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Bose–Einstein condensates: recent advances in collective effects/Avancées récentes sur les effets collectifs dans les condensats de Bose–Einstein

Kinetic models for superfluids: a review of mathematical results

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Presented by Guy Laval

Abstract

The mathematical contributions by X.G. Lu (J. Statist. Phys. 98 (5/6) (2000) 1335–1394) and by M. Escobedo et al. (Electronic J. Differential Equations, Monograph 4 (2003)) presented in this Note constitute the first stage in the understanding of the superfluid dynamics, especially of the Bose–Einstein condensation, by means of kinetic models. The Boltzmann–Nordheim equation, which is physically relevant to describe dilute quantum Bose gases, sets important mathematical problems. Nevertheless, under an unphysical truncation of the collision cross-section at low energies, it has been proved that the spatially homogeneous Cauchy problem is well-posed. Furthermore, relaxation towards equilibrium holds in a weak sense, with the appearance of a singularity in infinite time if the initial mass is supercritical, which corresponds to the formation of a Bose–Einstein condensate. ***To cite this article: L. Saint-Raymond, C. R. Physique 5 (2004).***

Résumé

Modèles cinétiques des superfluides : des résultats mathématiques. Les contributions mathématiques de X.G. Lu (J. Statist. Phys. 98 (5/6) (2000) 1335–1394) et de M. Escobedo et al. (Electronic J. Differential Equations, Monograph 4 (2003)) qui sont présentées dans cette Note constituent la première avancée dans la compréhension de la dynamique superfluide et notamment de la condensation de Bose–Einstein grâce aux modèles cinétiques. L'équation de Boltzmann–Nordheim, qui permet de décrire l'évolution d'un gaz quantique dilué constitué de bosons, pose de nombreux problèmes mathématiques. Néanmoins, sous une hypothèse non physique de troncature des collisions à basse énergie, on peut montrer que le problème de Cauchy homogène en espace est bien posé. De plus, le système relaxe vers l'équilibre (en un sens faible), avec apparition d'une singularité en temps infini si la masse initiale est supercritique : cela correspond à la formation d'un condensat de Bose–Einstein. ***Pour citer cet article : L. Saint-Raymond, C. R. Physique 5 (2004).***

Keywords: Bose–Einstein condensation; Kinetic equation; Cauchy problem; Relaxation towards equilibrium

Mots-clés : Condensation de Bose–Einstein ; Équation cinétique ; Problème de Cauchy ; Relaxation vers l'équilibre

1. Introduction

As for classical gases the only way to model quantum gases without any approximation (i.e., the only approach which is valid in all regimes) is the atomistic point of view. The Newton system of motion equations for N particles is then replaced by the linear Schrödinger equation for the N body wavefunction:

$$\mathrm{i}\hbar\partial_t\psi_N = H_N\psi_N, \tag{1}$$

where H_N is the Hamiltonian, or in other words, the energy operator of the system.

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Of course the terminology of 'gas' holds for a system which contains a large number $N \gg 1$ of particles (with an additional dilution condition of the form $Na^3/\Omega \ll 1$ where a denotes the typical molecular diameter and Ω is the volume of the system), and in general equation (1) cannot be studied as it stands.

1.1. Various approximations to get the qualitative behaviour of superfluids

The usual way to understand the qualitative properties of the gas is then to derive approximate models depending on the regime under consideration.

For superfluids at very low temperatures, we expect almost all bosons to be in the lowest energy state, which means that the gas is almost a pure Bose–Einstein condensate.

At zero temperature, for a Bose gas in a non-dissipative trap, the one body density is governed by the following equation, the so-called Gross–Pitaevskii model:

$$\mathrm{i}h\partial_t\psi = \left(-\frac{h^2}{2m}\Delta + U + g|\psi|^2\right)\psi, \tag{2}$$

where U is the trapping potential, and the cubic term describes the microscopic interaction between particles.

In order to study the interaction between the condensate and the normal component of the superfluid, that is the behaviour at very low but finite temperature, a natural idea is then to proceed by perturbative expansion around the pure condensate state, which is known as the Bogoliubov method [1]. This procedure leads actually to technical difficulties linked, in particular, to the fact that the spectrum of the fondamental Hamiltonian is discrete. Moreover, this perturbative method does obviously not allow one to catch certain phenomena, such as the formation of the condensate.

1.2. Three models to study the Bose–Einstein condensates at finite temperature

An alternative way of studying the Bose–Einstein condensate at finite temperature consists in modelling separately the condensate phase and the normal component of the superfluid.

As usual, the condensate is governed by a time dependent Gross–Pitaevskii equation, that is, an equation of the same type as (2) with coupling terms modelling the mass and energy exchanges with the non-condensate part of the fluid. This last component of the superfluid is considered as a gas of particles whose motion is classical. This means that we will use a classical model either at kinetic or at fluid level, taking into account the collisions between bosons which do not belong to the condensate, the mean field created by the condensate, and the mass exchange with the condensate.

