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The deconfinement and Hagedorn phase transitions in weakly coupled large N gauge theories

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Abstract

We demonstrate that weakly coupled, large N, d-dimensional $\mathrm{SU}(N)$ gauge theories on a class of compact spatial manifolds (including $S^{d-1} \times$ time) undergo deconfinement phase transitions at temperatures proportional to the inverse length scale of the manifold in question. The low temperature phase has a free energy of order one, and is characterized by a stringy (Hagedorn) growth in its density of states. The high temperature phase has a free energy of order N^2. These phases are separated either by a single first order transition that generically occurs below the Hagedorn temperature or by two continuous phase transitions, the first of which occurs at the Hagedorn temperature. These phase transitions appear to be continuously connected to the usual flat space deconfinement transition in the case of confining gauge theories, and to the Hawking–Page nucleation of AdS_5 black holes in the case of the $\mathbb{N} = 4$ supersymmetric Yang–Mills theory. Our analysis proceeds by first reducing the Yang–Mills partition function to a $(0 + 0)$-dimensional integral over a unitary matrix U, which is the holonomy (Wilson loop) of the gauge field around the thermal time circle in Euclidean space; deconfinement transitions are large N transitions in this matrix integral. ***To cite this article: O. Aharony et al., C. R. Physique 5 (2004).***

Résumé

Transitions de déconfinement et de Hagedorn dans le régime de couplage faible pour la limite de grand N des théories de jauge. Nous montrons que dans le régime de couplage faible, et dans la limite de grand N, les théories de jauge $\mathrm{SU}(N)$ en dimension d définies sur une classe de variétés spatiallement compactes (incluant $S^{d-1} \times$ temps) présentent des phases de déconfinement à des températures proportionnelles à l'inverse de l'échelle typique des variétés en questions. La phase de basse énergie a une énergie libre d'ordre un, et est caractérisée par une croissance de type Hagedorn de la densité d'états. La phase de haute température a une énergie libre d'ordre N^2. Ces deux phases sont séparées soit par une unique transition du premier ordre qui a lieu génériquement en dessous de la température de Hagedorn, soit par deux transitions de phase continues, la première ayant lieu à la température de Hagedorn. Ces transitions de phase sont connectées continûment à la transition de déconfinement traditionnelle des théories de jauge confinantes ayant lieu en espace plat et à la nucléation de Hawking–Page des trous noirs d'AdS_5 pour la théorie de Yang–Mills supersymétrique $\mathbb{N} = 4$. Dans notre analyse nous réduisons tout d'abord la fonction de

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partition de Yang–Mills à une intégrale matricielle en $(0,0)$ dimensions sur les matrices unitaires U, qui sont les boucles de Wilson d'holonomie de la théorie de jauge le long de la coordonnée temporelle thermale dans l'espace euclidien ; les transitions de déconfinement sont les transitions de grands N dans cette intégrale matricielle. ***Pour citer cet article : O. Aharony et al., C. R. Physique 5 (2004).***

Keywords: Hagedorn phase transition; Weakly coupled gauge theories

Mots-clés : Transitions de Hagedorn ; Couplage faible des théories de jauge

1. Introduction

Many non-Abelian gauge theories in $3+1$ dimensions are believed to exhibit the property of 'confinement' – all finite energy states are singlets of the gauge group (they carry no charge). Experimentally, this is known to be a property of QCD, the SU(3) gauge theory describing the strong nuclear interactions, and it is believed to be a general property of a large class of asymptotically free non-Abelian gauge theories. Studying this property theoretically is difficult, since it is not visible in perturbation theory (whose basic building blocks are charged states). Confinement can be seen in numerical simulations of various gauge theories, but there is still no good theoretical understanding of how it works (and no general way to predict whether a given theory confines or not), despite a lot of work on various theoretical models. In this work we will be interested in studying confinement in $\mathrm{SU}(N)$ gauge theories which have only fields in the adjoint representation, and we will specialize to this case from here on (except in some places where we will give more general results); this case includes theories such as the pure Yang–Mills theory and its supersymmetric generalizations.

Confinement is usually viewed as a strong-coupling phenomenon, associated with the fact that in asymptotically free gauge theories the coupling constant becomes strong at low energies. Thus, if we consider a gauge theory at a high temperature compared to its strong coupling scale Λ_{QCD}, we expect confinement to disappear since the temperature serves as an infra-red cutoff and the effective coupling at the scale of the temperature is weak. So, as we raise the temperature, we expect to have a *deconfinement phase transition* at a temperature of order Λ_{QCD}; for QCD this temperature is of order 170 MeV.[1] Since the coupling constant at the transition is strong, it is difficult to analyze theoretically the order of the phase transition, and in general the order is not known (except in some special cases like pure SU(3) gauge theory [1]).

