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The Cosmic Microwave Background/Le rayonnement fossile à 3K

The inflationary paradigm: predictions for CMB

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Abstract

The search for a causal explanation of the large scale properties of the universe supports the idea that a long period of accelerated expansion, called inflation, preceded primordial nucleosynthesis. The first consequence of inflation is that all pre-existing classical structures are washed out. In fact, in the simplest inflationary models, the primordial density fluctuations (the seeds of the large scale structures) only result from the amplification of quantum vacuum fluctuations. The properties of the spectrum so obtained are presented and compared to the CMB temperature fluctuations. The agreement is striking. ***To cite this article: R. Parentani, C. R. Physique 4 (2003).***

Résumé

L'inflation et ses prédictions concernant le fond diffus. La recherche d'une explication causale des propriétés de l'univers aux grandes échelles suggère qu'une longue période d'expansion accélérée, appelée inflation, ait précédé la nucléo-synthèse primordiale. La première conséquence de l'inflation est que toutes les structures pré-existantes sont effacées. En effet, dans les modèles d'inflation les plus simples, les fluctuations de densité primordiales (les germes des structures de grandes échelles) résultent uniquement de l'amplification de fluctuations quantiques du vide. Les propriétés du spectre ainsi obtenu sont présentées et comparées avec les fluctuations de température dans le fond diffus. La correspondance est remarquable. ***Pour citer cet article : R. Parentani, C. R. Physique 4 (2003).***

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Mots-clés : Inflation ; Théorie quantique des champs ; Anisotropies du fond de rayonnement cosmique ; Cosmologie ; Univers primordial

1. Introduction

This paper is conceived as a pedagogical introduction to the motivations for inflation, to its mathematical settings and to its implications concerning the Cosmic Microwave Background (CMB) anisotropies.

For the non-specialist, two remarks should be made from the outset. First, without recourse to fine-tuning, inflation has successfully passed the cross-checks based on recent observational data. Secondly, inflation is a rather conservative hypothesis since it rests, on one hand, on the set of cosmological observations conventionally interpreted, and on the other hand, on Einstein's equations, i.e., on the hypothesis that the action of General Relativity (GR), or a slight generalization thereof, governs the evolution of the space-time properties from cosmological scales, of the order of a thousand megaparsecs (1 Mpc $\simeq 10^{22}$ m), down to some microscopic scale which is close to the Planck length ($l_P \simeq 10^{-35}$ m), 'just' before the threshold of Quantum Gravity.

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Following these preliminary remarks, one should explain why the search for a causal explanation of the large scale properties of the universe calls for a long period of accelerated expansion (‘large scale’ and ‘long’ shall be defined in Sections 2.3 and 3.1). This need stems from the fact that the ‘standard model’ of cosmology [1,2], the hot big bang scenario, is *incomplete*: the causal structure from the big bang is such that no processes could have taken place to explain the homogeneity and the isotropy at large scales. The importance of these considerations can only be appreciated in the light of today’s understanding of cosmology.

2. The standard model of cosmology

The predictions of modern cosmology concern the evolution of *spatial averaged quantities* (temperature, densities...) and *local structures* (from density fluctuations to galaxies and clusters). Both aspects should be considered to appreciate the merits of inflation.

The CMB is extremely isotropic: the temperature fluctuations on the last scattering surface, at the decoupling time t_{dec}, have a relative amplitude of the order of 10^{-5}. By adopting GR, we learn that the (today visible) universe can at that time be described by a perturbed Robertson–Walker (RW) metric. Indeed the metric perturbations are also of the order of 10^{-5} since they are linearly related to the temperature fluctuations. We recall that the RW metric reads

$$\mathrm{d}\bar{s}^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Sigma_3^2, \tag{1}$$

where bar quantities describe smoothed out, spatial averaged, quantities. The spatial part of the metric $\mathrm{d}\Sigma_3^2$ describes isotropic and homogeneous 3-surfaces. The actual metric, ignoring gravitons (the two spin 2 fluctuation modes), can be written as

$$\mathrm{d}s^2 = -(1+2\Psi)\,\mathrm{d}t^2 + (1-2\Psi)a^2(t)\,\mathrm{d}\Sigma_3^2, \tag{2}$$

when the matter stress-tensor is isotropic, a mild hypothesis [3]. The local field $\Psi(t,x)$ acts as a Newtonian potential. Since primordial fluctuations are 10^{-5}, one can first adopt a mean field approximation and consider only spatial averaged quantities. In a second step, one can analyze the evolution of the fluctuations which ride on the background quantities formerly obtained.