We will actually distinguish three types of models (see Fig. 1):

- a time dependent Gross–Pitaevskii equation for the condensate (and possibly the particles of low energy) and a classical probabilistic description of the normal component (kinetic equation of Boltzmann type with corrections to catch the degeneracy of bosons) coupled by exchange terms;
- a time dependent Gross–Pitaevskii equation for the condensate (and possibly the particles of low energy) and a classical fluid description of the normal component (with a state relation taking into account the modified form of thermodynamic equilibria for bosons) coupled by exchange terms;
- two fluid models for the condensate and the normal component coupled by exchange terms as predicted by Landau [2]. In this last model the superfluidity of the gas is taken into account in the fact that the suprafluid phase does not transport any entropy (no heat flux), and slips into the normal component without any dissipation.

Note that all these models can be related through various asymptotics.

The kinetic equation for the non-condensate part of the superfluid can be obtained from the primary Schrödinger equation (1) using the BBGKY expansion in the low density limit ($h \to 0$, $N \to \infty$, $\lambda h^2 N/mkT\Omega \to 0$ where λ states for the diffusive length for the interaction between particles of low energy, and m, k, T denote as usual the mass of particles, the Boltzmann constant and the temperature) [3]. This kinetic equation describes the dynamics of the momentum distribution that is also the Wigner transform of the one-particle density.

The connection between kinetic and macroscopic fluid dynamics results from two types of properties of the collision operator: the operator C satisfies the usual conservation laws, as well as an entropy relation that implies the relaxation towards equilibrium (which are Planckian distributions for Bose gases). The macroscopic limits are obtained when the particles undergo many collisions over the scales of interest. Indeed, local equilibrium is reached everywhere, and the fluid is fully described by its moments. Such asymptotics have been extensively studied for classical perfect gases: the formal expansions have been derived by Hilbert [4] in inviscid regimes, then by Chapman and Enskog [5] in viscous regimes. An important mathematical literature is devoted to the rigorous proofs of these fluid limits. In the case of Bose gases, even the formal hydrodynamic limits

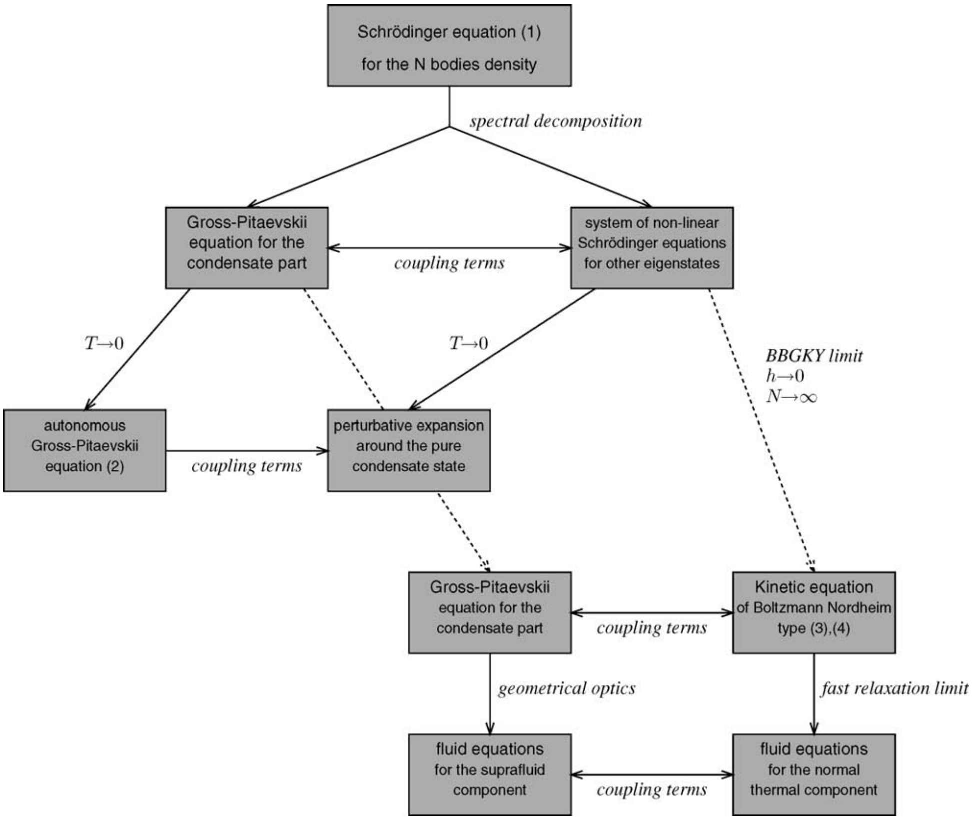

Fig. 1. The different models, and their interactions.

are not completely understood : a first work in this direction can be found in [6]. One of the difficulties is to deal with singular equilibria, another one is to obtain a relative velocity between the normal and the suprafluid components of the gas.

