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A screw dislocation near a non-parabolic open inhomogeneity with internal uniform stresses

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A B S T R A C T

We use conformal mapping techniques and analytic continuation to prove that the stress field inside a non-parabolic open inhomogeneity embedded in a matrix subjected to uniform remote anti-plane stresses can nevertheless remain uniform despite the presence of a screw dislocation in its vicinity. Furthermore, the internal uniform stresses inside the inhomogeneity are found to be independent of both the shape of the inhomogeneity and the presence of the screw dislocation. On the other hand, we find that the existence of the nearby screw dislocation exerts a significant influence on the non-parabolic shape of the inhomogeneity.

1. Introduction

It has been established that the internal stresses inside a non-elliptical inhomogeneity embedded in a matrix subjected to uniform remote anti-plane shear stresses can nevertheless remain uniform despite the additional influence of a screw dislocation in the vicinity of the inhomogeneity [1–4]. In all of the aforementioned analyses, the inhomogeneity is bounded by a closed curvilinear interface and referred to as 'non-elliptical' to signify that the action of the nearby screw dislocation causes a departure of the shape of the inhomogeneity away from the classical ellipse. Very recently, Wang and Schiavone [5] demonstrated the remarkable result that the stresses inside a parabolic inhomogeneity with an open interface are unconditionally uniform when the surrounding matrix is subjected to uniform anti-plane and in-plane stresses at infinity. It is of interest to note that the uniformity of stresses inside a parabolic inhomogeneity in the case of anti-plane elasticity is consistent with the observation that the flow within a parabolic inclusion is constant under a uniform incident flow [6,7].

In this paper, we investigate the uniformity of stresses inside a non-parabolic open inhomogeneity when the matrix is subjected not only to uniform remote anti-plane stresses but also to the action of a screw dislocation in the vicinity of the inhomogeneity. The term "non-parabolic open inhomogeneity" is adopted here to indicate that the open shape of the inhomogeneity is a departure from that of a conventional parabola due to the action of the nearby screw dislocation. Using the techniques of conformal mapping and analytic continuation [8,9], we show that the internal stresses remain uniform if the non-parabolic shape of the inhomogeneity and the location of the screw dislocation are chosen judiciously. In contrast to the case of a non-elliptical inhomogeneity interacting with a screw dislocation [1–4], the uniform stresses inside the

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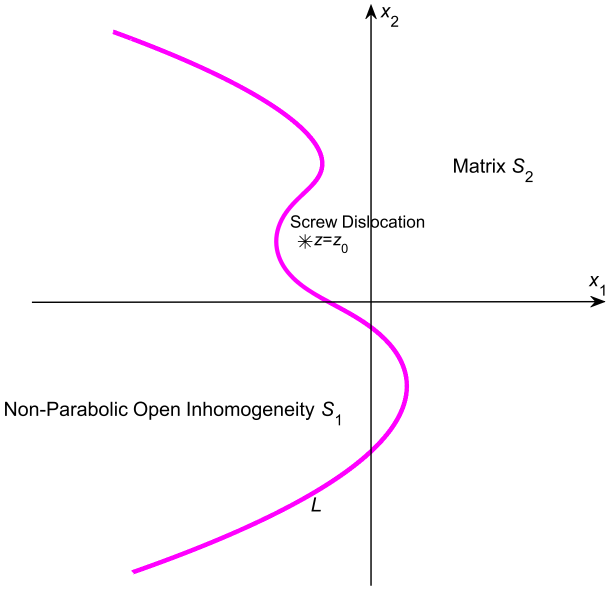

Fig. 1. A screw dislocation interacting with a non-parabolic open inhomogeneity with internal uniform stresses when the matrix is subjected to uniform remote anti-plane stresses.

non-parabolic open inhomogeneity are independent of the specific shape of the inhomogeneity. Detailed numerical results are presented to demonstrate the theory. As a final point, we mention that the practical importance of the uniformity property lies in the fact that a uniform interior stress distribution within the non-parabolic inhomogeneity is optimal since it eliminates the possibility of stress peaks which are well-known to be responsible for the failure of the inhomogeneity [10].