2.1. Background quantities and primordial nucleosynthesis

The evolution of the averaged metric Eq. (1) is governed by a single function of time, the scale-factor $a(t)$. The derivative of its logarithm defines $H = \partial_t a/a$ which enters in Hubble’s law $v = HR$. This law relates the velocity of a comoving galaxy to its proper distance $R(t) = a(t)d$, where d is its fixed comoving distance which is defined by the static line element $\mathrm{d}\Sigma_3^2$. In GR, a obeys the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{\kappa}{a^2}. \tag{3}$$

The averaged matter density is denoted $\bar{\rho}$ and includes a possible cosmological constant [4]. The last term in Eq. (3) arises from the curvature of the homogeneous 3-surfaces. Its relative contribution today is given by $\Omega_0^{\text{curv.}} = \kappa/(H_0^2 a_0^2)$ (where the subscript $_0$ means evaluated today) and it is observationally tightly constrained: $\Omega_0^{\text{curv.}} = 0.02 \pm 0.02$ [5]. This term is the integration constant of the ‘dynamical’ Einstein equation,

$$\frac{\partial_t^2 a}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\overline{P}), \tag{4}$$

and energy conservation $\mathrm{d}\bar{\rho} = -3(\bar{\rho} + \overline{P})\,\mathrm{d}\ln a$. Together with the density $\bar{\rho}$, the pressure $\overline{P}$ fully characterizes the matter stress tensor when imposing homogeneity and isotropy. Furthermore, when considering non-interacting components, $\bar{\rho}$ and $\overline{P}$ split into terms which obey equations of state of perfect fluids:

$$\overline{P_i} = w_i \bar{\rho}_i. \tag{5}$$

Dust, (cold) baryons and Cold Dark Matter (CDM) have $w = 0$, photons and (massless) neutrinos $w = 1/3$, whereas the cosmological constant has $w = -1$. Energy conservation then fixes the scaling of each component. For baryons, one gets $\bar{\rho}_b \propto a^{-3}$, and for radiation, $\bar{\rho}_{\text{rad}} \propto a^{-4}$. The 3-curvature term scales only as a^{-2} whereas the cosmological constant does not vary.

From these scalings and the present values of the densities, one immediately obtains the different stages of our cosmological history: The CMB temperature, $T_0 = 0.23$ meV, the effective number of massless fields [2], and the relative contribution of cold matter (baryons + CDM), $\Omega_0^{\text{cm}} = 0.27 \pm 0.04$ [5], fix the redshift of the transition from radiation to matter domination: $z_{\text{eq}} = a_0/a_{\text{eq}} - 1 \simeq 3200$. This equilibrium occurred before decoupling which occurred when $z_{\text{dec}} \simeq a_0/a_{\text{dec}} \simeq 1100$. At that time, the 3-curvature contribution obeyed $\Omega_{\rm dec}^{\rm curv.} < \Omega_0^{\rm curv.}/\Omega_0^{\rm cm} z_{\rm dec} < 10^{-4}$. Thus, for all processes which took place at or before decoupling, the spatial metric can be taken Euclidean and the term κ/a^2 in Eq. (3) can be dropped.

The next step consists in using *particle physics* to inquire about the processes which took place in the early universe, before $z_{\rm eq}$. The greatest success of this standard framework concerns the prediction of relative abundances of light elements by primordial nucleosynthesis [1,2,6]. This confirms the validity of extrapolating backwards in time Einstein's equations in a radiation dominated universe, for redshifts of the order of 10^9 and temperatures of the order of the MeV. Moreover, a detailed comparison of theoretical predictions and observations leads to a precise determination of the baryonic content of our universe (through the determination of n_b/n_γ, the number of baryon per photon, and the absolute normalization of the CMB thermal photons). Using the recent estimation of H_0, primordial nucleosynthesis gives $\Omega_{b0} = 0.042 \pm 0.005$, a value in agreement with that obtained from analyzing the CMB [7]. So, not only the predictions of the extrapolation can be observationally tested, but independent cross-checks agree. This puts the standard model of cosmology on extremely firm foundations.

2.2. Fluctuations and structures

So far, we have ignored the fluctuations around spatial averaged quantities. It is remarkable that the standard model, as it stands, also correctly predicts the evolution of local structures [1,2]. These structures are described by a perturbed metric, see Eq. (2), and a local matter density $\rho = \bar\rho(t) + \delta\rho(t,x)$. Their combined evolution is governed by matter interactions and by the *local* Einstein equations (and no longer by their restriction to RW metrics).

