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C. R. Physique 7 (2006) 4–21
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Superconductivity and magnetism/Supraconductivité et magnétisme

Conventional magnetic superconductors: coexistence of singlet superconductivity and magnetic order

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Available online 20 January 2006

Abstract

The basic physics of bulk magnetic superconductors (MS), related to the problem of the coexistence of singlet superconductivity (SC) and magnetic order, is reviewed. The interplay between the exchange (EX) and electromagnetic (EM) interaction is discussed. It is argued that the singlet SC and uniform ferromagnetic (F) order practically never coexist. In the case of their mutual coexistence the F order is modified into a domain-like or spiral structure depending on magnetic anisotropy. It turns out that this situation occurs in several superconductors such as $ErRh_4B_4$, $HoMo_6S_8$, $HoMo_6Se_8$ with electronic and in $AuIn_2$ with nuclear magnetic order. The latter problem is briefly discussed.

The coexistence of SC with antiferromagnetism (AF) is more favorable than with the modified F order. Very interesting physics is realized in AF systems with SC and weak-ferromagnetism which results in an very rich phase diagram.

A number of properties of magnetic superconductors in magnetic field are very peculiar, especially near the (ferro)magnetic transition temperature, where the upper critical field becomes smaller than the thermodynamical critical field.

The interesting physics of Josephson junctions based on MS with spiral magnetic order is also discussed. The existence of the triplet pairing amplitude $F_{\uparrow\uparrow}$ ($F_{\downarrow\downarrow}$) in MS with rotating magnetization (the effect recently rediscovered in SFS junctions) gives rise to the so called π-contact. Furthermore, the interplay of the superconducting and magnetic phase in such a contact renders possible a new type of coupled Josephson-qubits in a single Josephson junction. ***To cite this article: M.L. Kulić, C. R. Physique 7 (2006).***

Résumé

Supraconducteurs magnétiques conventionnels : coexistence de la superconductivité singulet et de l'ordre magnétique. Nous passons en revue la physique de base des supraconducteurs magnétiques massifs appliquée au problème de la coexistence de la supraconductivité singulet (SC) et de l'ordre magnétique. L'interelation entre l'interaction d'échange (EX) et l'interaction électromagnétique (EM) est discutée. Nous argumentons sur le fait que la SC singulet et l'ordre ferromagnétique (F) uniforme ne coexistent pratiquement jamais. En cas de coexistence mutuelle, l'ordre F est modifié en une structure spiralée dépendant de l'anistropie magnétique. Il s'avère que cette situation est réalisée dans plusieurs supraconducteurs, notamment dans $ErRh_4B_4$, $HoMo_6S_8$, $HoMo_6Se_8$ avec un ordre électronique et dans $AuIn_2$ avec un ordre magnétique nucléaire. Ce dernier problème est brièvement discuté.

La coexistence de la SC avec l'antiferromagnétisme (AF) est plus favorable qu'avec l'ordre F modifié. La physique des systémes AF avec SC et ferromagnétisme faible est très intéressante et produit un diagramme de phases très riche.

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doi:10.1016/j.crhy.2005.11.012

Les supraconducteurs magnétiques sous champ magnétique présentent certaines propriétés très particulières, notamment au voisinage de la température de transition (ferro)magnétique, température à laquelle le champ critique supérieur devient plus faible que le champ critique thermodynamique.

Nous discutons aussi l'intéressante physique des jonctions Josephson à base de MS avec ordre magnétique spiral. L'existence d'une amplitude d'appariement triplet $F_{\uparrow\uparrow}$ ($F_{\downarrow\downarrow}$) dans les MS avec aimantation tournante (effet récemment redécouvert dans les jonction SFS) donne lieu au « contact π ». En outre, l'interaction entre phases supraconductrice et magnétique dans un tel contact ouvre la voie à la réalisation de qubits Josephson couplés dans une jonction Josepson unique. ***Pour citer cet article : M.L. Kulić, C. R. Physique 7 (2006).***

Keywords: Superconductivity; Coexistence; Magnetic order; Triplet amplitude; π-Josephson contact; Qubits

Mots-clés : Superconductivité ; Coexistence ; Ordre magnétique ; Amplitude d'appariement triplet ; Contact Josephson π ; Qubits

1. Introduction

The physics of magnetic superconductors is a very interesting subject due to the pronounced competition of magnetic order and singlet superconductivity in bulk materials. The question of their coexistence was first raised theoretically in the pioneering work by Vitalii Lazarevich Ginzburg [1] in 1956, where only the *electromagnetic* (*EM*) *interaction* between magnetic moments and superconductivity was considered. However, the breakthrough in the physics of MS came after the discovery of *ternary rare earth* (*RE*) compounds such as borides $(RE)T_4B_4$ with transition elements $T = \text{Rh}$, Ir, chalcogenides $(RE)\text{Mo}_6\text{X}_8$ ($X = \text{S}$, Se), silicides $(RE)_2T_2\text{Si}_5$ and stannides $(RE)T_x\text{Sn}_5$ [2–4]. In most of them, type-II superconductivity is realized and in all of them the localized RE magnetic moments are *regularly distributed* in the crystal lattice. The basic crystallographic structure, for instance in $RE\text{Rh}_4\text{B}_4$, contains localized moments (LMs) placed far a way from the Rh and B blocks which deliver conduction electrons. Due to this spatial separation the conduction electrons rarely jump to magnetic ions, thus making the direct exchange interaction (EX) J_{sf} much smaller than in transition metallic magnets, i.e., $J_{\text{sf}}(\ll 10^3\ \text{K})$. In these systems the 4f rare-earth shell is responsible for localized moments, since the f-level lies much below the Fermi energy, $E_f \ll E_F$. A number of compounds belonging to the above families have shown coexistence of superconductivity and antiferromagnetism—*antiferromagnetic superconductors* (AFS), such as $(RE)\text{Rh}_4\text{B}_4$ ($RE = \text{Dy, Sm}, \ldots$). In most of them the Néel (AF) transition temperature T_N is smaller than the superconducting one T_c, i.e., $T_N \ll T_c$.

However, starting from the late seventies much research, both experimental and theoretical, was devoted to MS systems in which ferromagnetic (F) order and singlet SC compete due to their strong antagonistic characters. These systems are usually called *ferromagnetic superconductors* (*FS*) and they are the main subject of this review. It turned out that SC and the modified F order can coexist under certain conditions, since the F order is transformed (in the presence of superconductivity) into a spiral or domain-like structure—depending on the type and strength of magnetic anisotropy in the system [4,5]. In the RE ternary compounds this competition is rather strong and therefore these two orderings coexist in a limited temperature interval $T_{c2} < T < T_m$ (the re-entrant behavior), for instance in ErRh_4B_4 and HoMo_6S_8. The coexistence region in ErRh_4B_4 is narrow, with $T_m \approx 0.8$ K, $T_{c2} \approx 0.7$ K while at $T_{c1} = 8.7$ K superconductivity appears in the paramagnetic state. In HoMo_6S_8 the coexistence interval is even narrower with $T_m \approx 0.74$ K, $T_{c2} \approx 0.7$ K, while $T_{c1} = 1.8$ K—see [2,4,5]. In HoMo_6Se_8 where $T_{c1} = 5.5$ K, $T_m \approx 0.8$ the exchange interaction is weaker than in the previous two systems and the coexistence persists down to $T = 0$ K.

A new and very interesting research field in the physics of ferromagnetic superconductors was opened in 1997 by the Pobell's group in Bayreuth [6], which discovered the coexistence of *superconductivity and nuclear magnetic order* in the type-I superconductor AuIn_2. In this system SC appears at $T_{c1} = 0.207$ K and $T_m = 35$ μK. Although in this system there is a tendency to nuclear ferromagnetic order, superconducting electrons enforce the appearance of a spiral or domain-like nuclear magnetic ordering in the SC state below T_m [7], depending on the strength of the nuclear dipole–dipole interaction.

It turns out that not only the bulk properties of FS are exotic, but also Josephson junctions made of bulk MS with spiral ordering show potentially fascinating properties, such as π-contact [8], a combination of a magnetic analog of the Josephson effect for spin current with the ordinary Josephson effect for charge current [9]. In such a system a realization of two qubits in a single Josephson junction is potentially possible.

In the following we shall discuss mainly the *microscopic and macroscopic theory of ferromagnetic and antiferromagnetic superconductors* which takes into account the most relevant interactions between localized moments and conduction electrons—the *exchange* (EX) and *electromagnetic* (EM) interaction. This theory was elaborated by Buzdin, Bulaevskii, Panyukov and the present author—see [4] and references therein, and successfully applied to a number of systems. Due to the lack of space some interesting effects of magnetic field on magnetic superconductors will be discussed briefly. For this subject we refer the reader to [4].

We would like to point out here that in recent years there has been a huge activity in studying of hybrid heterogeneous magnetic superconductors such as S–F multilayers and S–F–S Josephson junctions. This field is not only of importance for fundamental solid state physics, but it is of enormous interest for applications in spintronics and quantum computing, especially after the experimental confirmation [10] of the remarkable theoretical prediction of the π-Josephson contact by Alexander Buzdin and coworkers [11,12]. This exciting, and for applications, important, field will be covered elsewhere in this issue. The physics of other magnetic superconductors such as heavy fermions, borocarbides $(RE)Ni_2B_2C$, cuprates $RuSr_2GdCu_2O_8$, ferromagnets with triplet SC such as UGe_2 will be discussed elsewhere in this issue.