In order to argue that deconfinement is a sharp phase transition we need to show that there are order parameters distinguishing the confined and deconfined phases. Two order parameters will be useful in this article. The canonical partition function for a field theory at finite temperature is equal to the Euclidean path integral of the theory with the time direction periodically identified with a periodicity $\beta = 1/T$. In such a theory one can consider the operator

$$W = \frac{1}{N}\,\mathrm{tr}\left(P\exp\left(i\oint A_t\right)\right) \tag{1}$$

which is the Wilson loop (in the fundamental representation) around the periodic time direction, sometimes called the Polyakov loop. The expectation value of this operator may be identified with $\exp(-\beta F_q)$ where F_q is the free energy of the theory in the presence of an external particle in the fundamental representation of the gauge group. Thus, in the confined phase $\langle W\rangle = 0$ (since F_q is infinite), but $\langle W\rangle$ is generally non-zero in a deconfined phase (and it can be computed in perturbation theory at high temperatures), so $\langle W\rangle$ is a good order parameter for the deconfinement phase transition. The gauge theory with a periodic time direction has a global $\mathbb{Z}_N$ symmetry (coming from $\mathrm{SU}(N)$ gauge transformations which are periodic only up to elements of the center of the gauge group), and W is charged under this symmetry. Thus, the deconfinement transition involves a spontaneous breaking of the global $\mathbb{Z}_N$ symmetry.

Another order parameter appears in the large N limit. States in the confined phase are made out of singlet particles like mesons or glueballs, whose properties do not change in the large N limit, so in this phase the free energy is of order one as $N\to\infty$. On the other hand, states of the deconfined phase can be described in terms of weakly coupled gluons, and the number of gluon species is N^2-1, so in the deconfined phase the free energy scales as N^2 in the large N limit. Thus, another order parameter for deconfinement is $\lim_{N\to\infty} F_{\mathrm{SU}(N)}(T)/N^2$, which also vanishes in the confined phase.

It would be nice if we could study the deconfinement phase transition perturbatively, but in infinite space this is impossible since it occurs at strong coupling. However, suppose that we compactify space on a manifold with characteristic scale R, such

[1] A similar transition is also expected to occur at large densities, and this is currently being investigated experimentally at RHIC. We will not discuss this here, even though it can be analyzed by similar methods to those we present below.

that $1/R \gg \Lambda_{QCD}$, and such that none of the fields have zero modes on the compact manifold (for instance, we can choose the manifold to be a three-sphere S^3). Then, the compactified theory has a mass gap of order $1/R$, and since the gauge coupling is weak at this scale (and at all higher scales) we can analyze this theory perturbatively. Thus, if the deconfinement phase transition persists in this finite volume case, we should be able to study it in perturbation theory (which we still have much better control over than non-perturbative computations, despite much progress in the latter in the last few years).

At first sight this idea sounds ridiculous, since there are no phase transitions at finite volume due to quantum fluctuations. And, correspondingly, the order parameter $\langle W \rangle$ that we defined always vanishes at finite volume, since one cannot put a single charge on a compact manifold due to Gauss' law. However, it turns out that even at finite volume one can still have phase transitions in the large N limit, in which the large number of degrees of freedom overcomes the quantum fluctuations. A large N deconfinement phase transition at finite volume could be characterized by the second order parameter we described above, or by $\langle |W|^2 \rangle$ which is also a good order parameter for deconfinement in the large N limit (though it does not vanish in the confined phase for finite values of N). These phase transitions are smoothed out for any finite value of N, but in the large N limit one obtains a sharp transition. For large volume, $1/R \ll \Lambda_{QCD}$, one expects this transition to be very similar to the deconfinement transition at infinite volume.

So, it is natural to ask whether the large N deconfinement phase transition persists to small values of R. We will show below that indeed it does, and that we can compute all the details of this transition (including the order of the phase transition) by a perturbative analysis.

The large N deconfinement transition is also interesting for another reason. 't Hooft has argued [2] that large N gauge theories (in the limit of large N with $\lambda \equiv g_{YM}^2 N$ fixed) are equivalent to string theories with the string coupling constant proportional to $1/N$. Indeed, the confined phase of gauge theories involves a 'stringy' spectrum of particles sitting on Regge trajectories, similar to the spectrum of weakly coupled string theories. On the other hand, the deconfined phase does not seem 'stringy' since it is described in terms of weakly coupled gluons. The transition between such phases is qualitatively similar to the *Hagedorn phase transition* which has been conjectured to occur in weakly coupled string theories at the Hagedorn temperature T_H, at which the contribution to the canonical partition function from very massive string states (whose density of states behaves as $\rho(E) \sim \exp(E/T_H)$) seems to diverge. Such a transition is unlikely to occur for string theories in flat space (for which the high-energy density of states seems to be governed by Schwarzschild black holes whose density of states grows too fast for the canonical ensemble to be well-defined), but it could occur for string theories in curved space, such as those which are dual to gauge theories, and it has been suggested that for large N gauge theories the Hagedorn and deconfinement phase transitions should be identified. The order parameter for the Hagedorn transition is a winding mode around the periodic time direction, which is similar to the Polyakov loop (1) described above.