When starting at $z \simeq 1100$ with the *primordial* fluctuations $\delta\rho/\bar\rho \simeq \Psi \simeq 10^{-5}$, gravitational instabilities lead to non-linear structures whose properties are in agreement with what is now observed. However, the best agreement is reached if one introduces some CDM whose density is about 10 times larger than that of baryons [8]. At first sight this could be considered as an ad hoc hypothesis to save the standard model, but this is not the case since there are now independent observations which are consistently interpreted as gravitational effects induced by CDM [4]. The most precise estimation of $\Omega_{\rm CDM0}$ $(= \Omega_0^{\rm cm} - \Omega_{b0} = 0.23 \pm 0.04)$ is presently obtained from the CMB anisotropies. (Given that $\Omega_{\rm CDM0}$ is known, the challenge with CDM is to identify what it is made from.)

2.3. The incompleteness of the standard model

Given these two successes, it is tempting to further extrapolate and to inquire what happened before nucleosynthesis, for redshifts larger than 10^{10} and energies larger than the MeV. Was there a period of baryo-genesis which could explain the baryon-antibaryon asymmetry, or a lepto-genesis?

In this inquiry one searches for processes by which to explain the expectation values of quantities which, for lack of something better, have been hitherto treated as initial conditions. Inflation might constitute a crucial step in this endeavor. The reason is the following. Irrespectively of the high energy local processes which took place, one inevitably encounters a problem which is non-local in character: Given the *causal* structure obtained by extrapolating $a(t)$, the solution of Eq. (4) driven by radiation, the *large scale isotropy* cannot be explained, as clearly seen from Fig. 1.

There is an interesting and complementary way to consider this incompleteness. It concerns the origin of *large scale anisotropies*. Since the RW metric is homogeneous, comoving wave vectors k are conserved. Hence, when treated linearly, the Fourier modes $\Psi_k = \int {\rm d}^3x\, {\rm e}^{{\rm i}kx}\Psi$ and $\delta\rho_k = \int {\rm d}^3x\, {\rm e}^{{\rm i}kx}\delta\rho$ evolve independently of modes with $k' \neq k$. Moreover the Hubble

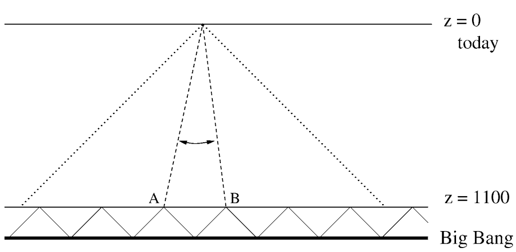

Fig. 1. The causal structure from the big bang. In a radiation dominated universe, there is a *space-like* singularity at $a = 0$ which is situated at a *finite* conformal time $\Delta\eta = \int_{t_{bb}}^t {\rm d}t/a$. Hence forward light cones from the big bang have, at time t, a finite (proper) size equal to $a(t)\Delta\eta(t)$. On the last scattering surface (*lss*), the proper distance between the points A and B is equal to $a\Delta\eta$. Hence the matter systems at A and B have never been in contact. Thus no (causal) process could possibly explain why the temperature fluctuation between A and B is only of the order of 10^{-5}. Moreover, the today visible universe (in dotted lines) encompasses about 5×10^4 disconnected patches on the *lss* with fluctuations within that range.

radius $R_H = 1/H$ $(c = 1)$ provides a natural length scale which coincides, in a radiation dominated universe, with the causal horizon $a\Delta\eta$ defined in the caption of Fig. 1. Because comoving wave vectors are conserved, it is appropriate to study the evolution of its comoving value:

$$d_H(t) = \frac{R_H(t)}{a(t)} = \frac{1}{\partial_t a}. \tag{6}$$

The relevance of d_H follows from the fact that modes behave very differently according to the relative value of their wave length $1/k$ and d_H: When $1/k < d_H$, modes oscillate. On the other hand, on super-horizon scales, i.e., when their wave length extends beyond the Hubble radius, modes are frozen (or decay).

These considerations become crucial when questioning the *origin* of the primordial fluctuations $\Psi_k \simeq \delta\rho_k/\bar{\rho} \simeq 10^{-5}$ on super-horizon scales (which we know exist since they determine the CMB large scale anisotropies for large scales, i.e., angles larger than 2 degrees, see Fig. 1). Since d_H always increased in a radiation dominated universe, as seen from Eqs. (6) and (4), the modes which were still outside d_H at decoupling must have been frozen since the big bang. Hence their amplitude can only be determined by *initial conditions* arbitrarily chosen.