2. Competition between SC and F order in FS

Here we shall be limited to those magnetic superconductors in which the magnetic ordering of the *localized 4f moments* (LM) is due to the indirect exchange interaction (RKKY) going via the conduction electrons. The characteristic exchange energy ($\theta_{\rm ex}$) is of the order of $\theta_{\rm ex} \approx N(0)h_0^2$, where $N(0)$ is the density of states at the Fermi level (per LM) and $h_0(= (g-1)nJ_{\rm sf}(0)\langle \hat{J}_z\rangle)$ is the maximal exchange field acting on conduction electrons. Here, g is the Lande factor, n is the density of localized magnetic moments (LMs), $J_{\rm sf}(0)$ is the direct exchange energy between conduction electrons and LMs, $\langle \hat{J}_z\rangle$ is the averaged total angular moment of the LM. Note that the exchange field acting on electrons is $h_{\rm ex}(\mathbf{r}) = h_0 S(\mathbf{r})$, where $S(\mathbf{r})(= \langle \hat{J}_z\rangle / J)$ is the localized spin normalized to one. Let us mention in advance that in a number of RE ternary compounds the exchange field h_0 is still rather large, i.e., $h_0 \sim 10^2$ K and $h_0 \gg \Delta_0 \lesssim 10$ K. We shall see below that in spite of the fact that h_0 is larger than the Clogston paramagnetic field h_p, i.e., $h_0 \gg h_p \approx 0.7\Delta_0$, there is a coexistence of SC and modified ferromagnetic order. In the presence of magnetic ordering, characterized by the magnetization $\mathbf{M}(\mathbf{r})$, there is an electromagnetic interaction between localized moments and (super)conducting electrons. The magnetization $\mathbf{M}(\mathbf{r}) = n\mu\mathbf{S}(\mathbf{r})$ creates a dipolar magnetic field $\mathbf{B}(\mathbf{r}) = \mathrm{rot}\,\mathbf{A}(\mathbf{r})$ which on the other side induces the screening current $\mathbf{j}_s$ of conduction electrons (the Meissner effect). The Fourier transformed $\mathbf{j}_s$ is related to $\mathbf{A}$ by the kernel $K_s(\mathbf{q})$, i.e., $\mathbf{j}_s(\mathbf{q}) = -K_s(\mathbf{q})\mathbf{A}(\mathbf{q})$. Having in mind those magnetic superconductors which are based on RE ternary compounds we shall discuss the physics in the mean-field approximation for both the SC and magnetic subsystem, i.e., in the Hamiltonian we replace $\hat{\mathbf{S}}(\mathbf{r}) \to \mathbf{S}(\mathbf{r}) = \langle \hat{\mathbf{S}}(\mathbf{r})\rangle$ and $g\psi(\mathbf{r})_\uparrow\psi(\mathbf{r})_\downarrow \to \Delta(\mathbf{r}) = g\langle\psi(\mathbf{r})_\uparrow\psi(\mathbf{r})_\downarrow\rangle$. Here, $\Delta(\mathbf{r})$ is the *singlet superconducting order parameter* and g is the electron–phonon coupling constant.

The total Hamiltonian of the system is given by

$$\hat{H} = \hat{H}_{\rm BCS} + \hat{H}_{\rm CF} + \hat{H}_{\rm imp} + \sum_i \bigl[-\mathbf{B}(\mathbf{r}_i)g\mu_B\hat{\mathbf{J}}_i + \hat{H}_{\rm CF}(\hat{\mathbf{J}}_i)\bigr] \tag{1}$$

$$\hat{H}_{\rm BCS} = \int d^3r\,\biggl\{\hat{\psi}^\dagger(\mathbf{r})\biggl[\hat{\varepsilon}\biggl(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\biggr) - \mu\biggr]\hat{\psi}(\mathbf{r}) + \hat{\psi}^\dagger(\mathbf{r})\hat{V}_{\rm ex}(\mathbf{r})\hat{\psi}(\mathbf{r}) + \frac{1}{2}\Delta(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\mathrm{i}\sigma_y\hat{\psi}^\dagger(\mathbf{r}) - \frac{1}{2}\Delta^*(\mathbf{r})\hat{\psi}(\mathbf{r})\mathrm{i}\sigma_y\hat{\psi}(\mathbf{r}) + \frac{|\Delta(\mathbf{r})|^2}{g_{\rm epi}}\biggr\}. \tag{2}$$

Here, $\hat{\varepsilon}(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A})$ is the band energy of electrons in magnetic field, $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z\}$ are Pauli matrices, while the *exchange field* acting on electrons is given by

$$\hat{V}_{\rm EX}(\mathbf{r}) = \begin{pmatrix} h_{\rm ex}^z(\mathbf{r}) & h_{\rm ex}^\perp(\mathbf{r}) \\ h_{\rm ex}^{\perp,*}(\mathbf{r}) & -h_{\rm ex}^z(\mathbf{r}) \end{pmatrix} \tag{3}$$

For the moment we go in advance by informing the reader that in the SC phase the ferromagnetic order is modified into a spiral or domain-like structure with the wave vector $\mathbf{Q}$. Which magnetic structure occurs depends on the magnetic anisotropy which is contained in the operator $\hat{H}_{\mathrm{CF}}$. If the magnetic anisotropy is small (or is of the easy-plane type) than the spiral structure is realized with $h^{\perp}(\mathbf{r}) = h\mathrm{e}^{\mathrm{i}Qz}$ and $h^{z}(\mathbf{r}) = 0$, while in the opposite case, with an easy axis anisotropy, one has $h^{\perp}(\mathbf{r}) = 0$ and $h^{z}(\mathbf{r}) = h^{z}(\mathbf{r} + \mathbf{L})$, $L = 2\pi/Q$. The effect of nonmagnetic impurities is described by $\hat{H}_{\mathrm{imp}}$ whose effect is characterized by the mean-free path l and time τ.

2.1. Sinus magnetic structure due to SC for $T \lesssim T_m$

In RE ternary magnetic superconductors in which the singlet SC and ferromagnetic order compete, the superconducting critical temperature, T_{c1}, is much higher than the magnetic one, i.e., $T_m \ll T_{c1}$. Before discussing the complete phase diagram we shall study the coexistence problem at temperatures near T_m, i.e., $T \lesssim T_m$, where the magnetic order parameter is small $S \ll 1$. In the case when the easy-axis magnetic anisotropy D is sufficiently large, then the sinus structure $\mathbf{S}(\mathbf{r}) \approx S(T)\sin \mathbf{Qr}$ appears below T_m (for small D a spiral order is favored—see Section 2.3). In that case $h_{\mathrm{ex}}(\mathbf{r}) = h_0|\mathbf{S}(\mathbf{r})| \ll h_0, \varDelta$ and the free-energy can be calculated by perturbation theory

$$F\{\mathbf{S}(\mathbf{r}), \varDelta(\mathbf{r}), \mathbf{A}(\mathbf{r})\} = F_M\{\mathbf{S}\} + F_S\{\varDelta\} + F_{\mathrm{Int}}\{\mathbf{S}, \varDelta, \mathbf{A}\} \tag{4}$$

Here, F_M and F_S are the magnetic and SC functional without mutual interaction, respectively. Near T_{m0} the free-energy has the form

$$F_M\{\mathbf{S}(\mathbf{r})\} = n\sum_q \left\{\frac{1}{2}\left[(T - T_{m0}) + \theta a^2 q^2\right]\left|\mathbf{S}(\mathbf{q})\right|^2 - D\left|S_z(\mathbf{q})\right|^2\right\} + \int \mathrm{d}^3 r\, \frac{(\mathbf{B} - 4\pi\mathbf{M})^2}{8\pi} \tag{5}$$

where $\mathbf{S}(\mathbf{q})$ is the Fourier transform of $\mathbf{S}(\mathbf{q})$. The last term in Eq. (5) is the magnetic energy for a given magnetization $\mathbf{M}(\mathbf{r}) = n\mu\mathbf{S}(\mathbf{r})$ and in the equilibrium the magnetic induction is given by $\mathbf{B} = 4\pi\mathbf{M}$. The characteristic energy for the EM interaction is given by $\theta_{\mathrm{em}} = (B^2/8\pi n) = 2\pi n\mu^2$ which is $\sim$1 K in the RE ternary compounds—see Table 1.

At temperatures near T_m ($\ll T_{c1}$) the exchange field is small, i.e., $h \ll \varDelta$, and the SC free-energy density ($F_S = \int \mathrm{d}^3 r\, \widetilde{F}_S$) has the form

$$\widetilde{F}_S\{\varDelta(\mathbf{r})\} = -\frac{1}{2}N(0)\varDelta^2 \ln\frac{e\varDelta_0^2}{\varDelta^2} \tag{6}$$

It is minimized for $\varDelta \approx \varDelta_0$ and weakly affected by magnetism and therefore we omit it from the analysis at T near T_m. The part F_{Int} describes the EX and EM interaction between SC and magnetic order (note $\mathbf{j}_s(\mathbf{q}) = -K_s(\mathbf{q})\mathbf{A}(\mathbf{q})$)

$$F_{\mathrm{Int}} = \sum_q \left\{\frac{\theta_{\mathrm{ex}}}{2}\frac{\chi_n(\mathbf{q}) - \chi_s(\mathbf{q})}{\chi_n(0)}\left|\mathbf{S}(\mathbf{q})\right|^2 + \frac{1}{2}K_s(\mathbf{q})\left|\mathbf{A}(\mathbf{q})\right|^2\right\} \tag{7}$$

Table 1
Basic parameters of the ferromagnetic superconductors $ErRh_4B_4$, $HoMo_6S_8$ and $HoMo_6Se_8$. The parameters are defined in the text

	$ErRh_4B_4$	$HoMo_6S_8$	$HoMo_6Se_8$
n [cm^{-3}]	$\sim 10^{22}$	$\sim 4\times 10^{21}$	$\sim 4\times 10^{21}$
μ [μ_B]	5.6	9.1	
$\tilde{a}$ [Å]	$\sim$1	2.5	2.7
$\lambda_L(0)$ [Å]	900	1200	
ξ_0 [Å]	200	1500	470
$\varDelta_0$ [K]	15.5	3.2	10
$N(0)^{-1}$ [K spinRE]	1850	3600	1754
v_F [cm s^{-1}]	1.3×10^7	1.8×10^7	1.8×10^7
θ_{ex} [K]	0.5–0.8	0.2	$0.1 < \theta_{\mathrm{ex}} < 0.34$
h_0 [K]	40	24	$\sim$10
τ_m^{-1} [K]	3	0.9	
θ_{em} [K]	1.8	1.3	$\sim$1.3
T_{c1} [K]	8.7	1.8	5.5
T_m [K]	0.8–1	0.7–0.74	0.53
T_{c2} [K]	0.7	0.65	no
$L = 2\pi/Q$ [Å]	90–100	200	70–100

where $\chi_n(\mathbf{q})$ and $\chi_s(\mathbf{q})$ are electronic susceptibilities in the normal and SC state, respectively. The EM Kernel $K_s(q)$ describes the screening effect of the dipole–dipole interaction by the superconducting electrons.