Can these two phase transitions really be identified? In order to answer this question we need to analyze examples of large N gauge theories whose string theory dual is known, but unfortunately only a few such examples are known. The best understood example is the AdS/CFT correspondence [3] between the $\mathbb{N} = 4$ supersymmetric Yang–Mills theory compactified on S^3 and type IIB string theory on $AdS_5 \times S^5$ (in global coordinates for AdS space). When the gauge theory is strongly coupled ($\lambda \gg 1$), its dual string theory is weakly curved and we can easily study it (in the large N limit when the dual string theory is weakly coupled), leading to the following results. In the microcanonical ensemble, there is a large range of energies for which the density of states of the string theory indeed grows exponentially with the energy, so the theory has a Hagedorn temperature proportional to the type IIB string scale; when translated into the gauge theory this temperature is proportional to $T_H \propto \lambda^{1/4}/R$. On the other hand, in the canonical ensemble, it was shown in [4] that the theory exhibits a phase transition between a gas of particles in AdS space and an AdS black hole at a temperature $T_{HP} = 3/2\pi R$, and Witten [5] has argued that this transition is a deconfinement transition in the gauge theory (by using the large N order parameters described above). Thus, in this example we find that we have deconfinement and we have Hagedorn behavior, but there seems to be no relation between them since $T_{HP} \ll T_H$. However, as we decrease the coupling constant λ, the two temperatures come closer together, suggesting that maybe at weak coupling the deconfinement transition could be related to the Hagedorn transition. This gives us an additional motivation for studying deconfinement transitions at weak coupling, to see whether they are related to Hagedorn transitions at weak coupling or not. Even though naively the string picture of large N gauge theories would not be expected to make sense at weak coupling (since the Feynman diagrams are not dense and do not resemble smooth worldsheets), we will find that even at weak coupling the compactified gauge theories exhibit a Hagedorn behavior, and that the two phase transitions are indeed related to each other in weakly coupled gauge theories.

Following these motivations, we turn to computing the partition function of compactified weakly coupled gauge theories. In the next sections we will summarize the results of [6] and justify the statements made above. More details and discussions (as well as a complete list of references) may be found in [6].

2. The partition function of free Yang–Mills theory

We are interested in weakly coupled gauge theories, so we start in this section by analyzing the extreme case of a free Yang–Mills theory (this case was first analyzed by Sundborg in [7]). Naively, this theory is completely trivial, but we define the free theory as the limit of the finite coupling theory in which we take the coupling constant to zero, and then even in the free limit we still have the Gauss law constraint telling us that the total charge should vanish (all states must be singlets of the global gauge group; note that this constraint does not imply confinement according to the order parameters for confinement that we described above). At finite volume this constraint is non-trivial and leads to effective interactions even when all interaction terms in the Lagrangian are set to zero. We will show that these ‘global interactions’ are sufficient to lead to non-trivial dynamics and to a deconfinement phase transition in free gauge theories (at finite volume).

There are two methods we can use to compute the canonical partition function $Z(T) = \sum_{\text{singlet states}} \mathrm{e}^{-E/T}$. The first is to sum over the Fock space of all possible multi-particle states and project onto the singlet states. The input to this computation is the partition function of single-particle states (in the theory on the compact manifold) in each representation R of the gauge group,

$$z_B^R(T) = \sum_{\substack{\text{bosonic single particle states}\\ \text{in representation } R}} \mathrm{e}^{-E/T},$$
$$z_F^R(T) = \sum_{\substack{\text{fermionic single particle states}\\ \text{in representation } R}} \mathrm{e}^{-E/T}, \tag{2}$$

which is easily computable (for instance, the contribution of a scalar field is simply related to the spectrum of the Laplacian operator on the compact manifold). Given these partition functions it is a straightforward combinatorical problem to compute the full multi-particle partition function and to project it onto singlets; the last step is most easily performed using the characters $\chi_R(U)$ of the gauge group, defined as the trace in the representation R of the group element U. A second method to compute the partition function is to simply compute the 1-loop Euclidean path integral of the free gauge theory with a periodic time direction (with appropriate gauge-fixings, and taking care of the Gauss law constraint). Both methods turn out to lead to precisely the same formula for the exact partition function of a free Yang–Mills theory with general matter content,

$$Z(T) = \int [\mathrm{d}U] \exp\left(\sum_R \sum_{n=1}^{\infty} \frac{1}{n} \left[z_B^R\left(\frac{T}{n}\right) - (-1)^n Z_F^R\left(\frac{T}{n}\right)\right] \chi_R(U^n) \right), \tag{3}$$

where $[\mathrm{d}U]$ is the measure on the $\mathrm{SU}(N)$ group manifold (in fact, this equation is valid for arbitrary gauge groups). In the first method the integral over U comes from the projection onto the singlet states, while in the second method U is the holonomy of the gauge field A_t (averaged over the compact manifold) around the periodic time direction (whose trace is related to the Polyakov loop (1)), which is the only zero mode in the Euclidean path integral (all other modes appear quadratically and can be easily integrated out).

Next, we would like to solve the matrix integral and compute expectation values of the Polyakov loop $\mathrm{tr}(U)$ to see if we have a deconfinement transition in this theory. We will now restrict to the case where all fields are in the adjoint representation of $\mathrm{SU}(N)$, in which we can write (in the large N limit) $\chi_{\mathrm{adj}}(U^n) = \mathrm{tr}(U^n)\,\mathrm{tr}(U^{-n})$ in terms of traces in the fundamental representation. As usual, to solve the unitary matrix integral we change variables to the eigenvalues of U, $\{\mathrm{e}^{\mathrm{i}\alpha_j};\ j = 1, \ldots, N;\ -\pi < \alpha_j \leqslant \pi\}$, and in the large N limit we can replace those by the eigenvalue distribution $\rho(\alpha) \equiv \frac{1}{N}\sum_i \delta(\alpha - \alpha_i)$, which becomes a continuous function in the large N limit.