3. Inflation

Inflation is the price to pay to reject this outcome. (Topological defects resulting from a phase transition could have been another possibility. This, however, is now ruled out by the detailed properties of CMB anisotropies [9,10].) From the above considerations, a necessary condition for allowing physical processes to have taken place is that there was a time when d_H was *larger* than today's Hubble scale d_{H0}. Therefore, between then and the beginning of the radiation era, d_H must have *decreased* tremendously. Thus from Eq. (6) there must have been a 'long' period of accelerated expansion, see Fig. 2.

At this point, inflation is simply a kinematic hypothesis which allows processes to have taken place. (Notice however that the dilution of scales associated with the *decrease* of d_H during inflation solves other problems of cosmology [11,12].) In order to proceed, several issues should be confronted. One should first dynamically realize inflation, i.e., find an 'engine' which could be responsible for it. Secondly one should identify the mechanism giving rise to the primordial spectrum [13]. The third issue concerns the *reheating* [14,15]: The collective process which liberates a lot of heat at the end of inflation thereby leading to the radiation dominated universe necessary for primordial nucleosynthesis. It should also be noticed that to a large extend these issues could be addressed separately. In particular, it turns out that the mechanism giving rise to the spectrum operates quite irrespectively of the particular dynamical realization one adopts. Hence, different dynamical models will end up with very similar spectra. Therefore, in view of our interest in the CMB, we shall focus on this mechanism and the spectrum it produces; and we shall restrict ourselves to the simplest inflationary model, that of a single massive scalar field. For a large panorama of inflationary models we refer to [16].

3.1. The inflationary background

To have a *successful* inflation requires two things. On one hand, the accelerated expansion should last long enough that today's Hubble scale d_{H0} be inside the Hubble radius at the beginning of inflation, see Fig. 2. To characterize the 'duration'

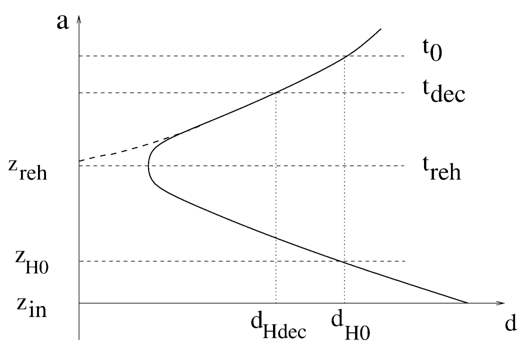

Fig. 2. The evolution of the Hubble radius d_H. A vertical line is a fixed comoving scale. The symbols d_{H0} and $d_{H\rm dec}$ designate the Hubble scale evaluated today and at the decoupling, respectively. The evolution of d_H with or without inflation splits near some high redshift $z_{\rm reh}$. The big bang branch is indicated by the dashed line whereas the inflationary branch is marked by the continuous line. $z_{\rm in}$, z_{H0}, and $z_{\rm reh}$ give the respective values of the redshift when inflation starts, when the scale d_{H0} exits $d_H(t)$, and at *reheating* when inflation stops. For inflation to be successful, one must have $d_H(t_{\rm in}) \gg d_{H0}$.

of inflation it is convenient to introduce the parameters $N_{\rm tot} = \ln(a_{\rm reh}/a_{\rm in})$ and $N_{\rm min} = \ln(a_{\rm reh}/a_{H0})$ which respectively give the (total) number of *e-folds* from the beginning of inflation to its end and the minimum number for inflation to be successful. On the other hand, even though inflation need not to be homogeneous, the patch which inflates should be sufficiently large and homogeneous that the gradients be negligible up to a scale larger than d_{H0}. If one of these conditions is not met, inflation would fail to explain the isotropy of the CMB up to d_{H0}.

To have an accelerated expansion in GR requires $\bar\rho + 3\overline{P} < 0$, see Eq. (4). To have inflation thus requires that this condition be satisfied for a long time but only in the above mentioned patch. This can be fulfilled by introducing a scalar field, called *inflaton*, which possesses an expectation value $\bar\phi$ which obeys three conditions: $\bar\phi$ must dominate all other contributions to ρ and P, it must slowly decay, and it must be homogeneous. In the patch, the inflaton can thus be written as:

$$\phi(t,x) = \bar\phi(t) + \delta\phi(t,x), \tag{7}$$

and it obeys the scalar field equation in a RW metric:

$$\partial_t^2\phi + 3H\partial_t\phi - \frac{\Delta\phi}{a^2} + m^2\phi = 0. \tag{8}$$

The local fluctuation $\delta\phi$ satisfies $\delta\phi \ll \bar\phi$ and Δ is the Laplacian. From Eq. (8) one finds that the slow decay of $\bar\phi$ follows $3H\partial_t\bar\phi \simeq -m^2\bar\phi$, i.e., $\bar\phi$ is dragged to zero at a rate given by $m^2/3H$.

