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A screw dislocation near a non-elliptical piezoelectric inhomogeneity with internal uniform electroelastic field

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ABSTRACT

Conformal mapping and analytic continuation are employed to prove the existence of an internal uniform electroelastic field inside a non-elliptical piezoelectric inhomogeneity interacting with a screw dislocation. We focus specifically on the case when the piezoelectric matrix surrounding the inhomogeneity is subjected to uniform remote anti-plane mechanical and in-plane electrical loading and a constraint is imposed between the remote loading and the screw dislocation. The constraint can be expressed in a relatively simple decoupled form by utilizing orthogonality relationships between two corresponding eigenvectors. The internal uniform electroelastic field is found to be independent of the presence of the screw dislocation; moreover, it can be expressed in decoupled form.

1. Introduction

The uniformity of stresses and strains inside elastic inhomogeneities has been discussed extensively in the literature by numerous investigators (see, for example, [1–14]). In particular, it was shown recently that the stress and strain fields inside elastic inhomogeneities can remain uniform despite the presence of nearby screw dislocations [15–17]. The electromechanical coupling phenomenon of piezoelectric materials has led to their widespread use as sensors and actuators in smart materials and structures. The study of piezoelectric solids with various kinds of defects is extensive, for example, we cite the works of Pak [18,19], Suo et al. [20], Ting [8], Lee et al. [21] and Wang & Schiavone [22].

In this paper, we show that, in the case of anti-plane shear deformations of a piezoelectric composite, the electroelastic field (stress, strain, electric displacement and electric field) inside a non-elliptical piezoelectric inhomogeneity interacting with a nearby screw dislocation in the matrix subjected to uniform remote anti-plane mechanical and in-plane electric loading can remain uniform subject to a particular constraint. For given material and geometric parameters, this constraint can be considered as two separate conditions on the remote loading and the screw dislocation after employing orthogonal relations for the two associated eigenvectors. The screw dislocation itself suffers jumps in anti-plane displacement and in electric potential across the slip plane, and is also subjected to an anti-plane line force and a line charge at the core. During the solution procedure, two either real or complex conjugate mismatch parameters, the magnitudes of which are smaller than unity, are introduced for the two-phase piezoelectric composite. The internal uniform electroelastic field, which

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is found to be independent of the presence of the screw dislocation, can be described in terms of the two mismatch parameters and a single geometric parameter.

2. Complex variable formulation

In the case of anti-plane shear deformations of a hexagonal piezoelectric material exhibiting 6 mm symmetry with its poling direction along the x_3-axis of a Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$, the general solution is given by [23]

$$\begin{bmatrix} u_3 \\ \phi \end{bmatrix} = \mathrm{Im}\{\mathbf{f}(z)\}, \tag{1}$$

$$\begin{bmatrix} 2\varepsilon_{32} + 2\mathrm{i}\varepsilon_{31} \\ -E_2 - \mathrm{i}E_1 \end{bmatrix} = \mathbf{f}'(z), \qquad \begin{bmatrix} \sigma_{32} + \mathrm{i}\sigma_{31} \\ D_2 + \mathrm{i}D_1 \end{bmatrix} = \mathbf{C}\mathbf{f}'(z), \tag{2}$$

where, respectively, u_3 and ϕ are the anti-plane displacement and electric potential; σ_{31} and σ_{32} are the Cartesian anti-plane shear stresses; D_1 and D_2 are the in-plane electric displacements; E_1 and E_2 are the in-plane electric fields; ε_{31} and ε_{32} are strain components; $\mathbf{f}(z)$ is a 2D analytic vector function of the complex variable $z = x_1 + \mathrm{i}x_2$; the 2×2 dimensionless real symmetric matrix $\mathbf{C}$ is defined by

$$\mathbf{C} = \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} C_{44} & e_{15} \\ e_{15} & -\epsilon_{11} \end{bmatrix}, \tag{3}$$

where C_{44}, e_{15} and ϵ_{11} are the elastic stiffness, the piezoelectric constant and the dielectric constant, respectively. Note that $\mathbf{C}$ defined by Eq. (3) is neither positive definite nor negative definite.

