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The tallest column problem: New first integrals and estimates

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ABSTRACT

We analyze the problem of finding the shape of the tallest column. For the system of equations that determine the optimal shape we construct a variational principle and two new first integrals. From the first integrals we are able to determine, analytically, the size of the cross-sectional area of the optimal column at the bottom, as well as the corresponding bending moment and curvature of the elastic line. Our result for critical load is compared with the results obtained by other methods.

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1. Introduction

In this work, we extend the results for the tallest elastic column in a constant gravity field. The problem of determining the shape of the lightest elastic column in a constant gravity field, or the tallest column problem, was treated in many publications, for example [1], [2], [3], [4], [8]. In [5], [6], Egorov proved the existence of the optimal design. In earlier analysis, the existence of an optimal design was assumed. Our main results in this note are a new variational principle for optimally designed column and two new first integrals for the equations, which determine the shape of the optimal column. From the first integrals we are able to find the values of the cross-sectional area, of the bending moment, and of the curvature of the elastic line of optimal column.

2. Formulation

Consider a column with an inextensible axis, positioned in a constant gravity field with built-in lower, and free upper end. Using the dimensionless variables, the system of equations describing the buckled state of the column becomes

$$(a^2\dot{\vartheta})^{\cdot} + \lambda\vartheta \int_0^t a(\xi)\,\mathrm{d}\xi = 0 \tag{1}$$

subject to

$$u(0) = 0; \qquad v(0) = 0; \qquad \lim_{t\to 0} a^2(t)\dot{\vartheta}(t) = 0; \qquad \vartheta(1) = 0 \tag{2}$$

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Here a denotes the cross-section, λ is the load parameter, ϑ is the angle between the column axis and the x-axis, and $\mathrm{d}(\cdot)/\mathrm{d}t = \dot{(\cdot)}$, see [7]. The volume of the column is

$$w = \int_0^1 a(t)\,\mathrm{d}t \tag{3}$$

The solution $\vartheta_0 = 0$ is a trivial one, valid for all λ. For loss of stability, it the necessary that there is a nontrivial solution to (1), (2). The *tallest column problem* is stated as: find $a(t) \geq 0$, which satisfies $\int_0^1 a(t)\,\mathrm{d}t = 1$, and such that the lowest λ in (1) has the highest value. For this problem, we have the following result.

Proposition 2.1. *Egorov [5], [6]. There exists a unique solution λ to the tallest column problem with $a \in C([0,1])$ and $a \geq 0, x \in (0,1)$, and $\vartheta \in C^1([0,1])$ $\vartheta \geq 0, t \in (0,1)$.*

The result of Egorov is the first rigorous proof of the existence and uniqueness of the optimal column. Concerning the existence of λ, see also [3]. We state this as, see [7].

Proposition 2.2. *[7]. For the lowest eigenvalue λ of (1), (2), the behaviour of a, b, and ϑ near $t = 0$ is*

$$a(t) \approx \frac{\lambda}{24} t^3, \quad b(t) \approx \frac{\lambda}{96} t^4, \quad \vartheta(t) \approx \frac{1}{t^2} \tag{4}$$

3. Optimal design *a*

Let $b(t) = \int_0^t a(\xi)\,\mathrm{d}\xi$ and let

$$\vartheta = x_1; \qquad m = a^2 \dot{\vartheta} = x_2; \qquad x_3 = b(t) \tag{5}$$

so that (1), (2) become

$$\dot{x}_1 = \frac{x_2}{a^2}; \qquad \dot{x}_2 = -\lambda x_1 x_3; \qquad \dot{x}_3 = a \tag{6}$$

and

$$x_1(1) = 0; \qquad x_2(0) = 0; \qquad x_3(0) = 0 \tag{7}$$

The *tallest column problem* states as follows: given λ determined as the lowest eigenvalue of the problem (1), (2), let us determine the control $a^*(t)$ that belongs to the admissible set of controls U such that

$$\min_{a \in U} I = \min_{a \in U} \int_0^1 a(t)\,\mathrm{d}t = \int_0^1 a^*(t)\,\mathrm{d}t \tag{8}$$

if the system is subjected to differential constraints (6), (7). Also, from constraint (3), we conclude that $\min_{a \in U} I = 1$. For U we take a set of continuously differentiable nonnegative functions defined on the interval $[0,1]$. By using the standard procedure of Optimal Control theory [9], the Pontryagin's function $\mathcal{H}$ is defined as