We are now in position to verify that $\bar\phi$ leads to inflation when several conditions are met. The energy density and the pressure are dominated by the contribution of $\bar\phi$ given by $\bar\rho_\phi, \overline{P}_\phi = (\partial_t\bar\phi)^2/2 \pm m^2\bar\phi^2/2$. When the decay is slow enough, the kinetic term is negligible. In this case one gets $\overline{P}_\phi/\bar\rho_\phi \simeq -1$ and hence an accelerated expansion, see Eq. (4). As long as the mass term is larger than the kinetic term, $\bar\phi$ thus acts as a (slowly decaying) cosmological constant. To characterize this decay, it is appropriate to introduce the *slow-roll* parameter $\epsilon = -\partial_t H/H^2 = -\mathrm{d}\ln H/\mathrm{d}\ln a \ll 1$. Algebra then gives $m^2/3H = \epsilon H$ for the decay rate, and $\overline{P}_\phi/\bar\rho_\phi = -1 + 2\epsilon/3$ for the equation of state. One also finds that $N_{\rm tot} = 1/2\epsilon = 2\pi G\bar\phi_{\rm in}^2$, thereby relating the number of e-folds to ϵ and to the initial value of the inflaton.

It remains to make contact with (micro)physics. It is generally believed that the reheating process has something to do with Grand Unification Theories [11,14]. If this is correct, $T_{\rm reh}$, the reheating temperature, should be close to GUT scale, near 10^{14} GeV. In this case, there must be at least 70 e-folds of inflation. This condition follows from: $N_{\rm tot} > N_{\rm min} \simeq \ln(d_{H0}/d_{H\rm reh}) \simeq \ln(a_0/a_{\rm reh}) \simeq \ln(T_{\rm reh}/T_0) \simeq 70$. (The first two $\simeq$ follow from scaling laws: $d_H \propto 1/a$ during inflation and $d_H \propto a$ during the radiation era.)

Before considering fluctuations, it should be stressed that the initial value of $\bar\phi$ obeys $\bar\phi_{\rm in} > N_{\rm min}^{1/2} M_P$, where the Planck mass is $M_P = G^{1/2}$ when $c = \hbar = 1$. That is, the slow-roll conditions send us *above* the Planck scale where there is no particular reason to believe that the settings we have used make sense. More complicated inflationary models based on two fields do not suffer from this disease [16], but the question of the nature of the inflaton is still open. There are basically two attitudes. The dominant attitude [11,14] is that inflation belongs to particle physics and occurs sufficiently below the Planck scale so that gravity can be safely treated by classical GR. This option raises a very difficult question [18] related to the cosmological constant problem: If gravity can all the way be treated classically, why the mechanism which screens the vacuum energy during the radiation era did not screen as well the inflaton potential energy during inflation? One should therefore not exclude the alternative possibility that the inflaton be merely a phenomenology which aptly characterizes, as, e.g., in R^2-inflation [13,17], the background and the fluctuations in a domain wherein quantum gravity (or stringy effects) could still play an important role. As far as the mechanism giving rise to the primordial spectrum is concerned, these alternatives are equivalent.

3.2. *The primordial spectrum*

To identify this mechanism, we need to consider the combined evolution of the inflaton and the metric perturbation, $\delta\phi$ and Ψ. Assuming we can work to first order in these fluctuations, GR gives us a set of equations for their Fourier components, $\delta\phi_k$ and Ψ_k, see [3] for details. Two important points should be mentioned. First the $0i$ Einstein equation gives

$$\frac{1}{a}\partial_t(a\Psi_k) = 4\pi G\partial_t\bar\phi\,\delta\phi_k. \tag{9}$$

From this constraint equation one learns that in the absence of matter density fluctuations, Ψ_k decays like $1/a$. Hence the metric fluctuations at decoupling are sustained by matter fluctuations. From the linearization procedure, one also learns that, during inflation, the *dominant* matter fluctuations are those of the inflaton, because the background energy $\bar\rho$ is dominated by $\bar\phi$. (When considering inflationary models with several scalar fields, a particular combination of their fluctuations, named *adiabatic*, drives the metric fluctuations Ψ_k. The other fluctuations, called iso-curvature, could nevertheless play some role [19].)