3. Internal uniform electroelastic field

Consider a domain in $\Re^2$, infinite in extent, containing a non-elliptical piezoelectric inhomogeneity, with electroelastic properties different from those of the surrounding matrix. Both the inhomogeneity and the matrix are hexagonal with their poling directions along the x_3-axis. Let S_1 and S_2 denote the inhomogeneity and the matrix, respectively, perfectly bonded through the non-elliptical interface L. The matrix is subjected to uniform remote anti-plane shear stresses $(\sigma_{31}^{\infty}, \sigma_{32}^{\infty})$ and in-plane electric displacements $(D_1^{\infty}, D_2^{\infty})$. In addition, a piezoelectric screw dislocation is applied at $z = z_0$ in the matrix. The screw dislocation suffers a jump b_3 in anti-plane displacement and a jump $\Delta\phi$ in electric potential across the slip plane. Meanwhile, the screw dislocation is subjected to an anti-plane line force f_3 and a line charge Q at its core. Throughout the paper, the subscripts 1 and 2 are used to identify the respective quantities in S_1 and S_2. In the remaining of this section, we will study the existence of an internal uniform electroelastic field inside the non-elliptical piezoelectric inhomogeneity in the presence of the nearby screw dislocation.

The boundary value problem for the two-phase piezoelectric composite has the form:

$$\begin{aligned} &\mathbf{f}_2(z) + \overline{\mathbf{f}_2(z)} = \mathbf{C}_2^{-1}\mathbf{C}_1\mathbf{f}_1(z) + \mathbf{C}_2^{-1}\mathbf{C}_1\overline{\mathbf{f}_1(z)}, \\ &\mathbf{f}_2(z) - \overline{\mathbf{f}_2(z)} = \mathbf{f}_1(z) - \overline{\mathbf{f}_1(z)}, \quad z \in L; \end{aligned} \tag{4a}$$

$$\mathbf{f}_2(z) \cong \frac{\hat{\mathbf{b}} - \mathrm{i}\mathbf{C}_2^{-1}\hat{\mathbf{f}}}{2\pi} \ln(z - z_0) + O(1), \quad z \to z_0; \tag{4b}$$

$$\mathbf{f}_2(z) \cong \mathbf{C}_2^{-1}\mathbf{w}z + \frac{\hat{\mathbf{b}} - \mathrm{i}\mathbf{C}_2^{-1}\hat{\mathbf{f}}}{2\pi} \ln z + O(1), \quad |z| \to \infty, \tag{4c}$$

where

$$\hat{\mathbf{b}} = \begin{bmatrix} b_3 \\ \Delta\phi \end{bmatrix}, \qquad \hat{\mathbf{f}} = \begin{bmatrix} f_3 \\ -Q \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} \sigma_{32}^{\infty} + \mathrm{i}\sigma_{31}^{\infty} \\ D_2^{\infty} + \mathrm{i}D_1^{\infty} \end{bmatrix}. \tag{5}$$

Equation (4a) describes the continuity of traction, normal electric displacement, displacement and electric potential across the perfectly bonded inhomogeneity-matrix interface L; Eq. (4b) describes the logarithmic singular behavior of $\mathbf{f}_2(z)$ at the location of the screw dislocation; Eq. (4c) gives the remote asymptotic behavior of $\mathbf{f}_2(z)$ due to the remote loading and the screw dislocation.