$$\mathcal{H} = a + p_1 \frac{x_2}{a^2} + p_2(-\lambda x_1 x_3) + p_3 a \tag{9}$$

where

$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = p_2 \lambda x_3, \qquad \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = -\frac{p_2}{a^2}, \qquad \dot{p}_3 = -\frac{\partial \mathcal{H}}{\partial x_3} = \lambda x_1 p_2 \tag{10}$$

subject to

$$p_1(0) = 0, \qquad p_2(1) = 0, \qquad p_3(1) = 0 \tag{11}$$

From $\min_{a \in U} \mathcal{H}$, we get $a = \left(\frac{2 p_1 x_2}{1 + p_3}\right)^{1/3}$. Since $p_1 = x_2$, $p_2 = -x_1$, we obtain:

$$a = \left(\frac{2 x_2^2}{1 + p_3}\right)^{1/3} \tag{12}$$

Further, $\frac{\partial^2 \mathcal{H}}{\partial a^2} = 6\frac{x_2^2}{a^4} \geq 0$; thus, $\mathcal{H}$ is minimum. We solve (10), (11) for p_3 to obtain:

$$p_3 = \lambda \int_t^1 x_1^2(\xi)\, \mathrm{d}\xi \tag{13}$$

so that

$$1 + \lambda \int_t^1 \vartheta^2(\xi)\, \mathrm{d}\xi = 2a\dot{\vartheta}^2 \tag{14}$$

Differentiating (14), it follows:

$$(a\dot{\vartheta}^2)^{\cdot} - \frac{\lambda}{2}\vartheta^2 = 0 \tag{15}$$

Summing up the above results, we conclude that the shape of the optimal column a may be determined (1) and (15),

$$(a^2(t)\,\dot{\vartheta}(t))^{\cdot} + \lambda\vartheta(t)\,b(t) = 0, \qquad (a(t)\,\dot{\vartheta}^2(t))^{\cdot} + \frac{\lambda}{2}\vartheta^2(t) = 0 \tag{16}$$

subject to

$$\lim_{t\to 0} a^2(t)\dot{\vartheta}(t) = 0; \qquad \vartheta(1) = 0 \tag{17}$$

and we determine a and λ. The results (16), (17) agree with the optimality conditions obtained in [6], [8], [2] and [7]. From (14), (12), and (7) it follows that

$$a(1)\dot{\vartheta}^2(1) = \frac{1}{2}; \qquad a(0) = 0 \tag{18}$$

4. First integrals and estimates of the solution to (16) and (17)

Since $\dot{b}(t) = a(t)$, the system (16) becomes

$$(\dot{b}^2\dot{\vartheta})^{\cdot} + \lambda\vartheta b = 0; \qquad (\dot{b}\dot{\vartheta}^2)^{\cdot} + \frac{\lambda}{2}\vartheta^2 = 0 \tag{19}$$

subject to

$$\lim_{t\to 0} \dot{b}^2(t)\dot{\vartheta}(t) = 0; \qquad \vartheta(1) = 0; \qquad w^* = b(1) = 1 \tag{20}$$

At the boundary conditions (20), we took $w^* = 1$ to recover the tallest column problem. The system (19), (20) is analyzed in [7]. Our main result states as follows.

Theorem 4.1. *The solution (b, ϑ) to the system (19), (20) has the following properties. For the functional*

$$J(W, \Theta) = \frac{1}{2}\int_0^1 \left(\dot{W}^2\dot{\Theta}^2 - \lambda\Theta^2 W\right) \mathrm{d}t \tag{21}$$

defined for $k = (W, \Theta)$

$$\mathcal{K} = \left\{k : k = (W, \Theta),\ W(0) = 0;\ W(1) = 1;\ \lim_{t\to 0} \dot{W}^2(t)\dot{\Theta}(t) = 0;\ \ \Theta(1) = 0\right\}$$

we have:

i) the functions (b, ϑ) give a stationary value to the functional (21), i.e.