2. A model kinetic equation for bosons: the Boltzmann–Nordheim equation

To study the dynamics of the non-condensate part of the superfluid and the formation of the condensate, we start by considering only the kinetic equation for bosons (the corresponding fluid equations being not really well defined).

2.1. Description of the dynamics of the momentum distribution

In kinetic theory, a monoatomic gas is represented as a cloud of like point particles and is fully described by its momentum distribution F. The phase space of kinetic theory is the set of $(x, v) \in \mathbf{R}^3 \times \mathbf{R}^3$ where x is the position variable while v is the velocity variable. The meaning of F is as follows: any infinitesimal volume $\mathrm{d}x\,\mathrm{d}v$ centered at (x, v) contains at time t about $F(t, x, v)\,\mathrm{d}x\,\mathrm{d}v$ particles. The interaction of particles through collisions is modelled by an operator C; this operator acts only on the variable v and is generally nonlinear. If there is neither external force, nor other interaction of particles, the evolution of the momentum distribution is given by an equation of Boltzmann type

$$\partial_t F + v \cdot \nabla_x F = C(F). \tag{3}$$

For bosons, Nordheim [7] has proposed a Boltzmann like quantum kinetic theory. The collision operator describing the microscopic interactions takes then into account the propensity of the bosons to occupy the same quantum state:

$$C(F) = \iint\limits_{\mathbf{R}^3\times\mathbf{S}^2} \big(F'F_1'(1+F)(1+F_1) - FF_1(1+F')(1+F_1')\big) b(v - v_1, \omega)\,\mathrm{d}\omega\,\mathrm{d}v_1, \tag{4}$$

where the notations F_1, F' and F_1' designate respectively the values $F(t,x,v_1)$, $F(t,x,v')$ and $F(t,x,v_1')$, with v' and v_1' given in terms of $v_1 \in \mathbf{R}^3$ and $\omega \in \mathbf{S}^2$ by the formulas

$$v' = v - (v - v_1)\cdot\omega\omega, \qquad v_1' = v_1 + (v - v_1)\cdot\omega\omega. \tag{5}$$

These formulas give all possible solutions to the system with unknowns v' and v_1'

$$v' + v_1' = v + v_1, \qquad |v'|^2 + |v_1'|^2 = |v|^2 + |v_1|^2, \tag{6}$$

in terms of the data v and v_1 and of an arbitrary unit vector ω. The relations (6) are the conservation of momentum and kinetic energy for each binary collision between gas molecules (of like mass). The collision kernel $b \equiv b(z,\omega)$ is, in general, an almost everywhere (a.e. in short) positive function defined on $\mathbf{R}^3 \times \mathbf{S}^2$ that encodes whichever features of the molecular interaction are relevant in kinetic theory; it satisfies the symmetries

$$b(v - v_1, \omega) = b(v_1 - v, \omega) = b(v' - v_1', \omega), \tag{7}$$

for a.e. $(v, v_1, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$. These properties of the collision kernel b together with the identity $\mathrm{d}v\,\mathrm{d}v_1\,\mathrm{d}\omega = \mathrm{d}v'\,\mathrm{d}v_1'\,\mathrm{d}\omega$ imply that the following relation holds

$$\begin{aligned}\int C(F)(v)\phi(v)\,\mathrm{d}v = \frac{1}{4}\iiint &\big(F'F_1'(1+F)(1+F_1) - FF_1(1+F')(1+F_1')\big)\\ &\times \big(\phi(v) + \phi(v_1) - \phi(v') - \phi(v_1')\big)b(v - v_1, \omega)\,\mathrm{d}\omega\,\mathrm{d}v_1.\end{aligned} \tag{8}$$

Although this model does not contain all the physically relevant features of superfluids, it has raised the interest of physicists [3,8,9] because it simultaneously shares many similarities with the classical Boltzmann model and seems to take into account the specificity of bosons which present a degeneracy at very low temperatures. By (8), we have, at least formally, the local conservation of mass

$$\partial_t \int F\,\mathrm{d}v + \nabla_x \cdot \int vF\,\mathrm{d}v = 0, \tag{9}$$

the local conservation of momentum

$$\partial_t \int vF\,\mathrm{d}v + \nabla_x \cdot \int v \otimes vF\,\mathrm{d}v = 0, \tag{10}$$

the local conservation of energy

$$\partial_t \int \frac{1}{2}|v|^2 F\,\mathrm{d}v + \nabla_x \cdot \int \frac{1}{2}|v|^2 vF\,\mathrm{d}v = 0, \tag{11}$$

as well as the entropy inequality

$$H\big(F(t)\big) - \iiint_0^t D\big(F(s)\big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \geqslant H\big(F^0\big), \quad t > 0, \tag{12}$$

where the entropy is defined for all nonnegative measurable function $f \equiv f(v)$ by