After minimization of $F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r}), \mathbf{A}(\mathbf{r})\}$ with respect to $\mathbf{A}(\mathbf{r})$ one obtains $F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r})\}$ in the following form (see more below and in [4])

$$F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r})\} = n\sum_q\left\{\frac{1}{2}\left[(T-T_{m0})+\theta a^2q^2\right]\left|\mathbf{S}(\mathbf{q})\right|^2 - D_z\left|S_z(\mathbf{q})\right|^2 + \frac{\theta_{\mathrm{ex}}}{2}\frac{\chi_n(\mathbf{q})-\chi_s(\mathbf{q})}{\chi_n(0)}\left|\mathbf{S}(\mathbf{q})\right|^2 + \frac{\theta_{\mathrm{em}}}{2}\frac{4\pi K_s(q)|\mathbf{S}(\mathbf{q})|^2+(\mathbf{q}\mathbf{S}(\mathbf{q}))(\mathbf{q}\mathbf{S}(-\mathbf{q}))}{q^2+4\pi K_s(q)}\right\} \tag{8}$$

The length a is of the order of the lattice constant (magnetic stiffness) and the bare critical temperature T_{m0} and θ (note $T_{m0} \neq \theta$ [4]) take into account, in a subtle way, the indirect EX and direct dipole–dipole (EM) interaction between LMs—see [4]. $\theta_{\mathrm{em}} = 2\pi\mu^2$ characterizes the EM effects in the dipole–dipole interaction between LMs. $D_z(>0)$ is the magnetic anisotropy which orients spins along the z-axis.

Due to the singlet SC pairing $\chi_s(\mathbf{q})$ is reduced significantly at small wave vectors $q < \xi_0^{-1}$ where ξ_0 is the SC coherence length. In the *clean limit* ($l \to \infty$) and at $T = 0$ one has $\chi_s(\mathbf{0}) = 0$ which means that the *ferromagnetic order can not coexist with singlet superconductivity*. In Fig. 1 we show $\chi_{s,n}(\mathbf{q})$ schematically for the cases when the ferromagnetic (a) or antiferromagnetic order (b) is realized in the normal state. It is seen that a singlet superconductor behaves as a normal metal at large momenta, i.e., $\chi_s(q \sim k_F)$ is weakly affected by SC. Therefore AF competes with SC much less than the ferromagnetic order does. This analysis has been first made in the remarkable paper by Anderson and Suhl [17]. They were the first who predicted a nonuniform ('crypto-ferromagnetic') magnetic order below the Curie temperature with the period much smaller than ξ. This idea was further elaborated in a number of papers—see review [4], where all relevant interactions are included, such as the EX and EM interaction, effects of non-magnetic impurities and also of magnetic anisotropy.

We stress that at finite temperature one has $\chi_s(\mathbf{0}) \neq 0$ which is exponentially small in the singlet s-wave SC, while in d-wave SC one has $\chi_s(\mathbf{0}) \sim \chi_n(\mathbf{0})(T/\Delta_0)$. In the presence of the spin-orbit (SO) scattering $\chi_s(\mathbf{0})$ is also finite. The general expression for $\chi_s(\mathbf{q})$ is calculated in [18]

$$\chi_s(\mathbf{q}) = 1 - \pi T\sum_{\omega_n}\frac{1}{(1+u_\omega^2)(P(\omega,q)-1/2\tau_1)} \tag{9}$$

where $u_\omega = \omega/\Delta$ and

$$P(\omega,q) = \frac{1}{2}\frac{qv_F}{\arctan\{\frac{qv_F}{2\Delta}\sqrt{1+u_\omega^2+1/2\tau_-}\}} \tag{10}$$

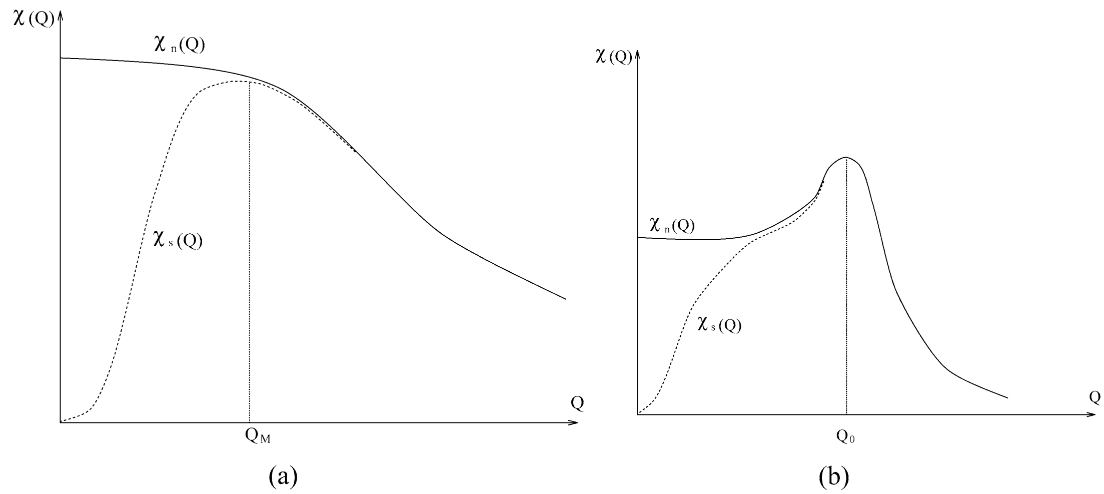

Fig. 1. Schematic spin susceptibility in the normal and SC state $\chi_{n,s}(\mathbf{q})$: (a) for the ferromagnetic order in the normal state—peak at $Q = 0$; (b) for the antiferromagnetic order—peak at Q_0.

Here, $\tau_1^{-1} = \tau_-^{-1} - (4/3)\tau_{so}^{-1}$, $\tau_-^{-1} = \tau^{-1} + \tau_{so}^{-1}$, $l_{so} = v_F \tau_{so}$ and $\omega_n = \pi T(2n+1)$. Later, we shall discuss the effect of the SO interaction on the coexistence phase. The effect of the exchange scattering is similar, i.e., $\chi_s(\mathbf{0})$ is finite for finite exchange scattering time τ_s. Concerning this point, as well as to other theoretical questions related to magnetic superconductors, we refer the reader to the review [3].

Since in the following we study the competition between SC and ferromagnetism at low temperatures it is sufficient to give a general expression for the EM kernel $K_s(q)$ in the clean limit

$$K_s(q) = \frac{3n_e \Delta}{mcqv_F} \int_0^1 \mathrm{d}x \frac{1-x^2}{x} \frac{\operatorname{arcsh}(\frac{xqv_F}{2\Delta})}{\sqrt{1+(\frac{xqv_F}{2\Delta})^2}} \tag{11}$$

The expression for $K_s(q)$ for finite mean-free path l is rather complicated and therefore omitted here. Some limiting cases of $K_s(q)$, which are relevant for real magnetic superconductors, will be studied below.

By knowing $\chi_s(\mathbf{q})$ and $K_s(\mathbf{q})$ one can minimize $F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r})\}$ with respect to the wave vector q. In such a way the equilibrium magnetic structure is obtained, which depends on the microscopic parameters a, ξ_0, λ_L (the London penetration depth), θ_{ex}, θ_{em}. From Eq. (8) we conclude that the EM interaction is minimized for $\mathbf{q} \cdot \mathbf{S}(\mathbf{q}) = 0$, i.e., the magnetic structure is *transverse*.

Let us analyze $\chi_s(\mathbf{q})$ and $K_s(q)$ in the interesting range of parameters. In the *clean limit* and for $q\xi_0 \ll 1$ one has

$$K_s(q) = \frac{1}{4\pi\lambda_L^2}$$
$$\chi_n(\mathbf{q}) - \chi_s(\mathbf{q}) = \chi_n(0)\left(1 - \frac{\pi^2 q^2 \xi_0^2}{30}\right) \tag{12}$$

while for $q\xi_0 \gg 1$ it holds

$$K_s(q) = \frac{3}{4\lambda_L^2 q\xi_0}$$
$$\chi_n(\mathbf{q}) - \chi_s(\mathbf{q}) = \chi_n(0)\frac{\pi}{2q\xi_0} \tag{13}$$

Based on Eqs. (12) and (13) and after the minimization of the free-energy F in Eq. (8) we obtain that just below the transition temperature $T_m = T_{m0}([1 - 3(\pi\theta_{ex}a/4\theta\xi_0)^{2/3}]$ a transverse ($\mathbf{Q}_s \perp S_z$) *sinus structure* $S_z(\mathbf{r}) \approx S\sin(\mathbf{Q}_s\mathbf{r})$ is realized. In case when $\xi_0^2 \ll a\lambda_L$ the wave vector Q_s is determined by the EX interaction and for $\theta_{ex}/\theta_{em} \gg (a/\xi_0)^2$ it is given by [19,20]

$$Q_s = \left(\frac{\pi}{4}\frac{\theta_{ex}}{\theta a^2 \xi_0}\right)^{1/3} \tag{14}$$

For $\theta_{ex}/\theta_{em} \ll (a/\xi_0)^2$ the EM interaction prevails with

$$Q_s = \left(\frac{1}{a\lambda_L}\right)^{1/2} \tag{15}$$

In the opposite limit $\xi_0^2 \gg a\lambda_L$ the EX interaction dominates for $\theta_{ex}/\theta_{em} \gg (a^2\xi_0/\lambda_L^3)^{2/5}$ which gives again [19, 20]

$$Q_s = \left(\frac{\pi}{4}\frac{\theta_{ex}}{\theta a^2 \xi_0}\right)^{1/3} \tag{16}$$

For $\theta_{ex}/\theta_{em} \ll (a^2\xi_0/\lambda_L^3)^{2/5}$ the EM interaction dominates which gives

$$Q_s \approx \left(\frac{1}{a^2\xi_0\lambda_L^2}\right)^{1/5} \tag{17}$$

From these expressions it is seen that for realistic parameters the wave-vector Q_s is determined by the EX interaction—it is independent of the EM parameter λ_L, while the EM interaction (with λ_L dependence of Q) is dominant only for an extremely small EX interaction ($\theta_{ex} \ll \theta_{em}(a/\xi_0)^2$ or $\theta_{ex} \ll \theta_{em}(a^2\xi_0/\lambda_L^3)^{2/5}$), i.e., for $(\theta_{ex}/\theta_{em}) \ll$

10^{-4}–10^{-5}. However, in typical ferromagnetic superconductors, such as $ErRh_4B_4$, $HoMo_6S_8$, $HoMo_6Se_8$, $AuIn_2$, the *EX interaction dominates* since $\theta_{ex} > 0.1\theta_{em}$ and $a \ll \xi_0 \lesssim \lambda_L$.

In reality, nonmagnetic impurities are always present and the knowledge of $\chi_s(\mathbf{q}, l)$ and $K_s(q, l)$ as a function of the mean-free path l is needed. The corresponding calculations show that if $(l^5/a^2\xi_0\lambda_L^2) \ll 1$ and for $\theta_{ex}/\theta_{em} \gg a^2\xi_0/l^3$ one has [19,20]

$$Q_s = \left(\frac{\pi}{4}\frac{\theta_{ex}}{\theta a^2\xi_0}\right)^{1/3} \tag{18}$$

while

$$Q_s \approx \frac{\theta_{ex}}{\theta}\left(\frac{1}{la^2\xi_0}\right)^{1/4} \tag{19}$$

for $(a^2\xi_0/l^3) \gg \theta_{ex}/\theta_{em} \gg (l^2/\lambda_L^2)$, a^2l/ξ_0^3.