$$\delta J(b, \vartheta) = 0 \tag{22}$$

ii) the value of the functional (21) on the solution to (19), (20) is zero,

$$J(b, \vartheta) = \frac{1}{2}\int_0^1 \left(\dot{b}^2\dot{\vartheta}^2 - \lambda\vartheta^2 b\right) \mathrm{d}t = 0 \tag{23}$$

iii) there exist two first integrals of the system (19), (20)

$$\frac{3}{2}\dot{b}^2\dot{\vartheta}^2+\frac{1}{2}\lambda\vartheta^2 b=2; \quad -5\vartheta\dot{b}^2\dot{\vartheta}+8\dot{b}\dot{\vartheta}^2 b=4t \tag{24}$$

iv) the values of the dependent variables at the bottom of the column are

$$\dot{b}(1)=a(1)=\frac{8}{3};\ \ \dot{\vartheta}(1)=\frac{\sqrt{3}}{4};\ \ a(1)^2\dot{\vartheta}(1)=m(1)=\frac{16}{3\sqrt{3}} \tag{25}$$

Proof. The Lagrangian $\mathcal{L}$ of the functional (21), is $\mathcal{L}=\frac{1}{2}\left(\dot{W}^2\dot{\Theta}^2-\lambda\Theta^2 W\right)$ so that $\delta J(b,\vartheta)=0$ that is i) holds. Next, we multiply $(19)_1$ by ϑ and integrate to obtain $\int_0^1\left(-\dot{b}^2\dot{\vartheta}^2+\lambda\vartheta^2 b\right)\,\mathrm{d}t=0$, which is ii). Since the Lagrangian $\mathcal{L}$ is not explicitly dependent on t, we have [9]:

$$\mathcal{H}=\frac{\partial\mathcal{L}}{\partial\dot{b}}\dot{b}+\frac{\partial\mathcal{L}}{\partial\dot{\vartheta}}\dot{\vartheta}-\mathcal{L}=\frac{3}{2}\dot{b}^2\dot{\vartheta}^2+\frac{\lambda}{2}\vartheta^2 b=const \tag{26}$$

so that, with $\vartheta(1)=0$, we have

$$\frac{3}{2}\dot{b}^2\dot{\vartheta}^2+\frac{\lambda}{2}\vartheta^2 z=\frac{3}{2}\dot{b}^2(1)\dot{\vartheta}^2(1) \tag{27}$$

By multiplying $(19)_1$ by ϑ and $(19)_2$ by $-2b$ and by adding the result, we get

$$[\vartheta(\dot{b}^2\vartheta)]^{\cdot}-\dot{b}^2\dot{\vartheta}^2-2[b(\dot{b}\dot{\vartheta}^2)]^{\cdot}+2\dot{b}^2\dot{\vartheta}^2=0 \tag{28}$$

Integration of (28) and the use of boundary conditions leads to

$$2\dot{b}(1)\dot{\vartheta}^2(1)=\int_0^1\dot{b}^2(t)\,\dot{\vartheta}^2(t)\,\mathrm{d}t$$

From $(18)_1$ it follows

$$\dot{b}(1)\dot{\vartheta}^2(1)=1/2$$

Therefore, (28) becomes

$$\int_0^1\dot{b}^2(t)\,\dot{\vartheta}^2(t)\,\mathrm{d}t=1 \tag{29}$$

Rewriting (27) as

$$3\dot{b}^2\dot{\vartheta}^2=3\dot{b}^2(1)\dot{\vartheta}^2(1)-\vartheta^2 b \tag{30}$$

integrating and using (23) and (29), we obtain:

$$\dot{b}^2(1)\dot{\vartheta}^2(1)=\frac{4}{3} \tag{31}$$

Combining (31) and (27), we get $(24)_1$.

To obtain the second first integral, we multiply $(19)_1$ by ϑ and $(19)_2$ by b and obtain:

$$(\dot{b}^2\dot{\vartheta}\vartheta)^{\cdot}=\dot{b}^2\dot{\vartheta}^2-\lambda\vartheta^2 b; \qquad (\dot{b}\dot{\vartheta}^2 b)^{\cdot}=\dot{b}^2\dot{\vartheta}^2-\frac{\lambda}{2}\vartheta^2 b \tag{32}$$

Using $(24)_1$ to determine $\dot{b}^2\dot{\vartheta}^2$ and by substituting the resulting expression in (32), it follows:

$$(\dot{b}^2\dot{\vartheta}\vartheta)^{\cdot}=\frac{4}{3}-\frac{4}{3}\lambda\vartheta^2 b, \qquad (\dot{b}\dot{\vartheta}^2 b)^{\cdot}=\frac{4}{3}-\frac{5}{6}\lambda\vartheta^2 b \tag{33}$$

In (33), we first eliminate the term $\lambda\vartheta^2 b$. Next, by integration, we have:

$$-5\dot{b}^2\dot{\vartheta}\vartheta+8\dot{b}\dot{\vartheta}^2 b=4t+D \tag{34}$$

where D is a constant. When (34) is evaluated at $t=0$, we get $D=0$ and obtain $(24)_2$.