For our purposes, it is sufficient to consider the following conformal mapping function [16]

$$z = \omega(\xi) = R\left(\xi + \frac{p}{\xi} + q \ln \frac{\xi - \bar{\xi}_0^{-1}}{\xi}\right), \quad \xi = \omega^{-1}(z), \ |\xi| \geq 1, \tag{6}$$

where R is a real scaling constant, p and q are two complex constants, $\xi_0 = \omega^{-1}(z_0)$. The branch cut for the logarithmic function appearing in Eq. (6) is chosen as the line segment connecting $\xi = 0$ and $\xi = \bar{\xi}_0^{-1}$ $(|\bar{\xi}_0^{-1}| < 1)$. Thus, the logarithmic function in Eq. (6) is analytic and single-valued for $|\xi| \geq 1$. Using the above mapping function, the exterior of the inhomogeneity is mapped onto $|\xi| \geq 1$, the inhomogeneity-matrix interface L is mapped onto the unit circle $|\xi| = 1$, the location of the piezoelectric screw dislocation $z = z_0$ is mapped onto $\xi = \xi_0$.

In order to ensure the uniformity of the electroelastic field incorporating stresses, strains, electric displacements and electric fields inside the piezoelectric inhomogeneity, it is sufficient that the analytic vector function defined inside the inhomogeneity take the following form

$$\mathbf{f}_1(z) = \mathbf{k}z, \quad z \in S_1, \tag{7}$$

where $\mathbf{k}$ is a two-dimensional complex vector to be determined.

By using Eq. (7) and imposing the continuity conditions for traction, normal electric displacement, displacement and electric potential across the perfect interface L in Eq. (4a), we arrive at

$$\mathbf{f}_2(\xi) = \mathbf{f}_2(\omega(\xi)) = \frac{(\mathbf{C}_2^{-1}\mathbf{C}_1 + \mathbf{I})\mathbf{k}}{2}\omega(\xi) + \frac{(\mathbf{C}_2^{-1}\mathbf{C}_1 - \mathbf{I})\bar{\mathbf{k}}}{2}\bar{\omega}\left(\frac{1}{\xi}\right), \quad |\xi| \geq 1. \tag{8}$$

Using Eq. (8) and satisfying Eqs. (4a) and (4c), we obtain the following relationships

$$(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{k} + \bar{p}(\mathbf{C}_1 - \mathbf{C}_2)\bar{\mathbf{k}} = 2\mathbf{w}, \tag{9}$$

$$Rq(\mathbf{C}_1 - \mathbf{C}_2)\mathbf{k} = 2(\mathbf{C}_2\hat{\mathbf{b}} + \mathrm{i}\hat{\mathbf{f}}). \tag{10}$$

The constant vector $\mathbf{k}$ can be uniquely determined from Eq. (9) as

$$\mathbf{k} = 2\left[\mathbf{C}_1 + \mathbf{C}_2 - |p|^2(\mathbf{C}_1 - \mathbf{C}_2)(\mathbf{C}_1 + \mathbf{C}_2)^{-1}(\mathbf{C}_1 - \mathbf{C}_2)\right]^{-1}\left[\mathbf{w} - \bar{p}(\mathbf{C}_1 - \mathbf{C}_2)(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\bar{\mathbf{w}}\right]. \tag{11}$$

Substitution of Eq. (11) into Eq. (10) yields the following constraint between the remote loading characterized by $\mathbf{w}$ and the piezoelectric screw dislocation characterized by $\hat{\mathbf{b}}$ and $\hat{\mathbf{f}}$:

$$\begin{aligned} &Rq\left[\mathbf{w} - \bar{p}(\mathbf{C}_1 - \mathbf{C}_2)(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\bar{\mathbf{w}}\right] \\ &\quad = \left[(\mathbf{C}_1 + \mathbf{C}_2)(\mathbf{C}_1 - \mathbf{C}_2)^{-1} - |p|^2(\mathbf{C}_1 - \mathbf{C}_2)(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\right](\mathbf{C}_2\hat{\mathbf{b}} + \mathrm{i}\hat{\mathbf{f}}). \end{aligned} \tag{12}$$

The coupling present in the above constraint makes it somewhat difficult to interpret. Consider the following eigenvalue problem:

$$\mathbf{C}_2\mathbf{v} = \lambda\mathbf{C}_1\mathbf{v}, \tag{13}$$

where λ denotes an eigenvalue and $\mathbf{v}$ the associated eigenvector.