2. Complex variable formulation for anti-plane elasticity

We first establish a Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$. Under anti-plane shear deformations of an isotropic elastic material, the two shear stress components σ_{31} and σ_{32}, the out-of-plane displacement w and the stress function ϕ can be expressed in terms of a single analytic function $f(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [11]

$$\sigma_{32} + \mathrm{i}\sigma_{31} = \mu f'(z), \qquad \phi + \mathrm{i}\mu w = \mu f(z) \tag{1}$$

where μ is the shear modulus, and the two stress components can be expressed in terms of the stress function as [11]:

$$\sigma_{32} = \phi_{,1}, \qquad \sigma_{31} = -\phi_{,2} \tag{2}$$

3. Uniform stresses inside a non-parabolic inhomogeneity

As shown in Fig. 1, we consider a screw dislocation with Burgers vector b located at $z = z_0$ in the vicinity of a non-parabolic open inhomogeneity (denoted by S_1) which is perfectly bonded to the surrounding matrix (denoted by S_2) through the interface L when the matrix is additionally subjected to uniform remote anti-plane stresses $(\sigma_{31}^{\infty}, \sigma_{32}^{\infty})$. Our objective is to investigate whether the internal stresses inside the inhomogeneity can still remain uniform through a judicious design of the non-parabolic open shape of the inhomogeneity and the location of the screw dislocation. In what follows, the subscripts 1 and 2 are used to identify the respective quantities in S_1 and S_2.

The boundary value problem for the two-phase composite has the following form:

$$\begin{aligned} &f_2(z) + \overline{f_2(z)} = \Gamma f_1(z) + \Gamma \overline{f_1(z)} \\ &f_2(z) - \overline{f_2(z)} = f_1(z) - \overline{f_1(z)}, \quad z \in L \end{aligned} \tag{3a}$$

$$f_2(z) \cong \frac{b}{2\pi} \ln(z - z_0) + O(1), \quad z \to z_0 \tag{3b}$$

$$f_2(z) \cong \frac{\sigma_{32}^{\infty} + \mathrm{i}\sigma_{31}^{\infty}}{\mu_2} z + O\left(z^{\frac{1}{2}}\right), \quad |z| \to \infty \tag{3c}$$

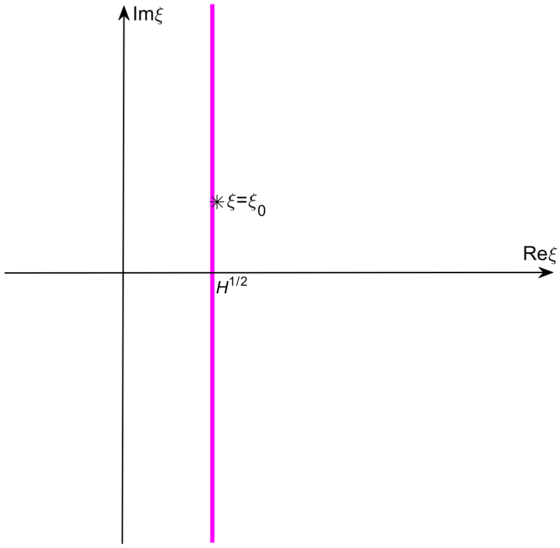

Fig. 2. The problem in the ξ-plane.

where $\Gamma = \mu_1/\mu_2$. Equation (3a) describes the continuity of traction and displacement across the perfect interface L; Eq. (3b) characterizes the logarithmic singular behavior of $f_2(z)$ at the location of the screw dislocation; Eq. (3c) specifies the asymptotic behavior of $f_2(z)$ at infinity due to the remote anti-plane loading.

We introduce the following conformal mapping function for the matrix region

$$z = \omega(\xi) = \xi^2 + p\ln\left(\xi + \bar{\xi}_0 - 2H^{\frac{1}{2}}\right), \quad \xi = \omega^{-1}(z),\ H > 0,\ \mathrm{Re}\{\xi\} \ge H^{\frac{1}{2}} \tag{4}$$

where $\xi_0 = \omega^{-1}(z_0)$ or conversely $z_0 = \omega(\xi_0)$, and p is a complex constant to be determined. The branch cut for the logarithmic function appearing in Eq. (4) is taken as a semi-infinite line connecting $\xi = 2H^{\frac{1}{2}} - \bar{\xi}_0$ and $\xi = \infty$ within the left half-plane $\mathrm{Re}\{\xi\} < H^{\frac{1}{2}}$. Thus, the mapping function in Eq. (4) is single-valued in $\mathrm{Re}\{\xi\} \ge H^{\frac{1}{2}}$. Indeed, using the mapping function in Eq. (4), the matrix S_2 is mapped onto the right half-plane $\mathrm{Re}\{\xi\} \ge H^{\frac{1}{2}}$ in the ξ-plane; the inhomogeneity-matrix interface L is mapped onto the vertical straight line: $\mathrm{Re}\{\xi\} = H^{\frac{1}{2}}, -\infty < \mathrm{Im}\{\xi\} < +\infty$ in the ξ-plane (see Fig. 2) and the screw dislocation located at $z = z_0$ is mapped onto the point $\xi = \xi_0$ (see Fig. 2). The information characterizing the screw dislocation has been embedded in the construction of the mapping function in Eq. (4). When $p = 0$ or when the screw dislocation is far from the interface, the interface L becomes a parabola described by