In terms of the eigenvalues, we can write the partition function (3) as

$$Z(T) = \int \left(\prod_{i=1}^{N} \mathrm{d}\alpha_i \right) \exp\left(-\sum_{i<j} V(\alpha_i - \alpha_j) \right), \tag{4}$$

in terms of a pair-wise potential for the eigenvalues given by

$$V(\theta) = -\ln\left(\sin\left(\frac{\theta}{2}\right)\right) - \sum_{n=1}^{\infty} \frac{1}{n} \left[z_B\left(\frac{T}{n}\right) - (-1)^n z_F\left(\frac{T}{n}\right)\right] \cos(n\theta). \tag{5}$$

The first term in (5) gives a repulsive force coming from the measure, while the second term gives an attractive force which grows as the temperature is increased (we are discussing theories which have a mass gap after the compactification, so $z_B(T)$ and $z_F(T)$ go to zero as $T \to 0$). Thus, we expect that the repulsive force should dominate at low temperatures, so the eigenvalues should be uniformly spread and $\rho(\alpha)$ should be a constant, while for high temperatures the eigenvalues will want to be come together.

The matrix model (3) can actually be solved exactly in this case, by a generalization of the methods used in [8] to solve the two-dimensional lattice gauge theory. However, for our purposes the form of the exact solution (which is quite complicated) will not matter, since a simpler analysis suffices to describe what happens near the phase transition. Let us change variables again, to the Fourier components of the eigenvalue distribution $\rho_n \equiv \int_{-\pi}^{\pi} \rho(\alpha)\, e^{in\alpha}\, d\alpha = \mathrm{tr}(U^n)/N$, which are complex variables obeying $\rho_{-n} = \rho_n^*$. These variables obey complicated constraints coming from the fact that the eigenvalue distribution must be non-negative everywhere, but the partition function turns out to be very simple in these variables (in the large N limit),

$$Z(T) = \int \left(\prod_{n=1}^{\infty} d^2\rho_n \right) \exp\left(-N^2 \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - z_B\left(\frac{T}{n}\right) + (-1)^n z_F\left(\frac{T}{n}\right) \right] |\rho_n|^2 \right). \tag{6}$$

At low temperatures the expressions in the square brackets are all positive, so in the large N limit this integral is sharply localized around the saddle point $\rho_n = 0$ (for all $n \neq 0$) corresponding to the uniform eigenvalue distribution, as expected. Moreover, it is easy to compute explicitly the partition function (6) in this phase, and we find the simple formula for the low-temperature partition function

$$Z(T) = \prod_{n=1}^{\infty} \frac{1}{1 - z_B(T/n) + (-1)^n z_F(T/n)}. \tag{7}$$

The free energy $F(T) = -T \log(Z(T))$ in this low-temperature phase is of order one in the large N limit.

As we raise the temperature, the analysis above breaks down at the temperature T_H for which

$$z_B(T_H) + z_F(T_H) = 1, \tag{8}$$

at which the mode ρ_1 in (6) becomes effectively massless (and later tachyonic). As we approach T_H, $Z(T)$ computed above diverges as $Z(T) \propto 1/(T_H - T)$. This is precisely the characteristic behavior of a system with a Hagedorn density of states, $\rho(E) \sim e^{E/T_H}$! Thus, we find that the low-temperature phase of this system is governed by a Hagedorn density of states, with a characteristic Hagedorn temperature T_H given by the simple equation (8). This equation may easily be solved for a given theory; for example, for the free pure Yang–Mills theory on S^3 we find $T_H = -1/R \log(2 - \sqrt{3})$, while for the $\mathbb{N} = 4$ supersymmetric Yang–Mills theory on S^3 we find $T_H = -1/R \log(7 - 4\sqrt{3})$.

Of course, the partition function of the free gauge theory is not really divergent, but rather the approximation we used in deriving (7) breaks down as we come very close to $T = T_H$. This reflects the fact that the Hagedorn density of states in the gauge theory is cut off at an energy of order N^2, and for much higher energies the density of states is actually similar to that of a free theory with no constraints (since the Gauss law constraint is negligible when we look at states with a very large number of particles). We can also see this directly by counting the gauge-invariant states in the theory, which are of the form $\mathrm{tr}(O_1 O_2 \cdots O_k)$ where O_i are creation operators for single-particle states in the adjoint representation. One can show that this leads to a Hagedorn spectrum with the partition function (7) and with the temperature T_H given above, in the approximation in which single-trace and multiple-trace states are independent. However, this approximation breaks down when we have more than N operators in the same trace, and this is the effect which smooths out the divergence of (7).

When we increase the temperature slightly above $T = T_H$, ρ_1 becomes tachyonic and goes to its maximal possible value; when all other $\rho_n = 0$ the maximal value for which the eigenvalue density is non-negative is $|\rho_1| = 1/2$, while the phase of ρ_1 is arbitrary (reflecting the $\mathbb{Z}_N$ global symmetry of the theory, which multiplies all the eigenvalues of U, and also ρ_1, by $e^{2\pi i/N}$). This gives an eigenvalue distribution $\rho(\alpha) = \frac{1}{2\pi}(1 + \cos(\alpha - \alpha_0))$. As we increase the temperature further, the other modes ρ_n also become non-zero allowing ρ_1 to increase further; this leads to a localized eigenvalue distribution with a gap. The behavior of the eigenvalue distribution is summarized in Fig. 1 (where we arbitrarily chose the distribution to be localized around $\alpha = 0$).