In the case when $(\theta_{ex}/\theta_{em}) \ll (a^2\xi_0/l^3)$, or $(\theta_{ex}/\theta_{em}) \ll l^2/\lambda_L^2$, the EM interaction dominates and

$$Q_s \approx \left(\frac{l}{a^2\xi_0\lambda_L^2}\right)^{1/4} \tag{20}$$

Let us stress some interesting properties of ferromagnetic superconductors:

(i) the ferromagnetic critical temperature T_{ferro} is strongly reduced in the presence of SC due to the formation of Cooper pairs in the SC state, i.e., one has $T_{ferro} = T_{m0}[1 - (\theta_{ex} + \theta_{em})/\theta] \ll T_m$. In fact this result is more general and holds also for the coexistence of SC and *itinerant ferromagnetism* (F) in the mean-field approximation (MFA). Namely, in systems where the pairing is due to the electron-phonon interaction and the EX interaction dominates the EM one, the singlet SC and ferromagnetism do not coexist in the MFA. In that sense a number of recent papers which claim that the itinerant F and SC coexist in the MFA should be abandoned [21]. However, in some *itinerant ferromagnets* such as Y_9Co_7 (with $T_F = 4.5$ K) the microscopic parameters favor spiral or domain magnetic structure in the SC state with $T_{c1} = 2.5$ K as it was proposed in [22];

(ii) in isotropic magnetic systems and near the critical temperature T_m the inverse scattering time due to magnetic fluctuations can diverge and thus destroy SC. However, this divergence is suppressed in the real RE ternary compounds due to the long-range dipole–dipole interaction. The interaction of SC with magnetic fluctuations is described by the free-energy contribution

$$F_{sc,fl} = \frac{\theta_{ex}}{2}\sum_{\mathbf{q}}\langle S_{z,q}S_{z,-q}\rangle\frac{\chi_n(\mathbf{q}) - \chi_s(\mathbf{q})}{\chi_n(0)} \tag{21}$$

where

$$\langle S_{z,q}S_{z,-q}\rangle \sim \frac{1}{\tau + a^2q^2 + (\theta_{em}/\theta)q_z^2/q^2} \tag{22}$$

with $\tau = (T - T_{m0})/\theta$. Due to the large dipole–dipole temperature with $\theta_{em} \sim \theta$ these fluctuations look four-dimensional, thus giving rather small value for the inverse scattering time $\tau_m^{-1} \sim \theta \ll T_{c1}$;

(iii) the relative strength of the EX and EM interaction is controlled by the parameter r

$$r = \frac{F_{Int}^{(EM)}}{F_{Int}^{(EX)}} = \frac{\theta_{em}}{\theta_{ex}}\frac{1}{Q^2\lambda_L^2} \tag{23}$$

In the RE ternary compounds the case $r \ll 1$ always occurs, due to the large value of $Q^2\lambda_L^2 \gg 1$. Therefore, practically in all RE ternary compounds the *EX interaction dominates* in the formation of magnetic structure, while the EM interaction makes it transversal—see exception in a weak ferromagnet below;

(iv) In the RE ternary compounds the ferromagnetism is stronger phenomenon than SC since the gain in the ferromagnetic energy (per LM and at $T = 0$ K) $E_m \approx N(0)h_0^2$ is larger than the gain in the SC condensation energy $E_c \approx N(0)\Delta_0^2$ since $h_0(\sim 10^2$ K$) \gg \Delta_0(\lesssim 10$ K$)$. Nevertheless, the ferromagnetic order is more 'generous' since it varies spatially in the SC state, while the SC order parameter is practically homogeneous. The reason for this peculiar phenomenon lies in the fact that the magnetic stiffness ($\sim a$) is much smaller than the superconducting stiffness ($\sim\xi_0$), since $a \ll \xi_0$.

2.2. Domain magnetic structure due to SC

By lowering T the higher order term in the free-energy, $\sim S^4(r)$, makes the change of the modulus of $\mathbf{S}(\mathbf{r})$ unfavorable. As a result, the sinus-structure is transformed, as it will be shown below, into the striped *domain structure* (DS)—see Fig. 2. The exchange field grows $h_{\mathrm{ex}} = h_0 S(T)$ by lowering T but if $h_{\mathrm{ex}}(T) < \Delta$ the mutual interaction of magnetism and SC can be treated by perturbation theory. In such a case the free-energy density is completed by the energy density of *domain walls*, QE_W/π, where E_W is the wall-energy per unit surface. In the case of sufficiently large magnetic anisotropy, $D_z > \theta$, the rotation of magnetic moments in the wall is unfavorable and the *linear domain wall* with $S_z(x) = S\,th(x/l_W)$, $S_x = S_y = 0$ is favored. Here, $l_W = a/\sqrt{\tau}$ is the domain-wall thickness [4]. The domain wall energy per unit surface is given by

$$E_W = \left(4\sqrt{2}/3\right) n\theta S^2 a \tau^{1/2} \equiv n\theta S^2 \tilde{a} \tag{24}$$

where $\tau = (T - T_{m0})/\theta$—see [4].

The free-energy density $\widetilde{F}_{\mathrm{DS}}$ in the DS phase is given by

$$\widetilde{F}_{\mathrm{DS}} = n\theta\left[\frac{1}{2}\tau S^2 + \frac{b}{4}S^4\right] + \frac{Q}{\pi}E_W + n\theta_{\mathrm{ex}}\frac{7\zeta(3)}{2\pi}\frac{S^2}{Q\xi_0} \tag{25}$$

It depends on S and Q only, since SC is practically unaffected by magnetism near T_m. Note, that if the anisotropy energy is small, i.e., $\tau > 2D_z/\theta$, the *rotating domain wall* is realized with the wall thickness $l_W \approx a(\theta/D_z)^{1/2}$ and the wall energy $E_W \sim nS^2 a(\theta D_z)^{1/2}$.

In the case of an extremely small anisotropy $(D_z/\theta) < (a/\xi_0) \sim 10^{-2}$–$10^{-3}$ the spiral structure occurs. Minimizing F_{DS} in Eq. (25) with respect to Q one obtains the equilibrium wave vector of the *striped DS phase*

$$Q_{\mathrm{DS}} \simeq 2\left(\frac{\theta_{\mathrm{ex}}}{\theta}\frac{1}{\tilde{a}\xi_0}\right)^{1/2} \tag{26}$$

By comparing Eqs. (14) and (26) one concludes that the period of the DS structure ($L = 2\pi/Q$) is larger than in the sinus phase. It is worth mentioning that: (i) the striped domain structure is due to SC and it is property of the bulk; (ii) the structure of the DS phase in the SC state is mathematically similar to the problem of the domain structure in a normal ferromagnetic plate with the magnetization perpendicular to the surface of the sample. In this case the role of F_{int} is played by the magnetic energy dissipated out of the plate. Generally, the domain structure is realized when the wall thickness $a/\sqrt{\tau}$ is much smaller then the domain thickness π/Q, thus implying that $\tau \gg (a/\xi_0)^{2/3} \sim 10^{-2}$.

At lower temperatures when the exchange field is large, i.e., $h_{\mathrm{ex}}(T) > \Delta$, the problem appears to be non-perturbative. Since the period L_D of the domain structure is much larger than a, i.e., $L_D \gg a$, the problem is studied

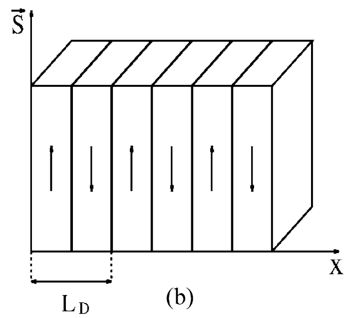

Fig. 2. The striped domain magnetic structure $\mathbf{S}(x) = S_z(x)\mathbf{e}_z$ with the period $L_D = 2\pi/Q_{\mathrm{DS}}$; $\mathbf{Q}_{\mathrm{DS}}$ is along the x-axis.

by the *quasiclassical ELO* equations. In the presence of nonmagnetic impurities these equations are solved for the striped domain structure given by

$$S_z(\mathbf{r}) = \frac{4S(T)}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\mathbf{Qr}}{2k+1} \equiv \sum_{\mathbf{q}} S_{z,\mathbf{q}}\mathrm{e}^{\mathrm{i}\mathbf{qr}} \tag{27}$$

By assuming that $\mathbf{Q}$ is along the x-axis the solution for the Green's function are searched in the form

$$f(\mathbf{v},x) = f_0(\mathbf{v}) + \sum_k f_k(\mathbf{v})\mathrm{e}^{\mathrm{i}kx} \tag{28}$$

and analogously for $g(\mathbf{v},x)$, where $k=(2m+1)Q$ and $|f_k| \ll |f_0|$, $|g_k| \ll |g_0|$. The calculations were done in [19,4] and here we present only the final result for the free-energy in the *dirty limit* ($l \ll \xi_0$). It turns out, in that case, that the interaction of the magnetic domain structure with SC is similar to the case of magnetic impurities with the inverse scattering time τ_m^{-1} and with $\tau_m\Delta > 1$, i.e., $\widetilde{F}_{\mathrm{DS}}$ is given by

$$\widetilde{F}_{\mathrm{DS}} = n\theta\left[\frac{1}{2}\tau S^2 + \frac{b}{4}S^4\right] + QE_W - \frac{1}{2}N(0)\Delta^2\ln\left(\frac{e\Delta_0^2}{\Delta^2}\right) + N(0)\frac{\pi\Delta}{2\tau_m}\left(1 - \frac{2}{3\pi\tau_m\Delta}\right) \tag{29}$$

where τ_m^{-1} is given by

$$\tau_m^{-1} = \sum_q\left\{\frac{\pi h_{z,q}h_{z,-q}}{v_F q}L_1(ql) + \frac{3\mathbf{B}_q\cdot\mathbf{B}_{-q}}{16\lambda_L^2 nN(0)v_F q^3}L_2(ql)\right\} \tag{30}$$

where $h_{z,q} = h_0 S_{z,\mathbf{q}}$. The functions L_1 and L_2 are given by

$$L_1(y) = \frac{2y\arctan y}{\pi(y - \arctan y)} \tag{31}$$

and

$$L_2(y) = \frac{2}{\pi}\left[\left(1 + \frac{1}{y^2}\right)\arctan y - \frac{1}{y}\right] \tag{32}$$