$$H(f) = \iint \big[f\log(f) - (1+f)\log(1+f)\big]\,\mathrm{d}v\,\mathrm{d}x \in [-\infty, 0] \tag{13}$$

while the dissipation term $D(f)$ is defined by

$$\begin{aligned}D(f) = \frac{1}{4}\iint &\left(\frac{f'f_1'}{(1+f')(1+f_1')} - \frac{ff_1}{(1+f)(1+f_1)}\right)\log\left(\frac{f'f_1'(1+f)(1+f_1)}{(1+f')(1+f_1')ff_1}\right)\\ &\times (1+f)(1+f_1)(1+f')(1+f_1')b(v - v_1, \omega)\,\mathrm{d}v_1\,\mathrm{d}\omega\,\mathrm{d}v.\end{aligned} \tag{14}$$

This last inequality shows furthermore that the equilibrium states for the Boltzmann–Nordheim collision integral, in other words the number densities $E \equiv E(v)$ such that $C(E) = 0$, are the so-called Bose–Einstein distributions, i.e., the distribution functions of the form

$$P_{(\beta,u,\mu)}(v) = \frac{1}{\mathrm{e}^{\nu(v)} - 1} \quad \text{with } \nu(v) = \beta(v - u)^2 - \mu, \tag{15}$$

for some $\beta \in \mathbf{R}^+$, $\mu \in \mathbf{R}^-$ and $u \in \mathbf{R}^3$.

2.2. Formal analysis of the formation of singularities

The fundamental properties of the Boltzmann–Nordheim model stated above show that any solution of (3) has to satisfy simultaneously the global conservation of mass, momentum and energy and the growth of entropy. In particular, in order that the dissipation of entropy

$$\iiint_0^t D(F)(s,x)\,\mathrm{d}x\,\mathrm{d}s$$

stays bounded as $t \to \infty$, any solution has to relax towards equilibrium.

This leads to an apparent contradiction, since, for a given temperature T (and a given mean velocity $u \in \mathbf{R}^3$), there exists a critical mass ρ_T (depending only on T by translation invariance) such that

$$\forall \mu \leqslant 0, \quad \int \frac{\mathrm{d}v}{\mathrm{e}^{\nu_{\mu,u,T}(v)} - 1} \leqslant \rho_T,$$

where $\nu_{\mu,u,T}$ is the quadratic function defined by

$$\nu_{\mu,u,T}(v) = \frac{1}{kT}(v-u)^2 - \mu$$

and k denotes the Boltzmann constant. The question is therefore to understand what happens for initial data such that $\rho > \rho_T$.

Note first that this problem is a specificity of the Boltzmann–Nordheim model for Bose gases; it does not arise in the case of the classical Boltzmann equation, neither for the Boltzmann–Nordheim model for Fermi gases. It expresses the degeneracy of Bose gases, i.e., its propensity to present a state of congestion especially at low temperatures. The contradiction above is therefore removed by considering all equilibria of the following form

$$E_{\rho,u,T}(v) = \frac{1}{\mathrm{e}^{\nu_{\mu,u,T}(v)} - 1} + (\rho - \rho_T)_+ \delta_{v-u}, \tag{16}$$

with the same mean velocity u for the singular part as for the normal component. The Dirac mass expresses the existence of a condensate phase in the system, or in other words the fact that a macroscopic part of the system has coherent oscillations. Indeed the distribution $E_{\rho,u,T}$ so defined has the prescribed density ρ and momentum ρu, and further satisfies

$$C(E_{\rho,u,T}) = 0,$$

which is obtained by a formal computation using the spherical symmetry. Such singular equilibria have been mathematically introduced by Caflisch and Levermore [10] for the Kompaneets equation, which is a simple model for photon/electron scattering.

Note that these equilibria are physically admissible insofar as the Dirac mass leads to an increase of the entropy and it does not modify the entropy dissipation. A natural question is then to determine if the formation of the singularity arises in finite time, that is, if some solutions of the kinetic equation may blow up in finite time. For the solutions of the Kompaneets equation with supercritical initial mass, it has been proved that singularities appear in finite time [11].

From a physical point of view we cannot expect that phase correlations with an infinite range set in after a finite time (which would imply that the information of phase propagates at infinite speed). Nevertheless, the growth of a singular part in the momentum distribution is an indication that a condensate is formed in some sense. In order to obtain a condensation in finite time, it seems then necessary to add some physics in the model. The theory proposed in [12] indicates that the phase correlation range of the condensate grows actually as the square root of the time.

From a mathematical point of view the first step consists in describing precisely how the particles pile-up near mean momentum before collapse: an argument of selfsimilarity developed in [9] allows one to predict a dynamical process with power law distributions. The mathematical study of the formation of such a singularity in finite time could involve arguments as in [13]. The next step would be to give sense to singular solutions, in particular to study the Boltzmann–Nordheim equation in a class of functions containing the generalized equilibria given by (16). This would require an important work of analysis to define the entropy and entropy dissipation functionals in such a class of functions.