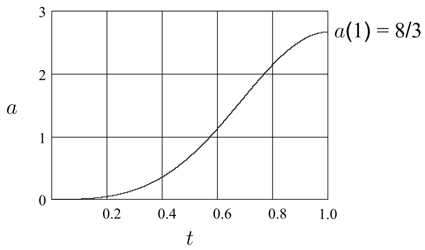

Fig. 1. Cross-sectional area of the optimally shaped column.

We derive now new estimates from the first integrals. From $(18)_1$ and $(24)_1$, we get:

$$a(1) = \frac{8}{3} \tag{35}$$

so that, with $m(1) = a^2(1)\,\dot{\vartheta}(1)$, we obtain:

$$\dot{\vartheta}(1) = \frac{\sqrt{3}}{4}; \quad m(1) = \frac{16}{3\sqrt{3}} \tag{36}$$

This completes the proof.

5. Numerical solution

The first integral $(24)_2$ may be written as $-5m\vartheta + 8\frac{m^2}{a^3}b = 4t$. This expression, together with $a^3 = \frac{2m^2}{1+\lambda p}$, where we used $x_4 = p$, leads to

$$\vartheta = \frac{4\left[b\left(1+\lambda p\right) - t\right]}{5m} \tag{37}$$

From (5) and (37), we obtain:

$$\begin{aligned} \dot{m} &= \lambda b \frac{4\left[b\left(1+\lambda p\right) - t\right]}{5m}; \qquad \dot{b} = \left[\frac{2m^2}{1+p}\right]^{1/3} \\ \dot{p} &= -\lambda \left\{\frac{4\left[b\left(1+\lambda p\right) - t\right]}{5m}\right\}^2 \end{aligned} \tag{38}$$

with

$$m(0) = 0; \qquad m(1) = \frac{16}{3\sqrt{3}}; \qquad b(0) = 0; \qquad b(1) = 1 \qquad p(1) = 0 \tag{39}$$

Also

$$\frac{3}{2}\frac{m^2}{\dot{b}^2} + \frac{\lambda}{2} b \left\{\frac{4\left[b\left(1+\lambda p\right) - t\right]}{5m}\right\}^2 = 2 \tag{40}$$

Note that the system (38), (39) is easy to solve since the function ϑ is eliminated. The problem that the variable ϑ introduced into the numerical scheme was that it is unbounded at $t = 0$, or $u = 1$, see (4), and yet it is specified at both ends of the column. After integration of (38), (39) the optimal cross-sectional area is determined from $a = \left[\frac{2m^2}{1+p}\right]^{1/3}$. We solved (38), (39) numerically with λ as a free parameter. The parameter $\lambda = 134.1935084471$ was chosen such that $b(1) = 0$. The value $m(1) = 0$ is satisfied with the error of the order 10^{-7}. The optimal cross-sectional area is shown in Fig. 1.

6. Conclusion

In this work, we treated the tallest column problem as an optimization problem. We derived the known equations, in the form given by [7]. Our main results are as follows.

- For the system of differential equations determining the cross-section of the tallest column, we formulated a variational principle given by (21) and two new first integrals in the form (24).

- From the first integrals, we determined physically important values of the cross-section. The curvature, and the moment at the bottom of the column

$$a(1) = \frac{8}{3}; \quad \dot{\vartheta}(1) = \frac{\sqrt{3}}{4}; \quad m(1) = \frac{16}{3\sqrt{3}}$$

- We solved the reduced system of equations and determined a new value of the critical load parameter

$$\lambda = 134{,}193\,508\,447\,1$$

  In [8], the value $\lambda_{K-N} = 134.19$, while in [7] the value $\lambda_{F-N} = 134.1944$ was determined.
- Our numerical solution agrees well with the asymptotic result obtained in [7].

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