The magnetic induction $\mathbf{B}_q$ is related to $\mathbf{S}_q$ by

$$\mathbf{B}_q = \frac{4\pi n\mu[q^2\mathbf{S}_q - \mathbf{q}(\mathbf{q}\mathbf{S}_q)]}{q^2 + K_s(q)(1 - 4/3\pi\tau_m\Delta)} \tag{33}$$

where the EM kernel $K_s(q)$ in the dirty limit has the form

$$K_s(q) = \frac{3\pi\Delta}{16v_F\lambda_L^2 q}L_2(ql) \tag{34}$$

Based on the free-energy in Eq. (29) we can study the coexistence problem in the whole temperature regions and for various Ql—see more in [19]. We summarize the main results:

(i) at $T = T_m$ the sinusoidal magnetic order appears with the wave vector $Q_s \sim (1/a^2\xi_0)^{1/3}$ where $\mathbf{Q_s}$ is perpendicular to $\mathbf{S}$, i.e., the structure is transverse;

(ii) by lowering the temperature the striped domain structure appears with $Q_{\mathrm{DS}} \sim (1/a\xi_0)^{1/2}$, which is also transverse and persists down to the temperature of the *first order phase transition* T_{c2} where the DS phase passes into the normal ferromagnetic state. At T_{c2} one has

$$F_{\mathrm{DS}}\{S_{\mathrm{DS}}(T_{c2}), \Delta(T_{c2}), Q_{DS,c2}\} = F_{FN}\{S_F(T_{c2}), 0, 0\} \tag{35}$$

where $Q_{\mathrm{DS},c2} \approx 1.8(\tilde{a}(T_{c2})\xi_0)^{-1/2} \sim (a\xi_0)^{-1/2}$, $\Delta(T_{c2}) = 0.85\Delta_0$ and $(S_{c2}^2/Q_{c2}) \approx 0.07(\Delta_0 v_F/h_0^2)$;

(iii) if $S_{\mathrm{DS}}(T_{c2}) > 1$ this means that the DS is stable down to $T = 0$ K—this situation occurs in systems with small EX interaction (which still dominates over the EM one), i.e., for $\theta_{\mathrm{ex}} < \theta_{\mathrm{ex}}^c \sim (T_{c1}^3/h_0^2)$;

(iv) in *dirty* SC with $(h\tau)^2 \ll 1$ there is a gap in the quasiparticle spectrum for $E < \Delta$ in the whole range of the existence of the domain phase. The calculations show, that in *clean* SC the *spectrum is gapless* for $h(T) \gg \Delta$ [5,4]. For instance, in the *DS phase* one has for $E \ll \Delta$

$$\frac{N(E)}{N(0)} = \frac{\pi h}{v_F Q} \frac{E}{\Delta} \ln \frac{4\Delta}{E} \tag{36}$$

while in the case of *spiral order*

$$\frac{N(E)}{N(0)} = \frac{\pi h}{2 v_F Q} \frac{E}{\Delta} \tag{37}$$

(v) the spin-orbit interaction decreases the value of $\chi_n(\mathbf{0}) - \chi_s(\mathbf{0})$ (this holds also for small q) and this effect is detrimental for the DS phase. However, the analysis in [23] shows that the spin-orbit scattering destroys the peak in $\chi_s(\mathbf{q})$ in very dirty systems only, in which case one has $l \sim a$.

We would like to point out that there have been many studies of ferromagnetic superconductors based on the phenomenological Ginzburg–Landau (G–L) theory which takes into account the EM interaction only [24,25]. Although very interesting, this phenomenology is inadequate in describing real materials, such as the above numbered RE ternary compounds where the EX interaction prevails in the formation of the oscillatory structure (with $Q \gg \xi_0^{-1}$, λ_L^{-1}) in the SC state.

2.2.1. Experimental situation

The important microscopic parameters of RE ternary compounds, which determine the properties of the coexistence phase of some ferromagnetic SC, are given in Table 1.

Based on the presented theory and by using the microscopic parameters from the Table 1, one concludes that in ferromagnetic superconductors $HoMo_6S_8$, $ErRh_4B_4$, $HoMo_6Se_8$ the superconductivity and oscillatory magnetic order coexist in some (narrow) temperature interval. It is seen also that the period L of the oscillatory magnetic structure (either sinus or domain-like) in all three compounds does not exceed the value $L\ (= 2\pi/Q) < 200\mathring{A}$. This important experimental result means that the energetics of the coexistence phase in the bulk sample is predominantly due to the EX interaction, while, as we said above, the EM interaction makes the structure transverse ($\mathbf{Q} \cdot \mathbf{S} = 0$). The latter property is due to the fact that in this situation the density of *magnetic charges* is zero, $\operatorname{div} \mathbf{M} = 0$, and the corresponding magnetic energy is also zero. The compound $HoMo_6Se_8$ is different from the other two, since in it SC and the oscillatory magnetic order coexist down to $T = 0$.

2.3. Domain magnetic structure in thin superconducting films

In the above calculations we have assumed that the thickness L of the sample is very large, i.e., $L \gg \xi_0$, Q_{DS}^{-1}, so that the dissipated magnetic energy (stray field) can be neglected. In case of thin films with $L \sim \xi_0$ the stray magnetic energy E_{st}, which exists around the domain walls and near the surface of the sample, must be added to the free-energy F_{DS} in Eq. (29) (or its simple version in Eq. (25)). The total free-energy $F_{\mathrm{tot}} = F_{\mathrm{DS}} + E_{\mathrm{st}}$ is given by [20]

$$\widetilde{F}_{\mathrm{tot}}/n = (\widetilde{F}_{\mathrm{DS}}/n) + E_{\mathrm{st}} = (\widetilde{F}_{\mathrm{DS}}/n) + 0.85\theta_{\mathrm{em}} \frac{S^2(T)}{QL} \tag{38}$$

The case where the ratio $r (= F_{\mathrm{Int}}^{(\mathrm{EM})}/F_{\mathrm{Int}}^{(\mathrm{EX})}) \ll 1$ the minimization of F_{tot} w.r.t. Q gives

$$Q_{\mathrm{tot}}^2 = Q_{\mathrm{DS}}^2 + Q_F^2 \tag{39}$$

where Q_{DS} is the wave vector of the DS phase without the stray magnetic energy and $Q_F \approx 1.6(\theta_{\mathrm{em}}/\theta \tilde{a} L)^{1/2}$ is the wave vector of the striped domain structure in the normal ferromagnetic state. From Eq. (39) it is seen that in a thin SC film the period of the DS ($d = 2\pi/Q_{\mathrm{tot}}$) is decreased due to the stray field. It comes from Eq. (38) that the transition temperature T_{c2} (for the first order phase transition $DS \to FN$(with domains) can be pushed to zero when $L < L_c = 3\xi_0(\theta_{\mathrm{em}} S_{c2}^4(L = \infty))/\theta_{\mathrm{ex}}(1 - S_{c2}^4(L = \infty))^2$. Some experiments in thin films of $HoMo_6S_8$ show such a thickness dependence of T_{c2}, where $T_{c2}(L) < T_{c2}(\infty)$ holds.

Let us mention that even in the normal ferromagnetic state, which occurs for $T < T_{c2}$, there is a possibility that SC exists in domain walls, as was shown in [26–28,4]. It seems that this situation happens in some pseudoternary compounds in which $h_0 \lesssim \Delta_0$.

2.4. Coexistence of nuclear magnetism and superconductivity

In 1997 Pobell's group from Bayreuth made an important discovery [6] by observing that *superconductivity and nuclear magnetism coexist* in $AuIn_2$ with $T_{c1} = 0.207$ K and $T_m = 35$ μK. At first glance this is not too surprising having in mind the smallness of the hyperfine interaction between conduction electrons and nuclear spins. However, Buzdin, Bulaevskii and the present author applied in 1997 [7] the above mentioned theory of magnetic superconductors [4] and obtained a surprising result, that the effective nuclear 'exchange' field (in fact the hyperfine contact interaction) is rather large $h_{\mathrm{hyp}} \approx 1$ K. At the same time the superconducting gap is $\Delta_0 \approx 0.6$ K, i.e., $h_{\mathrm{hyp}} > \Delta_0$! We point out that the *hyperfine interaction* has the same (mathematical) structure as the exchange interaction between the 4f LMs and conduction electrons

$$\hat{H}_{\mathrm{hyp}} = \int \mathrm{d}^3 r \sum_i A_{\mathrm{hyp}} \delta(\mathbf{r} - \mathbf{R}_i) \hat{\psi}^\dagger(\mathbf{r}) \sigma \hat{\mathbf{I}}_i \hat{\psi}(\mathbf{r}) \tag{40}$$

Here, A_{hyp} is the hyperfine interaction and the 'hyperfine exchange field' is given by $h_{\mathrm{hyp}} = n A_{\mathrm{hyp}} \langle \hat{\mathbf{I}}_i \rangle$, where $\hat{\mathbf{I}}_i$ is the nuclear spin. It turns out that the nuclear magnetism in $AuIn_2$, which shows a strong tendency toward ferromagnetism, competes rather strongly with SC. The estimation from experiment [6] gives: θ_{em} $(= 2\pi n_n \mu_n^2) \approx 1$ μK and θ_{ex} $(\approx N(0) h_{\mathrm{hyp}}^2) \approx 35$ μK, $\xi_0 \approx 10^5$ Å, $\lambda_L \approx 10^5$ Å, $l \approx 3.6 \times 10^4$ Å $(l < \xi_0)$. This set of parameters implies that the 'EX' (hyperfine contact) interaction is much stronger than the EM (dipole–dipole). The theory, which was originally invented for RE ternary compounds, is also applicable to this problem. It predicts, that if the nuclear magnetic anisotropy (which is due to the dipole–dipole interaction) is small, i.e., $(D/\theta_{\mathrm{ex}}) < 10^{-3}$, the spiral magnetic structure should be realized. In the opposite case $(D/\theta_{\mathrm{ex}}) > 10^{-3}$ the striped domain structure should be formed. The experiments in magnetic field [6] give evidence that SC and oscillating magnetic order coexist up to $T = 0$ K, i.e., the case $\theta_{\mathrm{ex}} < \theta_{\mathrm{ex}}^c$ occurs in $AuIn_2$, which is a type-I superconductor. Unfortunately, until now, there were no nuclear scattering measurements on $AuIn_2$ which could precisely resolve the nuclear magnetic structure below $T_m = 35$ μK.