3. Mathematical results for the Boltzmann–Nordheim model

3.1. Existence of global spatially homogeneous solutions

The mathematical theory of the Boltzmann–Nordheim equation is just at the start; in particular, it does not allow, at the present time, to consider some important features of the model such as the mass and energy exchanges between the condensate and non-condensate components of the superfluid, nor to have an idea of its hydrodynamic limits.

The difficulty which is encountered trying to proceed as for the classical Boltzmann equation is the nature of the nonlinearity. On the one hand, this nonlinearity is essentially cubic

$$C(F)=\iint_{\mathbf{R}^3\times\mathbf{S}^2}\big(F'F'_1(F+F_1)-FF_1(F'+F'_1)\big)b(v-v_1,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_1+\iint_{\mathbf{R}^3\times\mathbf{S}^2}(F'F'_1-FF_1)b(v-v_1,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_1,$$

which is much more complicated to control than the quadratic nonlinearity of the Boltzmann equation (which already requires a theory of renormalized solutions). On the other hand, the estimates given by the entropy and entropy dissipation are much weaker than the corresponding estimates coming from the classical Boltzmann equation: these functionals are sublinear, and do not provide any compactness. A crucial element to make a rigorous theory for the Boltzmann–Nordheim theory would be then to give a sense to the collision operator in a space of functions defined in terms of the entropy and entropy dissipation functionals (and containing in particular the general equilibria given by (16)).

In order to give sense to the kernel C, the few mathematical works [14–16] dealing with the Boltzmann–Nordheim model use three main simplifications: they first assume that the momentum distribution $f\equiv f(t,x,v)$ is independent on the space variable x and isotropic with respect to the velocity variable v.

(H0) The momentum distribution f depends only on the quantity $r=|v|$.
In view of this assumption, it is natural to introduce the following notation

$$\begin{aligned}&\widehat{w}(r,r_1,r',r'_1)=\iiint_{\mathbf{S}^2\times\mathbf{S}^2\times\mathbf{S}^2}w(r\sigma,r_1\sigma_1,r'\sigma',r\sigma+r_1\sigma_1-r'\sigma')\,\mathrm{d}\sigma\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma',\\&w(v,v_1,v',v'_1)\stackrel{\mathrm{def}}{=}b\left(v-v_1,\frac{v-v'}{|v-v'|}\right)|v-v_1|.\end{aligned}\tag{17}$$

Of course $\widehat{w}$ satisfies the same symmetry properties as w. Also the collision operator can be rewritten in a simpler way

$$C(F)=\iiint_{\mathbf{R}^3_+}\big(F'F'_1(1+F)(1+F_1)-FF_1(1+F')(1+F'_1)\big)\widehat{w}(r,r_1,r',r'_1)\delta_{r^2+r_1^2-r'^2-r_1'^2}\,\mathrm{d}r_1\,\mathrm{d}r'\,\mathrm{d}r'_1,\tag{18}$$

where the notations F_1, F' and F'_1 designate respectively the values $F(t,r_1)$, $F(t,r')$ and $F(t,r'_1)$.

Then, in order to give sense to the collision operator C, a strong (and unphysical) truncation assumption is made on the collision kernel b, which kills interactions between particles with low energy.

More precisely, the various assumptions can be stated as follows

(H1) $$\sup_{z\in\mathbf{R}^3}\frac{1}{1+|z|^s}\int_{\mathbf{S}^2}b(z,\omega)\,\mathrm{d}\omega<+\infty\quad\text{with } s=2,$$

which is the usual Grad cut-off for the Boltzmann collision operator (meaning that the singularity due to grazing collisions is made integrable);

(H2) $$\widehat{w}\in L^\infty\big(\mathbf{R}^4_+\big),\quad\text{where }\widehat{w}\text{ is defined by (17)},$$

which has to be understood as a truncation assumption near the origin, meaning more or less that

$$\exists B_0>0,\quad b(z,\omega)\leqslant B_0(\cos\theta)^2\sin\theta|z|^3.$$

Note that such a condition is satisfied for any cross-section w such that

$$\exists w_0>0,\quad w\leqslant w_0\min\big(|v'-v||v'_1-v|,1\big).$$

Under these assumptions, the Cauchy problem for the homogeneous equation

$$\partial_t F=C(F)\tag{19}$$

is globally well-posed in the space $\mathcal{M}_2$ of bounded measures with two bounded moments.

Theorem 3.1 [16]. *Let b be a collision kernel satisfying assumptions* (H1), (H2). *Consider an initial data $F^{\mathrm{in}}\in\mathcal{M}(\mathbf{R}^3)$ with radial symmetry*

$$F^{\mathrm{in}}(v)=g^{\mathrm{in}}\big(|v|\big)$$

for some bounded measure g^{in} defined on $\mathbf{R}^+$, and such that

$$\int g^{\mathrm{in}}(r) r^4 \, \mathrm{d}r < +\infty.$$

Then there exists a unique solution $F \in C([0,+\infty[, \mathcal{M}_2(\mathbf{R}^3))$ to (3).