We see that this system has a sharp phase transition at the Hagedorn temperature $T = T_H$, and based on our two order parameters for deconfinement, we can identify it also with a deconfinement transition. The free energy clearly changes at T_H

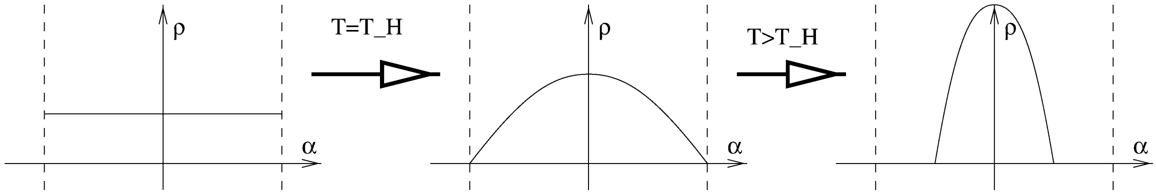

Fig. 1. A schematic graph of the distribution of the eigenvalues of the holonomy matrix in compactified free Yang–Mills theories as a function of the temperature.

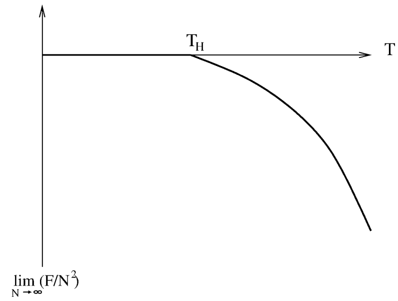

Fig. 2. The free energy of free pure Yang–Mills theory on S^3 as a function of temperature in the large N limit; the behavior for other free gauge theories is qualitatively similar.

from being of order one to being of order N^2; a graph of $\lim_{N\to\infty} F(T)/N^2$ based on the exact solution is drawn in Fig. 2. And, even though the expectation value of the Polyakov loop $\langle \mathrm{tr}(U)\rangle = \langle \rho_1 \rangle$ vanishes both for low temperatures and for high temperature, in the high temperature phase this vanishing comes from a sum over N different saddle points with different phases of ρ_1 (which is as close as we can get to a spontaneous breaking of the $\mathbb{Z}_N$ symmetry at finite volume), so in this phase $\langle |\rho_1|^2 \rangle$ is non-zero. Thus, we have shown that compactified free Yang–Mills theories have a deconfinement phase transition in the large N limit, which happens precisely at their Hagedorn temperature.

3. The partition function of weakly coupled gauge theories

Next, we wish to turn on a small coupling constant λ, and see how the results of the previous section change. In our first method of computation, this requires computing the change in the energy of all the states in the theory (in a conformal theory on S^3 this can be mapped to the anomalous dimensions of the operators), which is very complicated even at leading order in λ (see [9]). Luckily, our second method of computation, via the Euclidean path integral, is more suitable at finite coupling. It turns out that at finite coupling we can still integrate out all the modes appearing in the Euclidean path integral except for U, but this induces additional interactions for U, which we can compute order by order in perturbation theory. At k-loop order we find that the effective action for U (which at one-loop order involved only terms of the form $\mathrm{tr}(U^n)\,\mathrm{tr}(U^{-n})$) is deformed by terms of the form $\lambda^{k-1}\,\mathrm{tr}(U^{n_1})\,\mathrm{tr}(U^{n_2})\cdots\mathrm{tr}(U^{n_{k+1}})/N^{k-1}$ which involve up to $k+1$ traces (the number can be smaller if some of the n_i vanish); the terms we wrote here arise from planar diagrams, non-planar diagrams also contribute terms with a smaller number of traces, but their contributions are down by powers of N^2. The coefficients of all of these terms can be computed by computing the k-loop vacuum diagrams in the Euclidean Yang–Mills theory, with a non-trivial background holonomy of A_t given by the unitary matrix U.

Even at low orders in λ, the resulting matrix model is quite complicated, and it is not known how to solve it exactly. However, we are mostly interested in the behavior of the theory near its phase transition point. As we saw in our analysis of the free theory, this is dominated by the mode $\rho_1 = \mathrm{tr}(U)/N$ which is light near the transition. Thus, the region near the transition can be described by an effective action for ρ_1 in which we integrate out all other modes, and from our discussion above this effective action takes the form (up to second order in λ)

$$S_{\mathrm{eff}} = N^2\big[a(T_H - T)|\rho_1|^2 + b\lambda^2|\rho_1|^4\big], \tag{9}$$

where a and T_H for $\lambda = 0$ may be computed from the results of the previous section (and they then receive perturbative corrections in λ), while b (the leading non-quadratic term at weak coupling) may be computed from 2-loop and 3-loop vacuum diagrams in the Yang–Mills theory. Terms of third order in ρ_1 are not allowed by the global $\mathbb{Z}_N$ symmetry.

The action (9) is the standard Landau–Ginzburg action describing a phase transition whose order parameter is a complex scalar. As usual, the behavior of the theory near its phase transition depends on the sign of the quartic coefficient b. For $b < 0$, as we increase the temperature the effective potential for $|\rho_1|$ (which is bounded between zero and a half) goes from the top graph on the left side of Fig. 3 to the bottom graph. We see that at some critical temperature T_0 which is smaller than T_H (since the quadratic term there is still positive), the global minimum of the action (or the free energy) shifts from being at $\rho_1 = 0$ to being at $|\rho_1| = 1/2$. Thus, in this case we have a first order (discontinuous) phase transition at a temperature $T_0 < T_H$, as drawn in Fig. 3.