The study of the coexistence of SC and nuclear magnetic order is of enormous importance also for fundamental physics. These systems give an opportunity for studying the coexistence problem in cases when the electronic temperature (T_e) is different from the nuclear one (T_n), i.e., $T_e \neq T_n$. However, probably the most interesting problem is the coexistence of SC and nuclear magnetism in the case of *negative nuclear temperatures* ($T_n < 0$ K). Unfortunately, the famous Bayreuth laboratory was closed recently and further development of this fascinating field has stopped.

3. Antiferromagnetic superconductors (AFS)

3.1. Coexistence of antiferromagnetism and superconductivity

An evident experimental fact in RE ternary compounds is that superconductivity coexists with the antiferromagnetic (AF) order much more easily than with the modified ferromagnetic order.

As in the case of spiral (or domain-like) magnetic order the antiferromagnetic order influences SC in two ways: (a) there is a *splitting of energy levels* of conduction electrons due to the exchange field generated by localized moments. As a result, the gap opens at a small part of the Fermi surface only, thus lowering the total density of states [13,14,16]; (b) the *magnetic scattering* of conduction electrons on spin fluctuations above the Néel temperature T_N and on spin waves below T_N is pair-breaking for Cooper pairs [15].

In the case (a) one can say that the effect of the exchange field in real AF ternary compounds is rather small. The reason is that the effective exchange field in AFS varies rapidly in space on the scale of the lattice constant. (The AF wave vector is of the order $Q_{\mathrm{AF}} \sim a^{-1}$.) As a result the exchange field averaged over the volume of the Cooper pair ξ_0^3, as well as over a^3, is zero. This means that the electronic spin susceptibilities at Q_{AF} in the normal and SC state take practically the same value

$$\frac{\chi_n(\mathbf{Q}_{\mathrm{AF}}) - \chi_s(\mathbf{Q}_{\mathrm{AF}})}{\chi_n(0)} \approx \frac{\Delta}{v_F Q_{\mathrm{AF}}} \sim \frac{T_c}{E_F} \ll 1 \tag{41}$$

This implies that AF and SC influence each other weakly, i.e., the interacting part of the free-energy $F_{\rm Int}^{\rm (EX)}$ in AFS is very small. The theory [4,16] shows, that the decrease of the SC order parameter due to the AF exchange field is small, i.e., it $\delta\Delta/\Delta_0 \approx (h/v_F Q_{\rm AF})\ln(h/\Delta_0) \ll 1$ in clean systems and $\delta\Delta/\Delta_0 \approx T_N/T_c \ll 1$ in dirty systems, since $T_N \ll T_c \ll h \ll v_F Q_{\rm AF}$ and $\Delta_0 \ll h$. These results are confirmed in a number of RE ternary compounds, in which the Neel temperature $T_N(\approx N(0)h^2)$ is in most cases (much) smaller than T_c [2]. Due to the same reason, the EM interaction is small since $\delta K_s(Q_{\rm AF}) \sim a^3/(\lambda_L^2\xi_0)$, i.e., $F_{\rm Int}^{\rm (EM)}$ $(\ll F_{\rm Int}^{\rm (EX)})$.

The magnetic scattering (b), although pair-breaking, is not very harmful for SC in real AF ternary compounds since in these systems the inverse life time τ_m^{-1} is small, i.e., $\tau_m^{-1} \sim T_N \ll T_c$. However, the effect of the magnetic scattering on the upper critical field H_{c2} depends on the strength of the scattering. In case when $\tau_m^{-1} \sim T_N \ll T_c$, which is for instance realized in $TmRh_4B_4$, the H_{c2} curve is weakly affected by magnetic (exchange) scattering. In cases where $\tau_m^{-1} \sim T_c$, for instance in $SmRh_4B_4$, this scattering changes H_{c2} significantly—see more in [3,14,16].

Concerning the role of the *nonmagnetic scattering*, already the above analysis on the decrease of Δ tells us that nonmagnetic impurities (characterized by the life-time τ) increase the depairing effect of the exchange field—the breakdown of the Anderson theorem [29,14]. In the case when $T_N \ll T_c$, the effect of nonmagnetic impurities is like that of magnetic impurities with the inverse scattering time

$$\tau_m^{-1} = \frac{\pi h^2}{2v_F Q_{\rm AF}\sqrt{1+(h\tau)^2}} \tag{42}$$

For $h\tau \ll 1$ one obtains $\tau_m^{-1} \sim T_N \approx N(0)h_0^2 \ll T_c$ since $(1/v_F Q_{\rm AF}) \approx N(0)$. This means that in this case the pair-breaking effect of impurities is rather small [29]. A very interesting situation occurs in systems with $T_N \gg T_c$. Even in such a case the exchange field does not suppress T_c significantly, since the theory [4,16] predicts that $(\delta T_c/T_{c0}) \sim (h/E_F)(\ln h/E_F) \ll 1$. However, in the presence of nonmagnetic impurities T_c is renormalized appreciably and SC disappears for the mean-free path $l < l_c \approx 10\xi_0(h/v_F Q_{\rm AF}) \sim \xi_0 T_N/h$. In that respect there is one very interesting AFS compound $Tb_2Mo_3S_4$ with $T_N = 19$ K and $T_c = 0.8$ K. In this case one expects (naively) that SC should disappear due to the strong magnetic scattering. However, it turns out that in this compound the magnetic anisotropy, in conjunction with the large momentum $J = 9$, strongly suppress this pair-breaking effect, thus giving rise to superconductivity.

Finally, it is worth of mentioning that, generally speaking, the SC order parameter Δ is coordinate dependent, i.e., $\Delta(\mathbf{r}) = \Delta + \delta\Delta(\mathbf{r})$. The theory shows that $\delta\Delta(\mathbf{r}/\Delta \sim (h/v_F Q_{\rm AF}) \ll 1$. This means that pairing effects with non-zero momentum in AFS based on real RE ternary compounds can be neglected. That is the reason that the results [30] based on the uniform pairing (with $\Delta(\mathbf{r}) = \Delta$)and those in [13,14], where the non-uniform pairing is also included, are practically the same [4,16].

3.2. Weak ferromagnetism in antiferromagnetic superconductors

In the case of the competition of SC and the ferromagnetic order in the RE ternary compounds, the theory predicts that in the presence of an appreciable EX interaction SC can coexist only with spiral and DS (or sinus) order—depending on the strength of magnetic anisotropy. The realization of other phases are much less probable. It turns out that in *AF superconductors with weak ferromagnetism* (*WF*)—of the Moriya–Dyalozhinski type, the phase diagram is much richer than in the case of ferromagnetic superconductors. For instance, the *Meissner phase* ($M \neq 0$, $B = 0$) and the *spontaneous vortex state* [31] can be realized in these systems. We discuss this problem briefly by studying the simplest AF order with two sublattice, where $\mathbf{l} = \mathbf{S}_1 - \mathbf{S}_2$ is the AF order parameter. In systems which allow WF there is an additional term in the free-energy $F_{\rm WF} = \mathbf{D}[\mathbf{S_1} \times \mathbf{S_2}]$ which is responsible for the spin canting. If for instance $\mathbf{l}$ is along the xy-plane and $\mathbf{D}$ is so oriented that it allows the appearance of the weak ferromagnetism $\mathbf{m} = \mathbf{S}_1 + \mathbf{S}_2$ ($\mathbf{M} = n\mu\mathbf{m}$) in the xy-plane, then $F_{\rm WF}$ is given by

$$\widetilde{F}_{\rm WF} = \beta n\theta_{\rm ex}(m_x l_y + m_y l_x) \tag{43}$$

Since in most systems $m \sim 10^{-3}l$ this immediately implies that the parameter $\beta \ll 1$. In that case and when $T_N \ll T_c$ the interaction part $F_{\rm int}$ of the total free-energy ($F = F_m + F_s + F_{\rm int}$) is given by Eq. (7), while the magnetic system is described by F_m

$$F_m = \int d^3r\, n\theta_{\rm ex}\big[a_l\mathbf{l}^2 + \frac{c}{4}\big(\mathbf{l}^2\big)^2 + b\mathbf{m}^2 + a^2(\nabla\mathbf{l})^2\big] + \int d^3r\left[\beta n\theta_{\rm ex}(m_x l_y + m_y l_x) + \frac{(\mathbf{B} - 4\pi\mathbf{M})^2}{8\pi}\right] \tag{44}$$

By minimizing F with respect to $\mathbf{A}$, $\mathbf{l}$, m and $\mathbf{q}$ we get possible phases in AFS with WF [31,4,16]. The resulting free-energy is similar to the case of ferromagnetic superconductors with an effective magnetic stiffness $a_{\text{eff}} = (ab/\beta) \gg a$. It turns out that if $\beta \gg a/\xi_0$ the EX interaction dominates in the formation of the magnetic structure, and the sinus structure ($l \sim \sin \mathbf{Qr}$ and $m \sim \sin \mathbf{Qr}$) is realized at T_N, while for $(a/\lambda_L) < \beta \ll a/\xi_0$ the EM interaction prevails in the formation of the sinusoidal structure. If $\beta < (a/\lambda_L)\sqrt{2\theta_{\text{em}}/\theta_{\text{ex}}}$ than the nonuniform structure is unfavorable and the so called *Meissner state* (first proposed by Vitalii Ginzburg in 1956) is realized. It is characterized by $\mathbf{M} = const$ and by the magnetic induction (averaged over the sample) $\langle \mathbf{B} \rangle = 0$ ($B = 4\pi M \exp\{-z/\lambda_L\}$) in the bulk sample, which is due to the SC screening current on the surface of the sample. By lowering the temperature, the sublattice magnetizations $|\mathbf{S}_{1,2}|$ grow and it is necessary to take into account higher order terms in F. As a result one obtains that for $\beta \gg \sqrt{a/\xi_0}$ the EX interaction dominates again and the striped DS phase is realized, while for $\sqrt{a/\lambda_L} \ll \beta \ll \sqrt{a/\xi_0}$ the striped DS phase is realized due to the EM interaction. However, by lowering the temperature the domain wall energy grows and it my happen that a *spontaneous vortex state* (with $4\pi M > H_{c1}$—the lower critical field) is realized for $\beta \ll \sqrt{a/\xi_0}$ and for the AF vector $l > l_c \sim (H_{c1}/M(0))(\beta\lambda_L^2/\tilde{a}^2)^{1/3}$, $\tilde{a} = a[(T_N - T)/T_N]^{1/2}$. From the known RE ternary compounds a good candidate for such a behavior is the body centered tetragonal (b.c.t.) system $ErRh_4B_4$.

4. Magnetic superconductors in a magnetic field

There are a number of interesting effects of the magnetic field H, either in the coexistence phase or above the magnetic transition temperature T_m where $\mathbf{S}(T > T_m) = 0$. We discuss some of them briefly here, and for more details see [4,23].