The proof of this result is based on standard arguments once we are able to give sense to the collision operator $C(f)$ for any f in the class of functions under consideration. The first result in that direction is due to Lu [14]:

Lemma 3.2. *Denote by C^+* (*resp. C^-*) *the gain part* (*resp. the loss part*) *of the collision operator. Then, under assumptions* (H1), (H2) *on the collision kernel b, the following functional inequality holds*:

$$\forall F \in L^1_2, \quad \left\|C^+(F)\right\|_{L^1} + \left\|C^-(F)\right\|_{L^1} \leqslant C_B \|F\|^2_{L^1_2} + C_{\tilde{w}} \|F\|^3_{L^1}, \tag{20}$$

where L^1_2 denotes the space of inegrable functions with two integrable moments.

This crucial a priori bound is obtained estimating separately each term of the collision integrand.

The quadratic terms may be defined thanks to assumption (H1) and they are bounded by the first term in the right-hand side of (20).

The cubic terms are defined making one more integration in the representation formula (18) (on one of the variables r, r', r'_1) and they are bounded by

$$\|\widehat{w}\|_{L^\infty} \|F\|^3_{L^1}.$$

This estimate can actually be extended for bounded measures, as proved by Escobedo and Mischler [15]. Their method is inspired by the work of Povzner [17], it consists in defining the various terms by duality.

For instance, in order to define the cubic term

$$\iiint_{\mathbf{R}^3_+} F' F'_1 F \hat{w}(r, r_1, r', r'_1) \delta_{r^2 + r_1^2 - r'^2 - r_1'^2} \, \mathrm{d}r_1 \, \mathrm{d}r' \, \mathrm{d}r'_1$$

we consider its distributional bracket with any test function $\phi \in C^\infty_c(\mathbf{R}_+)$

$$\int \phi(r) \iiint_{\mathbf{R}^3_+} F' F'_1 F \widehat{w}(r, r_1, r', r'_1) \delta_{r^2 + r_1^2 - r'^2 - r_1'^2} \, \mathrm{d}r_1 \, \mathrm{d}r' \, \mathrm{d}r'_1 \, \mathrm{d}r = \iiint \mathrm{d}F(r_1) \, \mathrm{d}F(r_2) \, \mathrm{d}F(r_3) B[\phi](r_1, r_2, r_3),$$

where the quantity $B[\phi]$ is defined as follows

$$B[\phi](r_1, r_2, r_3) = \widehat{w}\Big(r_1, r_2, r_3, \sqrt{r_1^2 + r_2^2 - r_3^2}\Big) \frac{\sqrt{r_1^2 + r_2^2 - r_3^2}}{2} H_{r_1^2 + r_2^2 - r_3^2} \phi(r_1),$$

where H denotes the Heaviside function. The condition

$$B[1] \in C\big(\mathbf{R}^3_+\big)$$

guarantees that this cubic term is well defined for any isotropic measure F; it holds if

$$b \in C\big(\mathbf{R}^3 \times \mathbf{S}^2\big)$$

which is implied by (H1), and if moreover

$$\lim_{(r_1, r_2, r_3) \to 0} \widehat{w}\Big(r_1, r_2, r_3, \sqrt{r_1^2 + r_2^2 - r_3^2}\Big) \sqrt{r_1^2 + r_2^2 - r_3^2} = 0$$

which is a consequence of (H2).

Similar computations for the other terms lead finally to

$$\forall F \in \mathcal{M}_2, \quad \int C^+(F) + \int C^-(F) \leqslant C_B \bigg(\int \big(1 + |v|^2\big) \, \mathrm{d}F \bigg)^2 + C_{\tilde{w}} \bigg(\int \mathrm{d}F \bigg)^3$$

which allows one to establish Theorem 3.1 [16].

Remark 1. A refined version of (20) has been used by Lu [14] to establish a global existence result when $s = 0$ in (H1). For $s = 1$, Lu also proves an existence result making the additional assumption that b has the particular shape

$$b(z,\omega) = |z|^{\gamma}\zeta(\theta)$$

with $\gamma \in [0,1]$, $\zeta \in L^1$.

3.2. Relaxation towards equilibrium

In the L^1 framework, Lu has obtained some results on the long-time behaviour of such solutions: the relaxation towards equilibrium holds at least in a weak sense.

Theorem 3.3 [14]. *Let b be a collision kernel satisfying assumptions* (H1), (H2). *Consider an initial data $F^{\text{in}} \in L^1(\mathbf{R}^3)$ with radial symmetry*

$$F^{\text{in}}(v) = g^{\text{in}}\big(|v|\big)$$

for some function $g^{\text{in}} \in L^1_4(\mathbf{R}^+)$. Denote by $g \in C([0,+\infty[, L^1(\mathbf{R}^3))$ the corresponding solution of (3).

