2]^{\mathrm{T}}[\alpha_2][Z_{22}]^{-1}[U_2] \end{aligned}$$

with $[Z_{ij}] = [\alpha_i][\mathbb{I}_i][Z][\mathbb{I}_j]^{\mathrm{T}}[\alpha_j]$.

This last system is solved by a Krylov method (e.g., [23]). Observe that the main cost of a matrix–vector product is the computation of $[Z_{11}]^{-1}[Z_{12}][\mathbb{I}_2][X]$ and $[Z_{22}]^{-1}[Z_{21}][\mathbb{I}_1][X]$ which can be interpreted as the determination of the currents relative to a scattering problem set on Γ_1 and Γ_2 respectively. Finally, note that this formulation can be extended in an obvious way to equations also involving magnetic currents.

3.2. Validation of the method

To validate the above method, we consider the above example of a box cavity with thick walls. Adding to the left-hand side of (5) the Rumsey reaction in the exterior domain Ω_{N+1}, we obtain the following well-posed system of integral equations

$$\sum_{p=1}^{N+1} \int_{\partial\Omega_p} (\mathbf{E}_p \cdot \mathbf{J}'_p - \mathbf{H}_p \cdot \mathbf{M}'_p)\,\mathrm{d}S = 0$$

where now the reaction in Ω_{+1} includes the incident wave

$$\mathbf{E}_p = \mathbf{E}^{\mathrm{inc}} + \frac{1}{2}\mathbf{n}_p \times \mathbf{M}_p + \mathrm{i}kZ\,T_p\mathbf{J}_p + K_p\mathbf{M}_p$$

$$\mathbf{H}_p = \mathbf{H}^{\mathrm{inc}} - \frac{1}{2}\mathbf{n}_p \times \mathbf{J}_p + \mathrm{i}kZ^{-1}\,T_p\mathbf{M}_p - K_p\mathbf{M}_p$$

After the elimination of the dofs internal to the cavity, these equations are reduced to a linear system whose unknowns are the dofs of the electric currents on the opening and the exterior of the cavity and the magnetic ones on the opening of the cavity only. In Fig. 6, we have depicted the meshes used to remove the dofs at the interior to the cavity. In the same figure, we have also indicated the zone where the surface Γ_1 at the opening of the cavity and the surface Γ_2 related to the exterior side of the cavity walls overlap each other.

In Fig. 7, we have reported the reduction of the residual during the iterations for several kinds of overlapping zones. The overlapping area is indicated by a thick line on a schematic representation of the box cavity. Case (a) corresponds to an overlapping zone so narrow that the method reduces to a Krylov iterative solution of the full system using the corresponding block Jacobi algorithm as a preconditioner. The algorithm is then poorly convergent. Surprisingly enough, case (b) shows that putting the overlapping area far away from the zone where complicated phenomena are occurring slows down the convergence. Cases (c) and (d) show that convergence becomes rather independent of the overlapping zone once this zone is correctly designed and located.

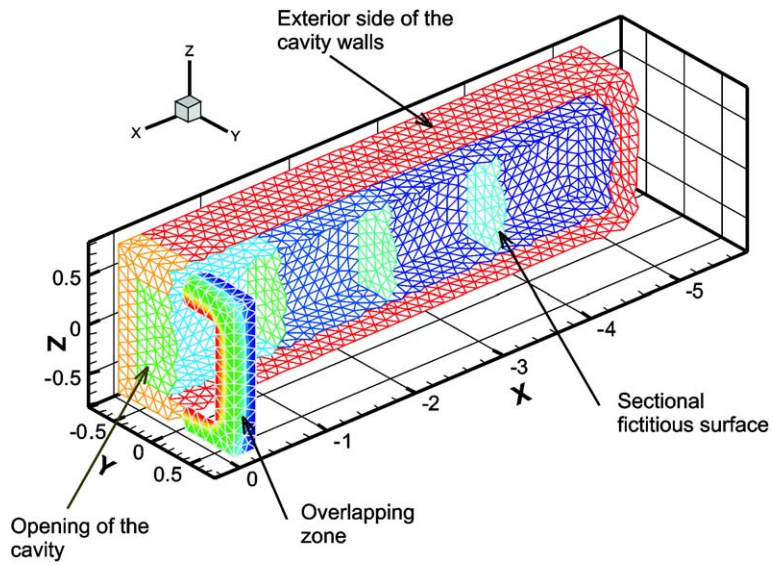


Fig. 6. The box cavity and the various meshes involved in the solution procedure.

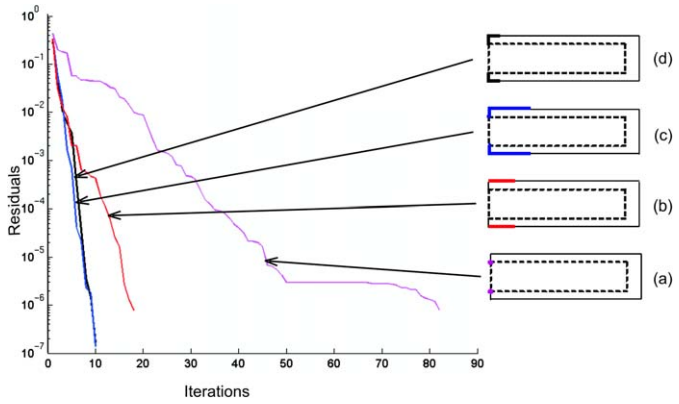


Fig. 7. History decreasing of the residual for various overlapping areas.