Let λ_1 and λ_2 be the two eigenvalues from Eq. (13) with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively (explicit expressions for λ_1, λ_2, $\mathbf{v}_1$ and $\mathbf{v}_2$ can be found in the appendix). It is relatively straightforward to verify the following orthogonality relationships for the two eigenvectors with respect to the two real symmetric matrices $\mathbf{C}_1$ and $\mathbf{C}_2$:

$$\begin{aligned} &\boldsymbol{\Phi}^T\mathbf{C}_1\boldsymbol{\Phi} = \mathrm{diag}\begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix}, \\ &\boldsymbol{\Phi}^T\mathbf{C}_2\boldsymbol{\Phi} = \mathrm{diag}\begin{bmatrix} \lambda_1\delta_1 & \lambda_2\delta_2 \end{bmatrix}, \end{aligned} \tag{14}$$

where δ_1 and δ_2 are constants, and

$$\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}. \tag{15}$$

We further introduce the following transforms:

$$\begin{aligned} &\widehat{\mathbf{w}}' = \begin{bmatrix} \widehat{w}_1' & \widehat{w}_2' \end{bmatrix}^T = \boldsymbol{\Phi}^T\mathbf{w}', \qquad \widehat{\mathbf{w}}'' = \begin{bmatrix} \widehat{w}_1'' & \widehat{w}_2'' \end{bmatrix}^T = \boldsymbol{\Phi}^T\mathbf{w}'', \\ &\widehat{\mathbf{b}} = \begin{bmatrix} \widehat{b}_1 & \widehat{b}_2 \end{bmatrix}^T = \boldsymbol{\Phi}^T\mathbf{C}_2\hat{\mathbf{b}}, \qquad \widehat{\mathbf{f}} = \begin{bmatrix} \widehat{f}_1 & \widehat{f}_2 \end{bmatrix}^T = \boldsymbol{\Phi}^T\hat{\mathbf{f}}, \end{aligned} \tag{16}$$

where $\mathbf{w}'$ and $\mathbf{w}''$ are the real and imaginary parts of $\mathbf{w}$. It is seen from Eq. (16) that both $\widehat{\mathbf{w}}'$ and $\widehat{\mathbf{w}}''$ are complex-valued when the two eigenvectors are complex.

Using Eqs. (14) and (16), the constraint in Eq. (12) can be rewritten in the following simpler decoupled form

$$Rq = \frac{(K_1^{-1} - |p|^2K_1)(\widehat{b}_1 + \mathrm{i}\widehat{f}_1)}{\widehat{w}_1' + \mathrm{i}\widehat{w}_1'' - \bar{p}K_1(\widehat{w}_1' - \mathrm{i}\widehat{w}_1'')} = \frac{(K_2^{-1} - |p|^2K_2)(\widehat{b}_2 + \mathrm{i}\widehat{f}_2)}{\widehat{w}_2' + \mathrm{i}\widehat{w}_2'' - \bar{p}K_2(\widehat{w}_2' - \mathrm{i}\widehat{w}_2'')}, \tag{17}$$

where the two either real or complex conjugate mismatch parameters K_1 and K_2 are defined by

$$K_1=\frac{1-\lambda_1}{1+\lambda_1},\qquad K_2=\frac{1-\lambda_2}{1+\lambda_2},\quad |K_1|<1,\ |K_2|<1. \tag{18}$$

The magnitudes of K_1 and K_2 are smaller than one since the two eigenvalues always have positive real parts (see the Appendix). The constraint in Eq. (17) can be considered as two separate conditions on the remote loading characterized by $\widehat{\mathbf{w}}'$ and $\widehat{\mathbf{w}}''$ and on the screw dislocation characterized by $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{f}}$ for given material and geometric parameters K_1, K_2, p and Rq.