Then, for all sequence of times t_n going to $+\infty$, there exist a mass $\rho \leqslant \rho^{\text{in}}$, an energy $E \leqslant E^{\text{in}}$ and a subsequence of (t_n) such that

$$g(t_n) \to P_{\beta,0,\mu}$$

in renormalized sense where $\beta \geqslant 0$ and $\mu \leqslant 0$ are uniquely defined by

$$\int P_{\beta,0,\mu}\,\mathrm{d}v = \rho, \qquad \int P_{\beta,0,\mu}|v|^2\,\mathrm{d}v = 2E$$

(*this very weak notion of convergence does not take into account the singular part of the limit*).

Moreover, if the initial mass is subcritical, the previous convergence holds in weak L^1 (*since there is no singular part in the limit*).

These asymptotics are established by means of a rather weak concept of convergence, the biting-weak convergence, introduced by Chacon, in the form proposed by Ball and Murat [18]. The study is based on the following equivalent formulations of Eq. (19)

$$\partial_t\beta(F) = \beta'(F)C(F),$$

where β denotes appropriate functions of $C_c^\infty(\mathbf{R}^+)$.

The proof is rather technical and will not be detailed here for the sake of simplicity.

Remark 2. For the spatially homogeneous Boltzmann–Compton equation (which describes the photon–electron scattering by means of a collision operator being more or less a 'linearized' version of the Boltzmann–Bose collision operator), a more detailed study of the asymptotic behaviour of the solutions can be made: indeed, it has been proved by Escobedo and Mischler [15] that the equation may be split in a system of two equations for the regular part $f\,\mathrm{d}v$ and the singular part μ of the measure $\mathrm{d}F$ (with respect to the Lebesgue measure).

Unfortunately such a decomposition does not hold for the general isotropic solutions F of the original Boltzmann–Nordheim equation (19) unless the singular part reduces to a single Dirac mass. Indeed, following [15], we split the collision operator in many parts, and we define in particular

$$Q(\mu) = \iiint_{\mathbf{R}^3_+} \mu'\mu_1\mu_1'\widehat{w}(r,r_1,r',r_1')\delta_{r^2+r_1^2=r'^2+r_1'^2}\,\mathrm{d}r_1\,\mathrm{d}r'\,\mathrm{d}r_1'.$$

If μ is not a single Dirac mass then $\operatorname{supp}(\mu)\setminus\{0\}$ is strictly contained in $\operatorname{supp}(Q(\mu))$. Then there exists μ singular such that $Q(\mu)$ has a regular part which is not equal to zero.

In the particular case where we assume that for every time $t > 0$, $\mu(t,v) = \alpha(t)\delta(v)$, Eq. (19) may be split into a coupled system of equations for the pair (f,α). Nevertheless, because of the truncation hypothesis, such a case is not very interesting.

As was said at the beginning of this paragraph, the mathematical results on the Boltzmann–Nordheim model are not satisfactory from a physical point of view.

First of all the truncation on the collision kernel prevents the solutions of (3) from blowing-up in finite time: if the initial distribution is integrable with respect to the Lebesgue measure, no singular measure can appear in finite time, which implies in particular that the mass of the non-condensate component is conserved. The convergence towards equilibrium has then to be understood as a long time behaviour. In order to take into account the interactions between particles of low energy which are expected to produce the blow-up, it would be necessary to work in a different functional framework to be determined.

Before being able to deal with such a difficult analysis, it would be interesting to better understand the qualitative properties of the solutions built by Lu. A natural problem is indeed to describe precisely the long time behaviour of the solutions: we actually expect the concentration to take place with a power law profile. Two preliminary questions would have to be answered: can we prove that

- a solution with supercritical mass converges to the corresponding generalized equilibrium in the sense of measures?
- the non-singular part of a solution with supercritical mass converges to the corresponding Boltzmann–Bose distribution, say in L^1-norm?

In the case where the initial distribution contains already a condensate, the correlation range is infinite and the equation exhibits transfer of mass between the condensate and non-condensate components of the fluid. A natural question is then to describe precisely the dynamics associated with this transfer of mass, and to understand how it is modified by taking into account the interactions between particles of low energy.

4. Back to the modelisation

In view of the previous section, the mathematical theory of the Boltzmann–Nordheim equation seems to give very few results, and to be at the present time of no help to understand the physics of the Bose–Einstein condensation.

In order to further study this equation, we have actually to distinguish two types of difficulties, the first ones coming from technical points or mathematical methods, and the second ones being inherent to the model (depending on its domain of validity).