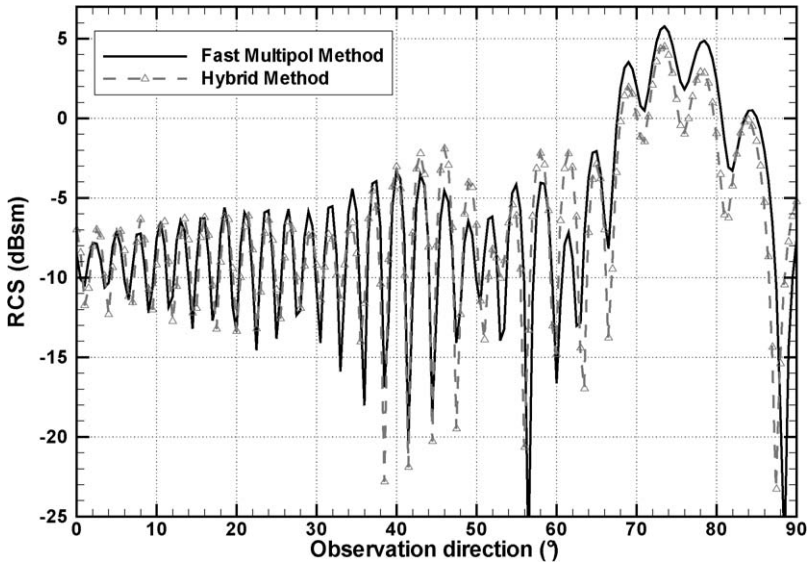


Fig. 8. Plots of the RCS computed by a Boundary Integral Equation solved by the FMM and the present coupling method.

The results of several tests reported in [26] show that the error on the RCS is less than 0.5% as compared to a direct solution.

3.3. Overlapping boundary decomposition and hybridization of a BEM with an asymptotic method

Of course, the above Boundary Decomposition Method (BDM) can be used as a solution process. Instead of solving one large size problem, one has to invert only moderately sized linear systems during the iterations. However, our main motivation here is to deal with the cavity by an exact method and to treat the exterior of the structure by means of an asymptotic method. Assume that the equations related to Γ_1 are dealt with through an exact solving and those related to Γ_2 by means of an asymptotic technique. Roughly speaking, this mainly amounts to computing $[Z_{12}]([\mathbb{I}_2][X])$ and $[Z_{22}]^{-1}[Z_{21}]([\mathbb{I}_1][X])$ approximately by means of a high frequency method, respectively $[Z_{12}]^{\mathrm{HF}}[X_2]^{\mathrm{HF}}$ and $([Z_{22}]^{-1}[Z_{21}])^{\mathrm{HF}}$. We refer for details to Balin's PhD thesis [26].

In Fig. 8, we compare the monostatic RCS of the COBRA cavity within a fuselage performed using the hybrid approach with the one obtained by means of the EFIE solved by the MLFMM. The frequency is 10 GHz. This amounts to considering that the length of the fuselage is 40 wavelengths. The results clearly demonstrates the reliability of the approach. Moreover the hybrid method used only 20% of the computation time that is necessary for the MLFMM

solution. It is worth noting that the MLFMM is used here within its general domain of validity contrary to the hybrid method which can tackle much more higher frequency cases.

4. Concluding remarks

The above brief review of some applications of DDMs to solve problems related to radiation or scattering of time-harmonic electromagnetic waves more efficiently clearly demonstrates the power and the potential of such techniques.

For the scattering problem related to a large structure with a deep cavity, we have seen how these techniques yield solvers much more efficient than the standard approaches. For instance, the substructuring procedure restores the suitability of the CFIE formulation in the context of an iterative solution. This makes it possible to address problems at higher frequencies which otherwise would be too time and memory consuming. In the same manner, the techniques, adapted from overlapping DDMs, enable the development of very efficient hybrid procedures. For the scattering cavity problem, the hybrid technique only uses about 20% of the computation time related to the fastest direct approach.

However, in our opinion, some related questions need further investigation.

As iterative procedures, the DDMs basically require a stopping criterion. Usually, this criterion is a reduction of the initial residual either for the coarse problem or for the direct equations. In many cases, this criterion tests the accuracy reached in the computation of the solution to the discrete system. As a result, many costly iterates are computed without any impact on the true accuracy, that is, relatively to the solution without any error coming from the discretization. Some possible remedies could be the use of a posteriori errors estimates like in adaptive FEM (e.g., [35]) or to take advantage of a balance of energy as was carried out in [36].

As indicated above, one of the main techniques in the use of DDMs for time-harmonic electromagnetic problems is to express the matching conditions through, roughly speaking, a combination of the electric field and the electric currents. However these quantities are naturally in duality instead of being in the same functional space. Apart from the FEM where this kind of condition can be dealt with directly by means of the variational formulation, even if some extra regularity has to be assumed a priori on the trial and test functions, it is generally hard to mimic this relationship at the discrete level in a BEM solution (e.g., [37]). In the same way, it is also hard to perform the rotation by $\pi/2$ around the normal to the surface to pass from fields to currents at the FEM level (see [38] and [39]). In our opinion, this issue still remains a pending difficulty which requires to be handled from a completely new point of view.

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