For example, when $p=0$, Eq. (17) reduces to the following one

$$\frac{\widehat{b}_1+\mathrm{i}\widehat{f}_1}{\widehat{w}_1'+\mathrm{i}\widehat{w}_1''}=RqK_1,\qquad \frac{\widehat{b}_2+\mathrm{i}\widehat{f}_2}{\widehat{w}_2'+\mathrm{i}\widehat{w}_2''}=RqK_2. \tag{19}$$

The internal uniform electroelastic field quantities of stress and electric displacement inside the piezoelectric inhomogeneity is given by

$$\begin{aligned}\begin{bmatrix}\sigma_{32}+\mathrm{i}\sigma_{31}\\ D_2+\mathrm{i}D_1\end{bmatrix}&=2\mathbf{C}_1\left[\mathbf{C}_1+\mathbf{C}_2-|p|^2(\mathbf{C}_1-\mathbf{C}_2)(\mathbf{C}_1+\mathbf{C}_2)^{-1}(\mathbf{C}_1-\mathbf{C}_2)\right]^{-1}\\ &\quad\times\left[\mathbf{w}-\bar{p}(\mathbf{C}_1-\mathbf{C}_2)(\mathbf{C}_1+\mathbf{C}_2)^{-1}\bar{\mathbf{w}}\right],\quad z\in S_1,\end{aligned} \tag{20}$$

which is independent of the complex constant q, in other words, independent of the existence of the piezoelectric screw dislocation.

The stresses and electric displacements are distributed within the matrix as follows

$$\begin{aligned}\begin{bmatrix}\sigma_{32}+\mathrm{i}\sigma_{31}\\ D_2+\mathrm{i}D_1\end{bmatrix}&=\left[\mathbf{I}-|p|^2(\mathbf{C}_1-\mathbf{C}_2)(\mathbf{C}_1+\mathbf{C}_2)^{-1}(\mathbf{C}_1-\mathbf{C}_2)(\mathbf{C}_1+\mathbf{C}_2)^{-1}\right]^{-1}\left[\mathbf{w}-\bar{p}(\mathbf{C}_1-\mathbf{C}_2)(\mathbf{C}_1+\mathbf{C}_2)^{-1}\bar{\mathbf{w}}\right]\\ &\quad-\left[(\mathbf{C}_1+\mathbf{C}_2)(\mathbf{C}_1-\mathbf{C}_2)^{-1}-|p|^2(\mathbf{C}_1-\mathbf{C}_2)(\mathbf{C}_1+\mathbf{C}_2)^{-1}\right]^{-1}\\ &\quad\times\left[\bar{\mathbf{w}}-p(\mathbf{C}_1-\mathbf{C}_2)(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{w}\right]\frac{\bar{\omega}'(\frac{1}{\xi})}{\xi^2\omega'(\xi)},\quad |\xi|\ge 1.\end{aligned} \tag{21}$$

Using Eq. (14) and (16), the expressions for stresses and electric displacements in Eqs. (20) and (21) can be concisely written into the following decoupled form

$$\begin{aligned}\widehat{\sigma}&=\frac{(1+K_1)[\widehat{w}_1'+\mathrm{i}\widehat{w}_1''-\bar{p}K_1(\widehat{w}_1'-\mathrm{i}\widehat{w}_1'')]}{1-|p|^2K_1^2},\\ \widehat{D}&=\frac{(1+K_2)[\widehat{w}_2'+\mathrm{i}\widehat{w}_2''-\bar{p}K_2(\widehat{w}_2'-\mathrm{i}\widehat{w}_2'')]}{1-|p|^2K_2^2},\quad z\in S_1;\end{aligned} \tag{22}$$