Let us first recall that an important part of the physics has been taken away to get the Boltzmann–Nordheim equation. First of all, this equation is expected to govern only the non-condensate part of the superfluid; in order to take into account the exchange of mass and energy with the condensate part it would be necessary to add coupling terms. Moreover, this equation is derived using the BBGKY hierarchy in the low density limit

$$\int f(t,x,v)\,\mathrm{d}v \ll 1,$$

considering velocities which are very large compared with the sound speed (computed with the Bogoliubov theory), so that it could be not relevant to study it in regimes where singularities arise (even before collapse).

4.1. From a physical point of view

To obtain a model which is more relevant from a physical point of view, the first step is to involve a mechanism (which is typically a quantum effect) that allows one to go from a singular L^1 distribution to a congestion state: as the formation of the condensate is predicted to occur through a solution with a finite time singularity, the rate of evolution of this solution diverges like the inverse of the time remaining until the singularity, which makes the kinetic theory invalid when this time scale becomes shorter than the period associated with free-particle motion by the Planck–Einstein correspondence.

Then, in the presence of a Bose–Einstein condensate, the model has to be modified and the resulting system of coupled equations has to compatible with the various hydrodynamic models, either the classical two-fluid hydrodynamics of Landau, or the hydrodynamic models out of equilibrium such as [6].

In [3], such a system is derived: it involves a nonlinear Schrödinger equation for the wave function of the condensate coupled with a kinetic equation for the normal (thermal) component of the superfluid. This system shows a possible exchange of mass between the two components through a kind of induced emission preserving the coherence of the condensate. This system is expected to give a relevant description of the superfluid in the low density limit since it is obtained more or less by a BBGKY expansion. The main features of the homogeneous model are the following:

- it takes into account Bogoliubov's renormalization for the energy spectrum;
- it is based on a decomposition of the Boltzmann–Nordheim collision operator as obtained in Remark 2.

Extended to the inhomogeneous case:

- it implies that the exchange of mass between the condensate and the normal component occurs without any modification in the phase of the wave function;
- it takes into account the frequency $\mathrm{d}\theta/\mathrm{d}t$ of the wave function in the conservation of energy, and the wave number $h\nabla\theta$ in the conservation of momentum for each binary collision, which modifies the definition of the cross-section.

4.2. From a mathematical point of view

The difficulties encountered in the mathematical study of the Boltzmann–Nordheim equation (at least in the homogeneous case) are essentially due to the presence of a cubic term in the collision operator. In other words, the problem comes from the lack of integrability of the momentum distribution. Of course, if the model is relevant to describe the dynamics of a Bose gas, we cannot expect to get estimates on the L^p-norms for $p > 1$.

A first approach inspired by the work of Pomeau et al. [3] would be to split the density in many components:

– a very regular component (bounded in some L^p for $p > 1$);
– another component regular with respect to the Lebesgue measure which models the flux towards mean momentum and which could play a crucial role in the formation of the singularity;
– a condensate part in the form of a Dirac mass,

and to write a formally equivalent system of coupled equations for these three components. Note that, in order that the collapse occurs in finite time, the coupling between the last two components has probably to be slightly modified.

The difficulty is then to do a similar analysis of the splitting as in Remark 2 without the unphysical assumption on the cross-section.

An alternative would be to modify directly the Boltzmann–Nordheim operator and to renormalize the various terms involved in the integrand. For instance, the operator C could be replaced by

$$\begin{aligned}\widetilde{C}(F) &= \iint\limits_{\mathbf{R}^3\times\mathbf{S}^2} \left(\frac{F'F_1'}{(1+F')(1+F_1')} - \frac{FF_1}{(1+F)(1+F_1)}\right) b(v-v_1,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_1 \\ &= \iint\limits_{\mathbf{R}^3\times\mathbf{S}^2} \frac{(F'F_1'(1+F)(1+F_1) - FF_1(1+F')(1+F_1'))}{(1+F')(1+F_1')(1+F)(1+F_1)} b(v-v_1,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_1,\end{aligned} \tag{21}$$

which has the same equilibrium states, and which can be easily defined for bounded measures.

From a certain point of view, such a renormalisation is not absurd, since the Boltzmann–Nordheim equation can be rigorously derived in the low density limit, under the stronger assumption

$$f \ll 1$$

(see [19]), and that in this limit

$$f \sim \frac{f}{1+f}.$$

Of course, such a model does not lead to a singularity in finite time (since the right-hand side $\widetilde{C}(F)$ is bounded in L^∞). However, it could be interesting to understand precisely the mechanism of relaxation for this simplified model.

Note that an equation of this type has been obtained by Laloë et al. in [20] or [21] using a phenomenological approach (without any link with the BBGKY hierarchy), the so-called free Wigner transform. The general idea is to replace the coupling introduced by the collisions by a singularity in the phase space. The model so obtained seems to present qualitative features which are relevant from a physical point of view (for instance, it allows one to establish the Bethe–Uhlenbeck formula for the contribution of the binary collisions to the equilibrium pressure).

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