For $b > 0$, the behavior of the effective potential is depicted on the left side of Fig. 4. For $T < T_H$, the only minimum is at the origin $\rho_1 = 0$. As we increase the temperature above T_H the minimum shifts away from the origin, with the value of $|\rho_1|$ at the minimum increasing smoothly with the temperature, until it reaches its maximal possible value $|\rho_1| = 1/2$ at a temperature

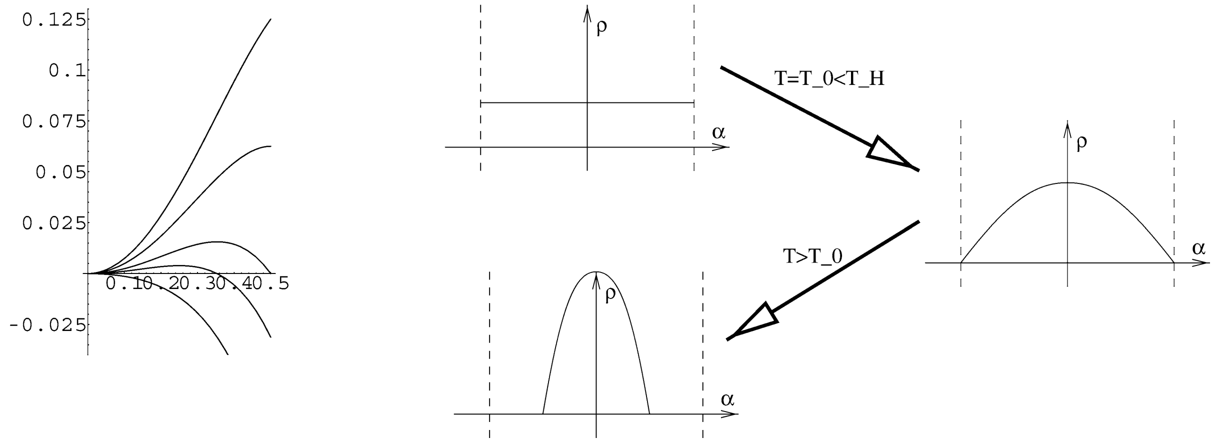

Fig. 3. The leftmost figure depicts the effective potential for $|\rho_1|$ for various temperatures when $b < 0$; the top graph corresponds to the lowest temperature. On the right the eigenvalue distribution is drawn (schematically) for various temperatures.

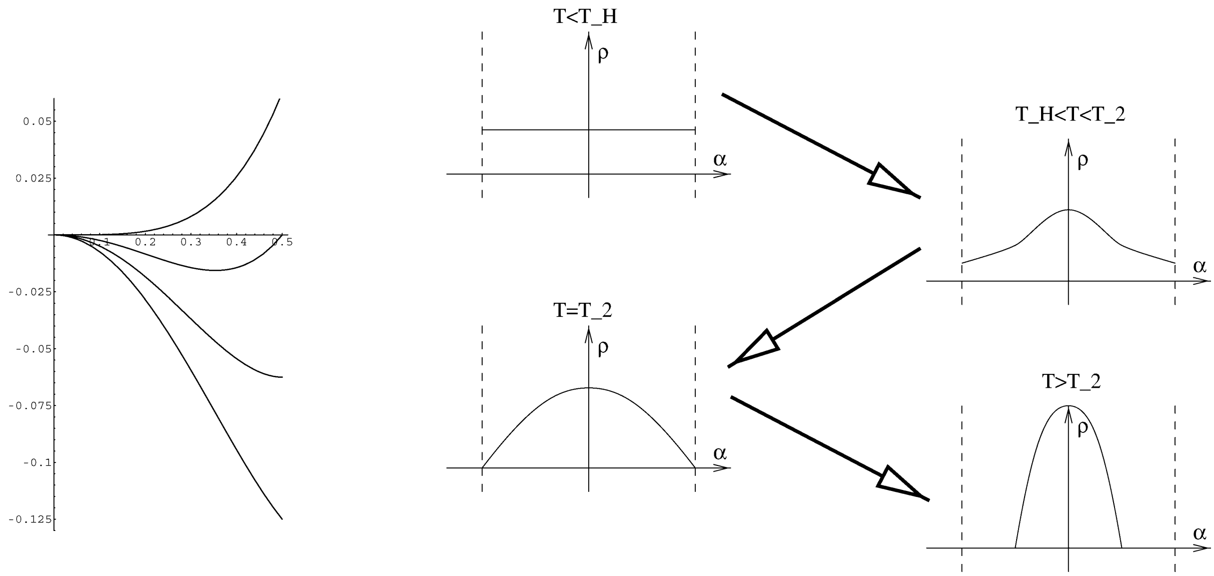

Fig. 4. The leftmost figure depicts the effective potential for $|\rho_1|$ for various temperatures when $b > 0$; the top graph corresponds to the lowest temperature. On the right the eigenvalue distribution is drawn (schematically) for various temperatures.

T_2. At this point another smooth transition takes place, in which the eigenvalue distribution changes from being non-uniform but everywhere non-zero to having a gap. The eigenvalue distributions for this case are depicted on the right-hand side of Fig. 4.

In order to know which of these two alternatives occurs in a given theory – namely, whether the transition is of first order or of second order, and whether it happens at the Hagedorn temperature or below it – we must compute the value of b in (9). This requires summing up all the diagrams contributing to the vacuum energy up to order λ^2. At the time of this article (September 2004) we are nearing the end of this computation for the case of the pure $\mathrm{SU}(N)$ Yang–Mills theory compactified on S^3. The contributing diagrams in this case, in a convenient choice of gauge and after integrating out some of the fields, are drawn in Fig. 5. More details will be given, together with the results of this computation, in [10].