The second important fact is that the modes v_k, defined by $v_k/a = \delta\phi_k + \Psi_k(\partial_t\bar\phi/H)$, are those of a canonical (and gauge invariant) field. Since all fluctuations obey linear equations it is crucial to identify which field is canonical because its *normalized* quantum mechanical fluctuations will be taken into account in the sequel. The modes v_k behave like harmonic oscillators with time dependent frequency:

$$\partial_\eta^2 v_k + (k^2 - \xi) v_k = 0, \tag{10}$$

where η is the conformal time ($\mathrm{d}\eta = \mathrm{d}t/a$) and where ξ is a function of the background solution approximatively given, in the slow-roll limit, by $2/\eta^2$. Hence, in the early past, for $\eta \to -\infty$, Eq. (10) reduces to a standard harmonic oscillator. However, when $k^2 = \xi$, which corresponds to $k \simeq 1/d_H(t)$, the mode exits the Hubble horizon and stops oscillating. One gets a growing ($\propto a$) and decaying mode ($\propto 1/a$). Thus v_k/a becomes constant. Straightforward algebra gives the following result. For large values of k, i.e., for $k \gg 1/d_H(t_{\mathrm{in}})$, and if v_k is a positive frequency mode of unit Wronskian at t_{in}, after horizon exit the *frozen* value of v_k/a is

$$\left|(v_k/a)^{\mathrm{fr}}\right|^2 = \frac{H_k^2}{2k^3} = \left|\delta\phi_k^{\mathrm{fr}}\right|^2 \bigl(1 + \mathrm{O}(\epsilon)\bigr), \tag{11}$$

where H_k is the value of $H(t)$ at horizon exit, when $k = 1/d_H(t)$.

Then Eq. (9) implies that Ψ_k also freezes out after horizon exit. The frozen value is given by $\Psi_k^{\mathrm{fr}} \simeq -\delta\phi_k^{\mathrm{fr}}/\bar{\phi}$, i.e., $|\Psi_k^{\mathrm{fr}}|^2 = 4\pi G\epsilon |\delta\phi_k^{\mathrm{fr}}|^2$. However, there is still a subtlety: Ψ_k^{fr} gets amplified when inflation stops. This follows from the existence of a conserved quantity [3] which is proportional to $\Psi_k(w + 5/3)/(w+1)$, where $w = \overline{P}/\bar{\rho}$. Hence when the equation of state changes from inflation to radiation, the amplification factor is $2/3\epsilon$.

In brief, from the combined evolution of the background variables $a, \bar{\phi}$ and the modes $\Psi_k, \delta\phi_k$, inflation tells us that in the radiation dominated era and before re-entry, the *primordial* spectrum of Ψ_k is given by

$$\left|\Psi_k^{\mathrm{prim}}\right|^2 = \frac{4}{9}\frac{4\pi G}{\epsilon_k}\left|(v_k/a)^{\mathrm{fr}}\right|^2. \tag{12}$$

We have added a subscript k to ϵ because, in general, it depends on time and hence on k through $k/a = H$. The primordial spectrum is thus given in terms of the equation of state and the frozen value of v_k/a evaluated during inflation, just after horizon exit. This is the first non-trivial outcome of inflation. From Eq. (10), the r.h.s. of Eq. (12) can be computed *if* we know the amplitude of v_k at the onset of inflation. Therefore, so far, we have only ‘postponed’ the problem of the initial conditions, from some high redshift before nucleosynthesis, to some still higher redshift before 70 e-folds of inflation, see Fig. 2.

3.3. *Initial conditions and QFT*

It thus remains to confront the question of the initial conditions of the inflaton fluctuations. This question only concerns the values of k which correspond to scales which are today visible. We shall call them the *relevant* scales. When inquiring about their initial conditions, one encounters a major surprise: *one must abandon classical settings.* The reason is the following. When a mode v_k is well inside the Hubble horizon, it behaves as a massless mode since its momentum $p(t) = k/a(t)$ is much larger than H. Hence, its energy density behaves like radiation and scales like $1/a^4$. Thus, *if* v_k were to possess a classical amplitude inside the Hubble horizon, one would reach an inconsistency, because when propagating backwards in time, its energy density would, after few e-folds, overtake the background density $\bar{\rho}_\phi$ since the latter is almost constant. But this would violate our assumption that there *is* inflation which implies that $\bar{\rho}_\phi$ *is* the dominant contribution. So, unless one fine tunes the number of *extra* e-folds, $N_{\mathrm{extra}} = N_{\mathrm{tot}} - N_{\mathrm{min}} = \ln(a_{H0}/a_{\mathrm{in}})$, the relevant modes cannot have classical amplitudes at the onset of inflation. By a similar reasoning, one shows that the curvature of 3-surfaces vanishes without fine tuning. In order words, inflation predicts that the universe is flat, i.e. $\Omega_0^{\mathrm{curv}} = 0$.