$$L:\ x_1 = H - \frac{x_2^2}{4H} \tag{5}$$

Thus when $p \neq 0$ and when the screw dislocation is in the vicinity of the inhomogeneity, the mapping function in Eq. (4) with $\mathrm{Re}\{\xi\} = H^{\frac{1}{2}}$, $-\infty < \mathrm{Im}\{\xi\} < +\infty$ describes a non-parabolic open interface. There are in total three parameters ξ_0, p and H in Eq. (4). However, through normalization, we find that $\xi_0 H^{-\frac{1}{2}}$ and p/H can be treated as two dimensionless non-trivial parameters describing the non-parabolic shape of the inhomogeneity.

In order to ensure that the mapping function in Eq. (4) is one-to-one for the matrix region, it is necessary that

$$\omega'(\xi) \neq 0 \quad \text{for } \mathrm{Re}\{\xi\} > H^{\frac{1}{2}} \tag{6}$$

which is equivalent to the following condition

$$\mathrm{Re}\{\xi_0\} - \mathrm{Re}\left\{\sqrt{\left(\xi_0 - 2H^{\frac{1}{2}}\right)^2 - 2\bar{p}}\right\} \ge 0 \tag{7}$$

The stresses inside the non-parabolic inhomogeneity are uniform if

$$f_1(z) = kz, \quad z \in S_1 \tag{8}$$

where k is a complex constant to be determined.

By enforcing the continuity condition across the perfect interface L in Eq. (3a) using Eq. (8) and with the aid of analytic continuation [8,9], we arrive at

$$f_2(\xi) = f_2\left(\omega(\xi)\right) = \frac{\Gamma+1}{2}k\omega(\xi) + \frac{\Gamma-1}{2}\bar{k}\bar{\omega}\left(2H^{\frac{1}{2}} - \xi\right), \quad \mathrm{Re}\{\xi\} \ge H^{\frac{1}{2}} \tag{9}$$

which can be written more explicitly in the following form

$$f_2(\xi) = \frac{k(\Gamma+1)+\bar{k}(\Gamma-1)}{2}\xi^2 - 2H^{\frac{1}{2}}\bar{k}(\Gamma-1)\xi + \frac{\bar{k}\bar{p}(\Gamma-1)}{2}\ln(\xi-\xi_0) + \frac{kp(\Gamma+1)}{2}\ln\left(\xi+\bar{\xi}_0-2H^{\frac{1}{2}}\right), \quad \mathrm{Re}\{\xi\} \geq H^{\frac{1}{2}} \tag{10}$$

Using Eq. (10) to satisfy the logarithmic singular behavior at the location of the screw dislocation in Eq. (3b) and the remote asymptotic behavior in Eq. (3c), we obtain the following relationships:

$$\frac{\bar{k}\bar{p}(\Gamma-1)}{2} = \frac{b}{2\pi}, \qquad \frac{k(\Gamma+1)+\bar{k}(\Gamma-1)}{2} = \frac{\sigma_{32}^{\infty}+\mathrm{i}\sigma_{31}^{\infty}}{\mu_2} \tag{11}$$

through which the two complex constants k and p can be uniquely determined as

$$k = \frac{\sigma_{32}^{\infty}+\mathrm{i}\Gamma\sigma_{31}^{\infty}}{\mu_1}, \qquad p = \frac{\mu_1 b}{\pi(\Gamma-1)(\sigma_{32}^{\infty}+\mathrm{i}\Gamma\sigma_{31}^{\infty})} \tag{12}$$

which clearly indicates that the constant p has the dimension of length. It is seen from Eq. (12) that the constant p is real valued only when $\sigma_{32}^{\infty} \neq 0$ and $\sigma_{31}^{\infty} = 0$; it is purely imaginary only when $\sigma_{31}^{\infty} \neq 0$ and $\sigma_{32}^{\infty} = 0$.

Thus, the internal uniform stress field inside the non-parabolic open inhomogeneity is explicitly given by

$$\sigma_{31} = \Gamma\sigma_{31}^{\infty}, \qquad \sigma_{32} = \sigma_{32}^{\infty}, \quad z \in S_1 \tag{13}$$

It can be seen quite clearly from the above that the internal uniform stress field is independent of the non-parabolic shape of the inhomogeneity and is also independent of the existence of the screw dislocation. However, the existence of the nearby screw dislocation exerts a significant influence on the non-parabolic open shape of the inhomogeneity described by the mapping function in Eq. (4) via the two complex parameters ξ_0 and p.