4.1. DS phase in magnetic field

In case of a bulk sample the applied magnetic field penetrates only on the length λ_L, thus affecting the surface of the sample only. However, in thin films the paramagnetic effect of the field is more important than the orbital one [4]. This problem was studied in the case of a thin (along the y-axis) film with the thickness $L_y < \xi_0$, when the magnetic field is parallel to the striped domains, i.e., $\mathbf{H} = H\mathbf{e}_z$—see Fig. 2. As a result, the magnetization $S_z(x)$ contains, besides the odd harmonics, also the zeroth-one as well as the even ones

$$S_z(x) = S\delta + \sum_{k=1}^{\infty} \frac{2S}{\pi k}\big\{\big[1 - (-1)^k \cos(\pi k\delta)\big]\sin(kQx) + (-1)^k \sin(\pi k\delta)\cos(kQx)\big\} \tag{45}$$

with $\delta = \mu H/2S\theta_{\text{ex}}$. This change of harmonics in $S_z(x)$ can be observed in magnetic neutron diffraction experiments. Eq. (45) tells us that domains with $\mathbf{M}$ parallel to $\mathbf{H}$ increase their thickness $d \to d(1+\delta)$, while the thickness of antiparallel domains is decreased, i.e., $d \to d(1-\delta)$. In the case where the zeroth component of the exchange field is sufficiently large, i.e., $\bar{h}(= h_0 S\delta) > \bar{h}_c = \Delta[1 - (1/\tau_m\Delta)^{2/3}]^{2/3}$, the DS phase is destroyed by the Zeeman effect, thus making $\Delta = 0$. For $\bar{h} < \bar{h}_c$ the parameters of the DS phase are renormalized, for instance $Q(H) < Q(0)$. In case when $\mathbf{H} = H\mathbf{e}_y$ (i.e., the field is orthogonal to the z-axis), then all domains have the same thickness and there is no redistribution of intensities of neutron peaks. However, there is only a decrease of intensities of $(2k+1)Q$ peaks by the factor $(1 - \delta_\perp^2)$ where $\delta_\perp = \mu H/S(\theta_{\text{ex}} + D_z)$ and D_z is the magnetic anisotropy.

4.2. MS in magnetic field at $T > T_m$

The effect of the exchange field on SC in a magnetic field is negligible for $T \lesssim T_{c1}$ since for $T_m \ll T_{c1}$ the magnetic susceptibility χ_m is very small. However, at T near T_m there is a significant increase of χ_m and accordingly an increase of the paramagnetic effect. This means that at temperatures $T \sim T_m$ the applied field strongly affect the superconductivity.

4.2.1. Thermodynamic critical field $H_c(T)$

We illustrate this effect by analyzing the change of the thermodynamical field $H_c(T)$ (for the transition N $\to$ MS) in magnetic superconductors. In that case the Gibbs energy density of the paramagnetic normal phase is equal to that of the SC phase, $\widetilde{G}_N(H_c) = \widetilde{G}_{\text{SC}}(H_c)$ where

$$\widetilde{G}_{\mathrm{SC}}(H_c)=\widetilde{F}_n(0)-\frac{H_{c0}^2}{8\pi} \tag{46}$$

$$\widetilde{G}_N(H_c)=F_n(0)-\frac{\mu H_c^2}{8\pi} \tag{47}$$

This gives the critical field

$$H_c(T)=\frac{H_{c0}(T)}{\sqrt{1+4\pi\chi_m(T)}} \tag{48}$$

where $(H_{c0}^2/8\pi)=N(0)\Delta^2/2$ is the SC condensation energy and the magnetic permeability is $\mu=1+4\pi\chi_m$ (here we neglect the conduction electron susceptibility χ_e since in MS one has $\chi_e\ll\chi_m$). In ferromagnetic superconductors for $T>T_m$ one has $\chi_m(T)\approx(\theta_{\mathrm{em}}T_{m0}/4\pi\theta)/(T-T_{m0})$ and

$$H_c(T)\sim\sqrt{T-T_{m0}} \tag{49}$$

It is seen from Eq. (49) that $H_c(T)$ is drastically reduced near $T_{m0}\ll T_c$, due to the divergence of $\chi_m(T)$. Very near to T_m the nonlinear effects of magnetic field start to dominate.

4.2.2. Upper critical field $H_{c2}(T)$

In the presence of the *external field* $\mathbf{H}_e$ and for T above T_{m0} the superconductivity is suppressed by the orbital effect of the field $\mathbf{B}=\mathbf{H}_i(1+4\pi\chi_m)$ and by the paramagnetic effect of the exchange field $\mathbf{h}$ (since the Zeeman effect of $\mathbf{B}$ is much smaller). Here, $\mathbf{H}_i=\mathbf{H}_e+\mathbf{H}_D$ where $\mathbf{H}_D$ is the demagnetization field. The critical field can be calculated by the same formula as for usual SC—see [32], where μ_B is replaced by $\tilde{\mu}_B=\mu_B+h_0M/n\mu H_i$ and the electron charge e by $\tilde{e}=e(1+4\pi M/H_i)$. In the pure limit, and for $T\ll T_{c1}$, one obtains the modified Gruenberg–Günther formula for $H_{c2}(T)$ [33]

$$H_{c2}(T)=\frac{\sqrt{2}}{1+4\pi\chi_m(T)}H_{c2}^*(0)\frac{f(\alpha)}{\alpha} \tag{50}$$

where $H_{c2}^*(0)$ is the upper orbital critical field in absence of magnetic moments. The function $f(\alpha)$ is calculated numerically in [33]. The parameter α describes the relative role of the orbital and paramagnetic effect

$$\alpha=\frac{2H_{c2}^*(0)h_0\chi_m(T)}{(1+4\pi\chi_m(T))n\mu\Delta_0} \tag{51}$$

In the RE ternary magnetic superconductors one usually has $h_0\gg\Delta_0$ and $4\pi M=4\pi n\mu$ is one order of magnitude smaller than $H_{c2}^*(0)$. This gives $\alpha\gg1$ in the region where $T\ll T_{c1}$. It is known that in pure superconductors for $\alpha>1.8$ [33] the Larkin–Ovchinikov–Fulde–Ferrell (LOFF) phase (due to paramagnetic effects) occurs [34,35]. In the LOFF state the SC order parameter oscillates spatially, being also zero at some points. For $\alpha\gg1$ one has $f(\alpha)\approx1$ and

$$H_{c2}(T)\approx1.5\frac{\Delta_0T_{m0}}{h_0\mu\theta}(T-T_{m0}) \tag{52}$$

$H_{c2}(T)$ depends linearly on $T-T_{m0}$ near T_{m0} and falls off much faster than of $H_c(T)$, i.e., $H_{c2}<H_c$. This leads to an very interesting effect that by the first order phase transition at $H_c(T)$ the system goes into the Meissner or vortex state depending on the relation between H_c and H_{c1}. Let us mention, that the lower critical field H_{c1} is very weakly affected by the exchange field. The theory [4] predicts the following dependence of H_{c1} on the EX and EM interaction

$$H_{c1}=\frac{\Phi_0}{4\pi\lambda_L^2}\ln\frac{\lambda_L\sqrt{p}}{\xi} \tag{53}$$

where p takes into account the screening effects due to the EX and EM interaction

$$p(T)=1-\frac{\theta_{\mathrm{em}}}{\theta_{\mathrm{em}}+\theta_{\mathrm{ex}}+\frac{\theta(T-T_{m0})}{T_{m0}}} \tag{54}$$

Note, that in the simplified theory, which is based on the EM interaction only (it assumes $\theta_{\mathrm{ex}}=0$), one obtains $p(T\to T_{m0})=0$. This gives that the effective penetration depth tends to zero, i.e., $\lambda_{\mathrm{eff}}=\lambda_L\sqrt{p}=0$ and the Ginzburg–Landau

(G–L) parameter $\kappa = (\lambda_{\text{eff}}/\xi) \to 0$. If this assumption would occur in RE compounds, then at temperatures near T_{m0} we would have a change from type-II to type-I superconductivity. This result is apparently incorrect in the RE ternary compounds, since $\theta_{\text{ex}} \sim \theta_{\text{em}}$, thus making p finite and the G–L parameter κ stays practically unchanged. So the change of the type of transition near T_{m0} is not due to the change of κ but it is due to the much faster temperature fall-off of $H_{c2}(T)$ than of $H_c(T)$. We shall not discuss further this interesting subject but refer the reader to [4] where various phases in the H–T phase diagram were analyzed. Depending on demagnetization effects several phases can be realized in the same material, such as Meissner-, vortex-, LOFF- or even intermediate-phase.

5. Josephson effect on magnetic superconductors

After the remarkable theoretical discovery by Buzdin and coworkers [11,12] of the possibility of π-Josephson junctions in the hybrid S–F–S structure, where F is a ferromagnet, the interest in Josephson junctions with magnetic degrees of freedom has grown dramatically—see this issue. In that sense it was a natural challenge to investigate this problem in magnetic superconductors.

5.1. π-contact due to triplet amplitude $F_{\uparrow\uparrow}$ $(F_{\downarrow\downarrow})$