4. Summary and future directions

To summarize, we have shown that weakly coupled large N gauge theories on compact spaces exhibit a Hagedorn density of states, $\rho(E) \sim \exp(E/T_H)$ for high energies (this behavior is cut off at an energy of order N^2 in the large N limit), and

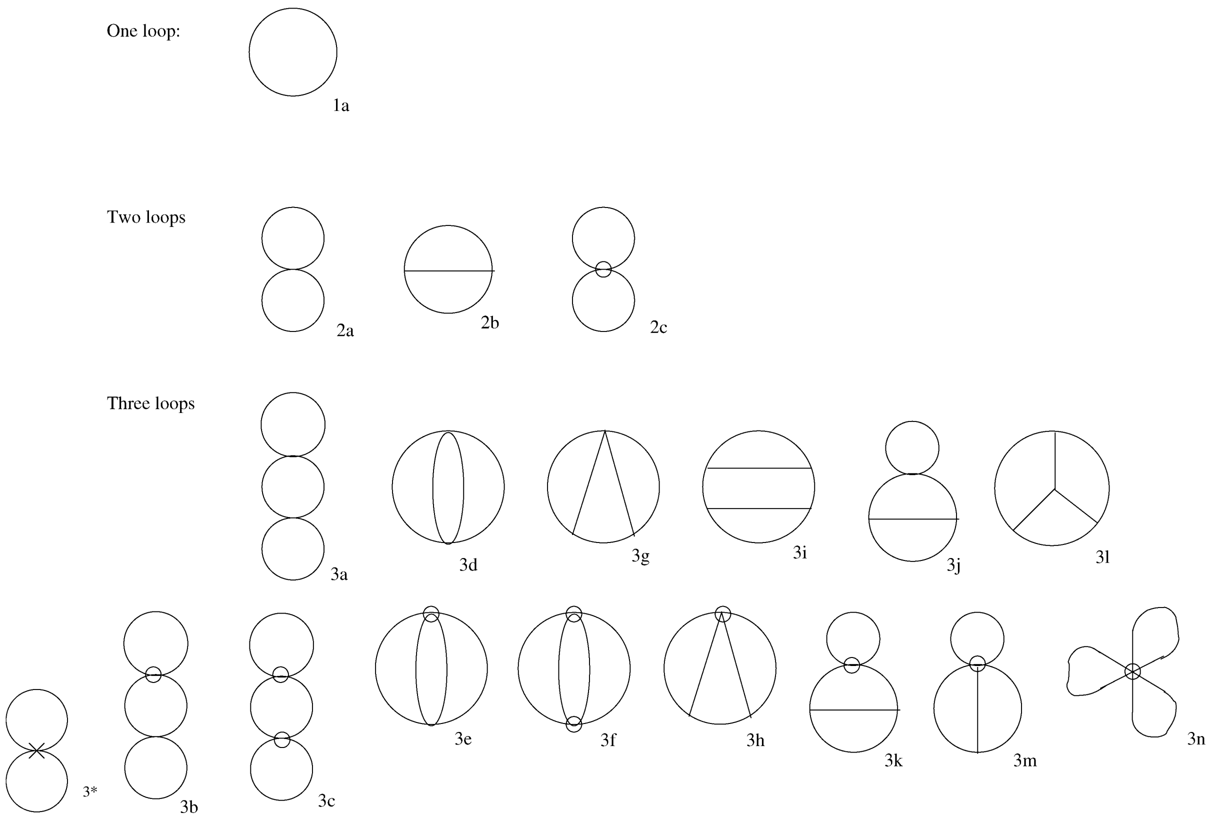

Fig. 5. All the diagrams contributing to the computation of b in the pure Yang–Mills theory on S^3. The solid lines are gauge field propagators, the circles denote vertices arising from integrating out the ghosts and some components of the gauge field, and the cross in the bottom left diagram denotes a double-trace counter-term of order g_{YM}^4 (which contributes at the same order as the 3-loop diagrams).

that they exhibit a deconfinement phase transition at a temperature which (like the Hagedorn temperature) is inversely related to the size of the compactification manifold. The deconfinement transition can either be a first order transition occurring below the Hagedorn temperature, or a second order transition at the Hagedorn temperature which is followed by another continuous transition at a higher temperature. The properties of the transition and of the stringy spectrum in specific theories may be computed in perturbation theory, through a unitary matrix model.

Our analysis applies both to asymptotically free theories compactified on manifolds much smaller than $1/\Lambda_{\mathrm{QCD}}$, and to compactified weakly coupled conformal field theories (such as $\mathbb{N} = 4$ supersymmetric Yang–Mills), as long as there are no zero modes appearing in the reduction of any of the fields on the compact manifold. In particular, our analysis applies to the large N pure Yang–Mills theory on a very small S^3. We expect that the transition we find there should be continuously related (by increasing the size of the S^3) to the deconfinement transition on a large S^3, which is a first order transition as in flat space. Depending on the sign of b, the simplest possibilities for the phase diagram of this theory as a function of the dimensionless parameter $R\Lambda_{\mathrm{QCD}}$ are depicted in Fig. 6.