One must therefore look for a quantum mechanical origin of the primordial spectrum. (If it turns out that the *largest* structures of our universe are of quantum origin, this would constitute the triumph of quantum mechanics which was elaborated from atomic and molecular spectra.) Using the settings of quantum field theory in curved space [20], one should re-address the question of the initial conditions. It is now formulated in terms of n_k, the occupation number of the quanta of the field operator $\hat{v}$. For relevant k, one finds that n_k must vanish at the onset of inflation. Indeed, the energy density carried by these quanta ($\propto n_k k^4/a^4$) would violate, as in classical settings, the hypothesis that the energy is dominated by $\bar{\rho}_\phi$. So one is left with the vacuum as the unique possibility. It is then remarkable that the vacuum energy contribution evades a potential inconsistency which would have otherwise ruined inflation: Because of the subtraction of the zero-point energy [20], the vacuum energy does *not* grow like $1/a^4$ and stays much smaller than $\bar{\rho}_\phi$. Hence vacuum is the only initial state of relevant modes which is consistent with inflation. Notice that nothing can, nor should, be said about (irrelevant) infra-red modes with $k \ll 1/d_{H0}$ because their energy density does not diverge and because they are viewed today as part of the homogeneous background.

There is a complementary way of understanding how inflation answers the question of initial conditions. In the homogeneous inflating patch, there is an energy scale: H. Hence, the initial occupation number of quanta of (proper) frequency $\omega_{\mathrm{in}} = k/a_{\mathrm{in}}$ must obey a Wien law $n(\omega_{\mathrm{in}}) \propto \mathrm{e}^{-\omega_{\mathrm{in}}/H}$ for $\omega_{\mathrm{in}} \gg H$. Now, if N_{extra} is larger than, say 10, one finds that all relevant modes *are* in their ground state because $\omega_{\rm in} > H z_{\rm in}/z_{H0} = H e^{10}$, where z_{H0} is the redshift when d_{H0} exits the Hubble radius. Concomitantly, d_{H0} had an initial proper size equal to $H^{-1} z_{H0}/z_{in} = H^{-1} e^{-N_{\rm extra}}$. So *unless* $N_{\rm extra} < \ln(M_p/H) \simeq 13$, our universe was inside a Planck cell. To appreciate the unavoidable character of this conclusion, we invite the reader to draw the 'trajectory' of the (proper) Planck length l_P in Fig. 2.

So, in inflation, the choice of the initial state does not follow from a principle (e.g., some symmetry) but it is fixed by the kinematics of inflation itself. This stems from the blue-shift effect encountered in a backward in time propagation which sends the frequencies $\omega = k/a$ of relevant modes way above H. (It is interesting to point out the analogy between these aspects and black hole physics. Classically black holes cannot radiate and this is deeply connected to the *no hair* theorem: Stationary solutions are characterized only by mass and angular momentum because multipoles are radiated away after a few e-folds [21], where the unit of time is here given by the Schwarzschild radius. So if after some e-folds some radiation is emitted by a black hole, it must be of quantum origin, i.e., it must be Hawking radiation [22,23]. Similarly, for the relevant scales in inflationary cosmology, the geometry would classically be bald, no hair, because, when $N_{\rm extra} \gg 1$, the pre-existing classical structures have been so much diluted that today they are still part of the homogeneous background. So if some structure is found, it must be of quantum origin. Moreover, in both cases, the resulting spectrum is determined by the geometry because the initial state of relevant modes is the ground state, see, however, [24–26].)