The tunnelling Josephson junction, based on magnetic superconductors, was studied in [8] by assuming that in the left-L and right-R bulk magnetic superconductors the *spiral magnetic ordering* occurs—see Fig. 3. The spiral magnetic order is characterized by the wave vector $\mathbf{Q}_{L,R}$ and the exchange fields $h_{\theta_{L(R)}} = h_{L(R)} e^{i\theta_{L(R)}}$, respectively, while the superconductivity in the banks is described by the order parameter $\Delta_{L,R} = |\Delta_{L,R}| e^{i\varphi_{L,R}}$. For reasons of simplicity it was assumed $|\Delta_L| = |\Delta_R| = \Delta$, $h_L = h_R = h$, $|\mathbf{Q}_L| = |\mathbf{Q}_R| = Q$ where $\mathbf{Q}_{L,R} = \chi_{L,R} Q \hat{\mathbf{z}}$ are orthogonal to the tunnelling barrier with the spiral helicity $\chi_{L(R)} = \pm 1$ for $\mathbf{Q}_{L,R}$ along $(+)$ or opposite $(-)$ to the z-axis. Note, that such a junction is characterized by the standard *superconducting phase* $\varphi = \varphi_L - \varphi_R \neq 0$ and additionally by the *magnetic phase* $\theta = \theta_L - \theta_R \neq 0$. We shall demonstrate below that besides the *singlet amplitude* $F_{\uparrow\downarrow}$ $(F_{\downarrow\uparrow})$ the new amplitude—*triplet pairing amplitude* $F_{\uparrow\uparrow}$ $(F_{\downarrow\downarrow})$, plays very important role in the Josephson effect [8]. (Note, that $F_{\uparrow\uparrow}$ $(F_{\downarrow\downarrow})$ was first introduced in [5] in the study of the spiral magnetic order in SC and first applied to the Josephson junction in [8]. Later on this effect was rediscovered by the Efetov's group in Bochum by studying S–F–F–S structures with rotating magnetization [36]. In that case $F_{\uparrow\uparrow}$ $(F_{\downarrow\downarrow})$ give rise for additional proximity effects—see elsewhere in this issue.) As a result the Josephson current $J_J(\varphi,\theta)$ contains two parts

$$J_J(\varphi,\theta) = (J_c^s - J_{-\chi}^t \cos\theta)\sin\varphi \tag{55}$$

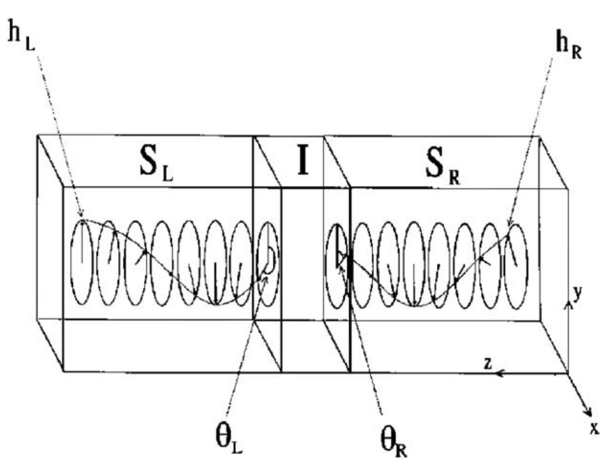

Fig. 3. The Josephson junction with the insulating contact. S_L and S_R are superconductors with spiral magnetic order. The exchange fields $\vec{h}_{L,R}$ at the surface make angles $\theta_{L,R}$with the y-axis. $\vec{Q}_{L,R}$ are along the z-axis.

where

$$J_c^s \sim T \sum_{\mathbf{k}_L,\mathbf{k}_R,\omega_n} |T_{\mathbf{k}_L,\mathbf{k}_R}|^2 F_{\uparrow\downarrow}^{\dagger}(\mathbf{k}_L,\omega_n) F_{\uparrow\downarrow}(\mathbf{k}_R,-\omega_n) \tag{56}$$

The current ($J_c^s \sim \Delta^2$) is due to the *singlet amplitude*, while the current $J_{-\chi}^t$

$$J_{-\chi}^t \sim -T \sum_{\mathbf{k}_L,\mathbf{k}_R,\omega_n} |T_{\mathbf{k}_L,\mathbf{k}_R}|^2 \big\{ F_{\uparrow\uparrow}^{\dagger}(\mathbf{k}_L,\omega_n)\big[F_{\uparrow\uparrow}^{\dagger}(\mathbf{k}_R,-\omega_n)\big]^* + F_{\downarrow\downarrow}^{\dagger}(\mathbf{k}_L,\omega_n)\big[F_{\downarrow\downarrow}^{\dagger}(\mathbf{k}_R,-\omega_n)\big]^*\big\} \tag{57}$$

is due to the *triplet amplitude*. The calculations for $J_{-\chi}^t$ give

$$J_{-\chi}^t \sim \Delta^2 h^2 \big[f_1 + (\chi_L \chi_R) f_2(\Delta, h)\big] \tag{58}$$

where the functions $f_{1,2}(\Delta, h)$ are calculated in [8], while $\chi = \chi_L \chi_R$ is the total helicity (in [36] renamed to chirality) of the junction. It was shown in [8] that in some parameter region the effects of the triplet amplitude dominate, i.e., $|J_{-\chi}^t| > J_c^s$, thus giving rise to the *π-Josephson junction*. If such a junction is placed in a superconducting ring with sufficiently large inductance L, i.e., with $L > L_c$, then a spontaneous current flows in the ring by producing the half-flux quantum in the hole of the ring [37].

From Eq. (55) it is obvious that by changing the magnetic phase θ and chirality χ one can tune the system from 0- to a π-junction. This new degree of freedom in the junction—the magnetic phase θ, first proposed in [8], opens a new possibility for *switching elements* and *quantum computing*. From the physical point of view the above model is a *paradigm* for analogous effects in S–F–F–S structures with rotating magnetization. In this case θ is the angle between magnetization in neighboring layers [9].

5.2. Combined superconducting and magnetic Josephson effect

In [9] the above model is developed further by including the tunnelling of electronic spins and their effect on the energy of the contact. Namely, in ferromagnetic superconductors with rotating magnetization (such as spiral) besides the standard Green's function $G_{\uparrow\uparrow}$ ($G_{\downarrow\downarrow}$), $F_{\uparrow\downarrow}$ ($F_{\downarrow\uparrow}$) for singlet SC other Green's functions $G_{\uparrow\downarrow}$ ($G_{\downarrow\uparrow}$) and $F_{\uparrow\uparrow}$ ($F_{\downarrow\downarrow}$) are important [8,9], since they can produce a *static spin current* J_{spin} through the junction (in absence of voltage),

$$J_{\text{spin}} = J_{\text{spin},G} + J_{\text{spin},F} \tag{59}$$

where

$$J_{\text{spin},G} \sim \sum |T|^2 (G_{\uparrow\downarrow,L} G_{\downarrow\uparrow,R} - G_{\downarrow\uparrow,L} G_{\uparrow\downarrow,R}) \sim h^2 \sin\theta \tag{60}$$

$$J_{\text{spin},F} \sim \sum |T|^2 \big\{ F_{\uparrow\uparrow}^{\dagger}(\mathbf{k}_L,\omega_n)\big[F_{\uparrow\uparrow}^{\dagger}(\mathbf{k}_R,-\omega_n)\big]^* - F_{\downarrow\downarrow}^{\dagger}(\mathbf{k}_L,\omega_n)\big[F_{\downarrow\downarrow}^{\dagger}(\mathbf{k}_R,-\omega_n)\big]^*\big\} \sim h^2 \Delta^2 \cos\varphi \sin\theta \tag{61}$$

The exact expression for $J_{\text{spin},G}$ and $J_{\text{spin},F}$ will be published elsewhere [9]. The energy of this combined *magnetic and superconducting Josephson junction* $E = E_{mJ}(\theta) + E_J(\varphi,\theta)$ must be an even function on φ and θ

$$E(\theta,\varphi) = -Ah^2 \cos\theta - \Delta^2 (B + C_\chi h^2 \cos\theta) \cos\varphi \tag{62}$$

The explicit form of $A(\Delta,h)$, $B(\Delta,h)$, $C(\Delta,h)$ is given in [9]. Note, that both the spin $J_{\text{spin}}(\theta,\varphi) \sim \partial E/\partial\theta$ and the charge $J_J(\varphi,\theta) \sim \partial E/\partial\varphi$ Josephson current depend on φ and θ. Thus, by tuning θ and φ one can tune these currents. For instance, one can obtain the superconducting π-junction—with spontaneous charge currents in the superconducting ring. Analogously, the magnetic π-contact can be realized, and accordingly, the spontaneous spin current in the ring from magnetic superconductors.

Another interesting aspects of the two-phase junction might be realized in small Josephson contacts. In such a case the smallness of the *charge* and *'spin' capacitance* brings the system in the *quantum regime*, thus giving a possibility for a novel Josephson qubit. In fact the latter consists from *two qubits*—the superconducting and magnetic one, which might be also of a potential interest for applications. By assuming that both qubits are two-level systems the operation of this system can be described by the pseudo-spin formalism [38] for the magnetic qubit $\hat{\boldsymbol{\tau}}_m = (\hat{\tau}_{m,x}, \tau_{m,y}, \hat{\tau}_{m,z})$

and the superconducting *charge qubit* $\hat{\boldsymbol{\tau}}_s = (\hat{\tau}_{s,x}, \tau_{s,y}, \hat{\tau}_{ms,z})$, the Hamiltonian of this two-bit systems has the form $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$ with the Hamiltonian $\hat{H}_0$ of single-qubits

$$\hat{H}_0 = -\frac{1}{2}\big[B_{m,z}(t)\hat{\tau}_{m,z} + B_{m,x}(t)\hat{\tau}_{m,x}\big] - \frac{1}{2}\big[B_{s,z}(t)\hat{\tau}_{s,z} + B_{s,x}(t)\hat{\tau}_{s,x}\big] \tag{63}$$

The interaction between these two qubits (the interqubit coupling) in a single Josephson junction is described by the XX coupling

$$\hat{H}_{\text{int}} = -J_{xx}^{ms}(t)\hat{\tau}_{s,x}\hat{\tau}_{m,x} \tag{64}$$

The physical conditions for the realization of this system, as well as possibilities of one- and two-qubit logic gates, will be studied elsewhere [9].

6. Conclusion

The rare earth ternary compounds are rich physical systems which allow the coexistence of singlet superconductivity and various magnetic orders, such as ferromagnetic, antiferromagnetic, weak ferromagnetism. It turns out that in these systems superconductivity and ferromagnetism practically never coexist. However, the modified ferromagnetic order in the form of a spiral or domain structure (depending on magnetic anisotropy) coexists with singlet superconductivity. This is realized in rare earths ternary compounds, as well as in $AuIn_2$ where a modified nuclear ferromagnetism—spiral or domain-like structure—coexists with singlet superconductivity.

Although the antiferromagnetic order and superconductivity coexist much more easily, these systems show a peculiar behavior in the presence of nonmagnetic impurities, which surprisingly act as pair-breakers. In the case when the antiferromagnetic order is accompanied by the weak ferromagnetism new coexistence phases appear—the Meissner and spontaneous vortex state. Magnetic superconductors show peculiar behavior in magnetic field. Near the magnetic critical temperature the upper critical field tends to zero faster than the thermodynamical field, thus giving rise to the first order transition. Various phases are possible in the H–T diagram depending on the purity and demagnetization effects of real samples. The lower critical field is weakly affected by the exchange field (which is due to localized moments).

Josephson junctions based on bulk ferromagnetic superconductors with spiral order are characterized by the superconducting and magnetic phase. This opens possibilities for a new kind of coupled qubits in a single Josephson junction. The triplet pairing amplitude, arising in system with rotating magnetization, gives rise to the π-Josephson junction which can be tuned by changing the magnetic phase and chirality.

Acknowledgements

I would like to thank Alexander Buzdin for collaboration, support and hospitality I have enjoyed during my stay, in summer 2005, at the University of Bordeaux when this review was written. I thank Igor Kulić for collaboration and support.

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