We are continuing to work on computing the sign of b in this theory, as well as in the $\mathbb{N} = 4$ supersymmetric Yang–Mills theory on S^3. One can show that in general both signs of b can occur, depending on the precise Lagrangian of the gauge theory, and it is not clear if there is any simple way to determine the sign of b without doing the full 3-loop computation. The generalization of the analysis of the free theory above to include fields in the fundamental representation was performed in [11]. Other interesting future directions are:

– Our results provide explicit formulas for the spectrum of the free string theories which are dual to free large N Yang–Mills theories. It would be interesting to understand these theories better and to try to find their worldsheet description. In particular for the case of the $\mathbb{N} = 4$ supersymmetric Yang–Mills theory on S^3, these results are supposed to apply to type IIB string theory on $AdS_5 \times S^5$ in the limit in which the radius of curvature (in string units) goes to zero. Note that even in this limit we find a finite Hagedorn temperature, so it seems that this limit does not involve tensionless strings. This limit of

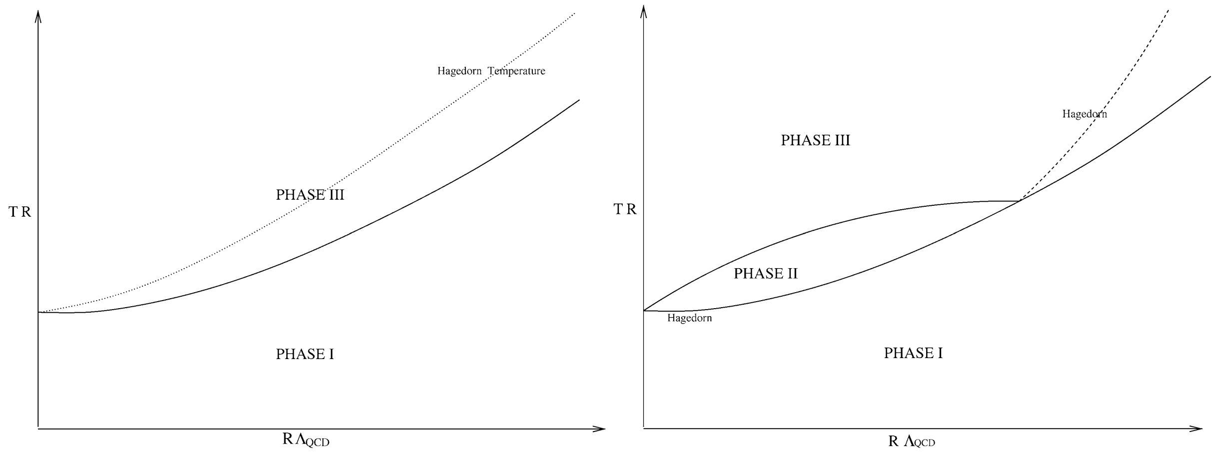

Fig. 6. The simplest possible phase diagrams for pure Yang–Mills theory on S^3. The solid lines are phase boundaries. Phase I (the confining phase) has a uniform eigenvalue distribution, phase II has a non-uniform but nowhere vanishing distribution, while phase III has a distribution with a gap. The left-hand diagram could apply if $b < 0$ in this theory, and the right-hand diagram if $b > 0$.

the AdS/CFT correspondence was discussed from other points of view in the talks at the conference by Gopakumar [12,13] and by Bianchi [14].

– When $b > 0$ we find an interesting intermediate phase, involving an eigenvalue distribution which is non-uniform but nowhere-vanishing (this phase also appears when $b < 0$ but it never dominates the canonical ensemble in that case). It would be interesting to understand this phase better and to learn how to distinguish it from the 'standard' deconfining phase which has a gapped eigenvalue distribution, both from the point of view of the Yang–Mills theory and from the point of view of possible string theory duals (where deconfined phases are generally mapped to black holes).
– There is mounting evidence that the $\mathbb{N} = 4$ supersymmetric Yang–Mills theory on S^3 may be integrable in the large N limit. It would be interesting to understand how this is reflected in the canonical ensemble, and whether the integrable properties of the theory (such as degeneracies of energy eigenstates) may be analyzed using our methods. Note that the canonical partition functions which we can compute perturbatively encode the full spectrum of perturbative anomalous dimensions in the case of conformal field theories on S^3, at least when the deconfinement phase transition is of second order.
– It would be interesting to try to generalize our analysis here to cases where the fields have zero modes, such as toroidal compactifications. In such cases perturbation theory breaks down since the zero modes are strongly interacting, so other methods must be used; some preliminary results for the case of $(1 + 1)$-dimensional Yang–Mills theories on a circle were recently presented in [15], and more will follow in [16]. In these cases the gauge field has also spatial holonomies, and there is an intricate phase structure involving both the spatial and the temporal holonomies. The spatial holonomies also have phase transitions similar to the ones described above, and at strong coupling these are mapped in the string theory duals to the Gregory–Laflamme black hole/black string transitions.
– Finally, it would be interesting to study to what extent one can indeed smoothly interpolate between the weak and strong coupling regimes, as we suggested in Fig. 6, and to what extent we can use our results to study (or generate models for) the deconfinement phase transition of QCD at infinite volume (at large N or for finite N). It is clear that some features differ between the regime of small $R\Lambda_{\mathrm{QCD}}$ and the regime of large $R\Lambda_{\mathrm{QCD}}$ (typical low-energy states in the former regime are uniformly spread out on the compact manifold, while typical low-energy states in the latter regime look like a gas of glueball particles), but it seems that no order parameters distinguish them so there is no obstruction to having a smooth interpolation between them. Unfortunately, there are no good methods to study the intermediate regime in which the size of the compactification manifold is of the same order as the QCD scale.

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