In addition, the stresses are non-uniformly distributed in the surrounding matrix as follows

$$\sigma_{32}+\mathrm{i}\sigma_{31} = \sigma_{32}^{\infty}+\mathrm{i}\sigma_{31}^{\infty} + \frac{(\sigma_{32}^{\infty}-\mathrm{i}\Gamma\sigma_{31}^{\infty})(1-\Gamma)\left[2H^{\frac{1}{2}} - \frac{\bar{p}}{2(\xi-\xi_0)} + \frac{p}{2(\xi+\bar{\xi}_0-2H^{\frac{1}{2}})}\right]}{\Gamma\left(2\xi + \frac{p}{\xi+\bar{\xi}_0-2H^{\frac{1}{2}}}\right)}, \quad \mathrm{Re}\{\xi\} \geq H^{\frac{1}{2}} \tag{14}$$

The prescribed uniform remote stresses in the matrix can be observed quite clearly from the above expression. In fact, the non-parabolic shape of the inhomogeneity and the location of the screw dislocation in Fig. 1 are obtained from Eq. (4) by choosing $\xi_0 = H^{\frac{1}{2}}(1.05+0.8\mathrm{i})$ and $p = H$. In this case, the mapping function for the matrix in Eq. (4) is indeed one-to-one by considering the fact that

$$\mathrm{Re}\{\xi_0\} - \mathrm{Re}\left\{\sqrt{\left(\xi_0-2H^{\frac{1}{2}}\right)^2 - 2\bar{p}}\right\} = 0.5157H^{\frac{1}{2}} > 0 \tag{15}$$

More non-parabolic shapes of the inhomogeneity can be found in Figs. 3 and 4. The internal stress field inside the inhomogeneity shown in Fig. 3 with real p is $\sigma_{31} = 0$, $\sigma_{32} = \sigma_{32}^{\infty}$; whilst that in Fig. 4 with purely imaginary p is $\sigma_{31} = \Gamma\sigma_{31}^{\infty}$, $\sigma_{32} = 0$. There is a sharp corner on the interface when choosing: (i) $\xi_0 = H^{\frac{1}{2}}(1.05+0.8\mathrm{i})$, $p = -0.15805H$ in the left and upper subplot of Fig. 3; (ii) $\xi_0 = 1.05H^{\frac{1}{2}}$, $p = -0.1H$ in the left and lower subplot of Fig. 3; (iii) $\xi_0 = 1.05H^{\frac{1}{2}}$, $p = 0.47H\mathrm{i}$ in Fig. 4. In all of these plots with a sharp corner on the interface, condition (7) is met in the case of equality. It can be easily verified from Eqs. (1) and (9) that the stresses in the matrix are bounded at these sharp corners.

When both ξ_0 and p are real valued, it is deduced quite simply from Eq. (7) that p and ξ_0 should satisfy the following restriction in order to ensure that the mapping in Eq. (4) is one-to-one in the matrix region

$$\xi_0 = \bar{\xi}_0 > H^{\frac{1}{2}}, \qquad p = \bar{p} \geq 2\left(H - H^{\frac{1}{2}}\xi_0\right) \tag{16}$$

In addition, when $p = \bar{p} = 2(H - H^{\frac{1}{2}}\xi_0)$ in Eq. (16), there is a sharp corner on the interface L. When the parameter ξ_0 is real valued but the parameter p is purely imaginary, the permissible range of $\mathrm{Im}\{p\}$ for different real values of $\xi_0 (> H^{\frac{1}{2}})$ is illustrated in Fig. 5. $\mathrm{Im}\{p\}$ and ξ_0 should lie within the region enclosed by the curve shown in Fig. 5. When $\mathrm{Im}\{p\}$ is chosen just as the value on the curve in Fig. 5, the interface L will possess a sharp corner. In addition, the permissible range of $\mathrm{Im}\{p\}$ increases as ξ_0 increases, as shown in Fig. 5.

The present method can be extended in a straightforward manner to the more general scenario in which there is an arbitrary number of screw dislocations in the matrix.