3.4. Predictions and observational data

Having understood how inflation fixes the initial condition of fluctuations, we can bring together the various results. Since v is a canonical field, the v.e.v. of $\hat{v}_k \hat{v}^\dagger_{-k}$ is equal to $1/2k$ ($\hbar = 1$) at the onset of inflation. Then Eq. (11) gives the v.e.v. after horizon exit. (In quantum optical terms, one would say that the freezing out of the modes at horizon exit induces a parametric amplification of vacuum fluctuations which leads to extremely squeezed two-mode states [27]. When considering the modes near horizon re-entry, the *two*-mode squeezing translates into properties of *coherence* which are specific to inflation, see below.) Using Eq. (12), the primordial power spectrum $P_k^{\rm prim}$, defined by the two-point function

$$\big\langle \Psi(t, x+y)\Psi(t, y)\big\rangle = \int_0^\infty \frac{dk}{k} \frac{\sin(kx)}{kx} P_k^{\rm prim}, \tag{13}$$

is given by

$$P_k^{\rm prim} = \frac{k^3}{2\pi^2}\big|\Psi_k^{\rm prim}\big|^2 = \frac{4}{9}\frac{4\pi G}{\epsilon_k}\left(\frac{H_k}{2\pi}\right)^2. \tag{14}$$

In Eq. (13), t should be in the radiation dominated era, well before horizon re-entry. When getting close to horizon re-entry, Ψ_k starts re-oscillating.

In brief, the simplest model of inflation *predicts* that the primordial spectrum should enjoy the following properties:

- The power spectrum Eq. (14) is *nearly scale invariant*. This results from the stationarity of the process: parametric amplification of vacuum fluctuations, k after k. In fact the k-dependence of $P_k^{\rm prim}$ only originates from the slow evolution of the background at horizon exit.
- The fluctuation spectrum forms a *Gaussian ensemble* – because the vacuum is a Gaussian state, and because one starts with quantum mechanics. This means that the *probability* of finding $\Psi_k^{\rm prim}$ with a given amplitude is Gaussian, and that the l.h.s. of Eq. (13) should be interpreted as an ensemble average. (Non-Gaussianities are not expected to develop in the early universe since the rms fluctuations are 10^{-5}.)
- The modes Ψ_k are *coherent* (in time) at horizon re-entry. By coherent one means the following. When $k/a \simeq H$, each Ψ_k starts to re-oscillate. Hence the general solution is governed by two independent variables. Inflation predicts that the combination representing the decaying mode vanishes, because it has had all the time to do so.

Observational data extracted from the CMB anisotropies tell us that the spectrum does enjoy the following properties:

- The power spectrum is nearly scale invariant: It has a rms amplitude $\simeq 10^{-5}$ (which fixes $H_k \simeq 10^{-6} M_P$) and it is parametrized by a spectral index, defined by $n_k = d\ln P/d\ln k$, which obeys [5] $n_k = 0.93 \pm 0.04$ and $dn_k/d\ln k = -0.03 \pm 0.02$ at a scale which represents 0.5% of today's Hubble radius $\simeq 4000$ Mpc.
- Non-Gaussianities have not been found, and the spread of the data (the *cosmic variance*) is compatible with standard deviations given the finite number of independent observables.

- Narrow 'acoustic' peaks are observed in the temperature fluctuations spectrum. These are by-products of the coherence of the Ψ_k at horizon re-entry. Moreover, when considering the *polarization* of the CMB, one finds an anti-correlation peak which is "the distinct signature of primordial adiabatic fluctuations" [10]. This peak excludes that cosmic strings or textures could be a relevant mechanism for primordial fluctuations.

The outcome of this comparison is that *the generic predictions of the* simplest *inflationary models are in accord with observational data.*

4. Developments and perspectives

It should be clear to the reader that several aspects have not been discussed in this paper. These include polarization, galaxy spectra (to have access to the spectrum at smaller scales) as well as gravitational waves and iso-curvature modes. By taking all these into account [28], a finer understanding of the predictions of inflation will be reached and one might envisage constraining inflationary models by observational data (already available and also to come).

In order to provide some guidelines in this direction, we present the following points:

- The primordial power spectrum can be *phenomenologically* expressed in terms of several slow-roll parameters which characterize the inflaton action and/or the evolution of the background [29]. It is hoped that these parameters, and therefore the inflaton potential, shall be observationally determined.
- Primordial non-Gaussianities can be obtained when iso-curvature modes are considered [30]. Theoretical studies might orient and ease observational search.
- Inflation also predicts that gravitational waves were produced. Their spectrum is given by $P_k^{\rm grav} = 36\epsilon_k P_k^{\rm prim}$ [29]. Simultaneous observations of $P_k^{\rm grav}$ and $P_k^{\rm prim}$ would allow us to perform consistency checks since they are characterized by different spectral indices. These checks will confirm (or rule out) inflation. Notice however that the lower value of $P_k^{\rm grav}$ might delay (or even exclude) the detection of primordial gravitational waves.

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