$$\begin{aligned}\widehat{\sigma}&=\frac{\widehat{w}_1'+\mathrm{i}\widehat{w}_1''-\bar{p}K_1(\widehat{w}_1'-\mathrm{i}\widehat{w}_1'')}{1-|p|^2K_1^2}-\frac{K_1[\widehat{w}_1'-\mathrm{i}\widehat{w}_1''-pK_1(\widehat{w}_1'+\mathrm{i}\widehat{w}_1'')]}{1-|p|^2K_1^2}\frac{\bar{\omega}'(\frac{1}{\xi})}{\xi^2\omega'(\xi)},\\ \widehat{D}&=\frac{\widehat{w}_2'+\mathrm{i}\widehat{w}_2''-\bar{p}K_2(\widehat{w}_2'-\mathrm{i}\widehat{w}_2'')}{1-|p|^2K_2^2}-\frac{K_2[\widehat{w}_2'-\mathrm{i}\widehat{w}_2''-pK_2(\widehat{w}_2'+\mathrm{i}\widehat{w}_2'')]}{1-|p|^2K_2^2}\frac{\bar{\omega}'(\frac{1}{\xi})}{\xi^2\omega'(\xi)},\quad |\xi|\ge 1,\end{aligned} \tag{23}$$

where

$$\begin{bmatrix}\widehat{\sigma}\\ \widehat{D}\end{bmatrix}=\boldsymbol{\Phi}^{\mathrm{T}}\begin{bmatrix}\sigma_{32}+\mathrm{i}\sigma_{31}\\ D_2+\mathrm{i}D_1\end{bmatrix}. \tag{24}$$

It can be seen from Eqs. (22) and (23) that the constants $\widehat{\sigma}$ and $\widehat{D}$ inside the piezoelectric inhomogeneity can be completely determined once the three parameters K_1, K_2 and p and the remote loading $\widehat{\mathbf{w}}'$ and $\widehat{\mathbf{w}}''$ are given. More precisely, $\widehat{\sigma}$ is determined only by K_1, p, $\widehat{w}_1'$, $\widehat{w}_1''$; whilst $\widehat{D}$ is determined only by K_2, p, $\widehat{w}_2'$, $\widehat{w}_2''$.

When $p=0$, Eqs. (22) and (23) reduce to

$$\widehat{\sigma}=(1+K_1)\left(\widehat{w}_1'+\mathrm{i}\widehat{w}_1''\right),\qquad \widehat{D}=(1+K_2)\left(\widehat{w}_2'+\mathrm{i}\widehat{w}_2''\right),\quad z\in S_1; \tag{25}$$

$$\widehat{\sigma}=\widehat{w}_1'+\mathrm{i}\widehat{w}_1''-K_1\left(\widehat{w}_1'-\mathrm{i}\widehat{w}_1''\right)\frac{\bar{\omega}'(\frac{1}{\xi})}{\xi^2\omega'(\xi)},\qquad \widehat{D}=\widehat{w}_2'+\mathrm{i}\widehat{w}_2''-K_2\left(\widehat{w}_2'-\mathrm{i}\widehat{w}_2''\right)\frac{\bar{\omega}'(\frac{1}{\xi})}{\xi^2\omega'(\xi)},\quad |\xi|\ge 1. \tag{26}$$

We have verified that the uniform electroelastic field in Eq. (25) agrees with the uniform field inside a circular piezoelectric inhomogeneity under uniform remote electromechanical loading in the absence of the screw dislocation [19]. The adoption of the transform in Eq. (24) makes the expression in Eq. (25) rather concise.

4. Conclusions

Within the framework of anti-plane piezoelectricity, we have shown that the internal electroelastic field inside a non-elliptical piezoelectric inhomogeneity interacting with a nearby screw dislocation can be maintained uniform when the constraint in Eq. (17) is satisfied. After the introduction of the transform in Eq. (24), the electroelastic field in the piezoelectric composite can be written concisely in the decoupled form in Eqs. (22) and (23).