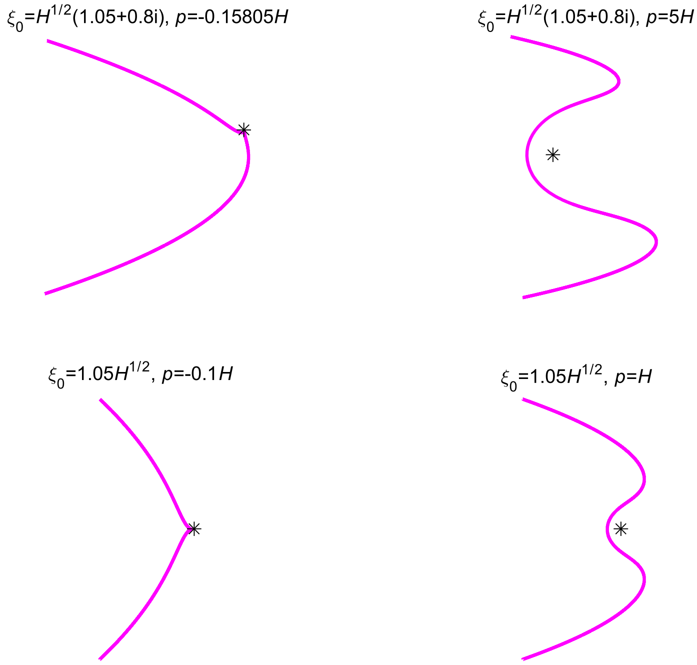

Fig. 3. The non-parabolic shape of the inhomogeneity and the location of the screw dislocation for different values of ξ_0 and $p=\bar{p}$. The star in each subplot indicates the location of the screw dislocation.

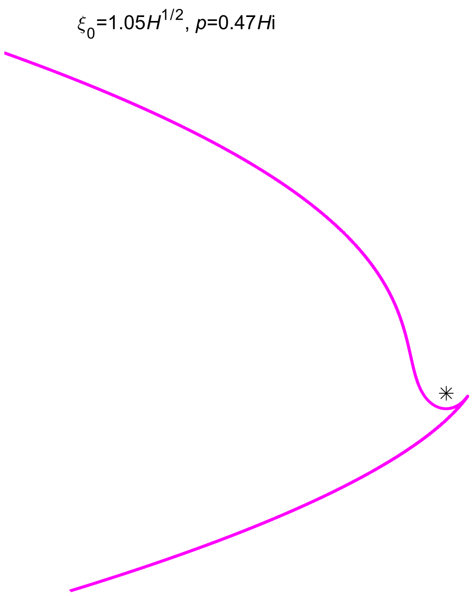

Fig. 4. The non-parabolic shape of the inhomogeneity and the location of the screw dislocation with $\xi_0=1.05H^{\frac{1}{2}}$ and $p=0.47H\mathrm{i}$. The star indicates the location of the screw dislocation.

4. Conclusions

We have demonstrated for the first time that the internal stresses inside a non-parabolic open inhomogeneity can indeed be maintained uniform when the matrix is simultaneously subjected to the action of a screw dislocation and uniform remote anti-plane stresses. Through the introduction of the conformal mapping function in Eq. (4), the internal uniform stress field in the inhomogeneity is quite simply determined by Eq. (13) and the exterior non-uniform stress distribution in the matrix is obtained in Eq. (14).

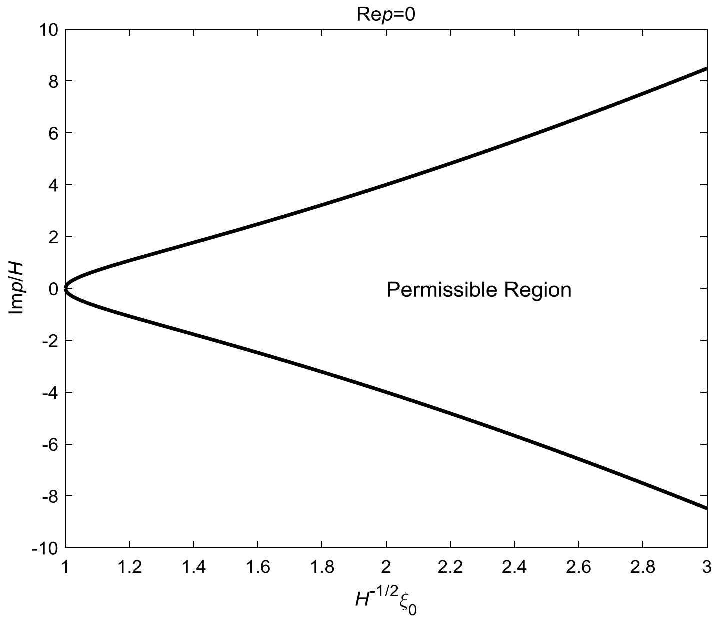

Fig. 5. The permissible range of a purely imaginary parameter p for different real values of $\xi_0(> H^{\frac{1}{2}})$.

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