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Appendix A

Let

$$\mathbf{C}_1=\begin{bmatrix} C_{44}^{(1)} & e_{15}^{(1)} \\ e_{15}^{(1)} & -\in_{11}^{(1)} \end{bmatrix}, \qquad \mathbf{C}_2=\begin{bmatrix} C_{44}^{(2)} & e_{15}^{(2)} \\ e_{15}^{(2)} & -\in_{11}^{(2)} \end{bmatrix}. \tag{27}$$

When $(C_{44}^{(1)}\in_{11}^{(2)}+C_{44}^{(2)}\in_{11}^{(1)}+2e_{15}^{(1)}e_{15}^{(2)})^2>4(C_{44}^{(1)}\in_{11}^{(1)}+e_{15}^{(1)2})(C_{44}^{(2)}\in_{11}^{(2)}+e_{15}^{(2)2})$, the two distinct eigenvalues of Eq. (13) are positive real and are given by

$$\lambda_{1,2}=\frac{\left(\begin{array}{l} C_{44}^{(1)}\in_{11}^{(2)}+C_{44}^{(2)}\in_{11}^{(1)}+2e_{15}^{(1)}e_{15}^{(2)} \\ \pm\sqrt{(C_{44}^{(1)}\in_{11}^{(2)}-C_{44}^{(2)}\in_{11}^{(1)})^2+4(C_{44}^{(1)}e_{15}^{(2)}-C_{44}^{(2)}e_{15}^{(1)})(\in_{11}^{(2)}e_{15}^{(1)}-\in_{11}^{(1)}e_{15}^{(2)})} \end{array}\right)}{2(C_{44}^{(1)}\in_{11}^{(1)}+e_{15}^{(1)2})}>0. \tag{28}$$

When $(C_{44}^{(1)}\in_{11}^{(2)}+C_{44}^{(2)}\in_{11}^{(1)}+2e_{15}^{(1)}e_{15}^{(2)})^2<4(C_{44}^{(1)}\in_{11}^{(1)}+e_{15}^{(1)2})(C_{44}^{(2)}\in_{11}^{(2)}+e_{15}^{(2)2})$, the two distinct eigenvalues of Eq. (13) are complex conjugates with positive real part and are given by

$$\lambda_{1,2}=\frac{\left(\begin{array}{l} C_{44}^{(1)}\in_{11}^{(2)}+C_{44}^{(2)}\in_{11}^{(1)}+2e_{15}^{(1)}e_{15}^{(2)} \\ \pm\mathrm{i}\sqrt{4(C_{44}^{(1)}e_{15}^{(2)}-C_{44}^{(2)}e_{15}^{(1)})(\in_{11}^{(1)}e_{15}^{(2)}-\in_{11}^{(2)}e_{15}^{(1)})-(C_{44}^{(1)}\in_{11}^{(2)}-C_{44}^{(2)}\in_{11}^{(1)})^2} \end{array}\right)}{2(C_{44}^{(1)}\in_{11}^{(1)}+e_{15}^{(1)2})},$$
$$\mathrm{Re}\{\lambda_1\}=\mathrm{Re}\{\lambda_2\}>0. \tag{29}$$

The two eigenvectors associated with the eigenvalues for the above two cases are

$$\mathbf{v}_1=\begin{bmatrix} \lambda_1 e_{15}^{(1)}-e_{15}^{(2)} \\ C_{44}^{(2)}-\lambda_1 C_{44}^{(1)} \end{bmatrix}, \qquad \mathbf{v}_2=\begin{bmatrix} \lambda_2 e_{15}^{(1)}-e_{15}^{(2)} \\ C_{44}^{(2)}-\lambda_2 C_{44}^{(1)} \end{bmatrix}. \tag{30}$$

When the two eigenvalues are real, the two eigenvectors are also real. When the two eigenvalues are complex conjugates, we have from Eq. (30) that $\mathbf{v}_2=\bar{\mathbf{v}}_1$.

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