C. R. Mecanique 347 (2019) 710–715

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# Non-global existence of solutions to pseudo-parabolic equations with variable exponents and positive initial energy

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ARTICLE INFO

*Article history:*
Received 17 May 2018
Accepted after revision 21 September 2019
Available online 9 October 2019

*Keywords:*
Non-global existence
Variable exponents
Positive initial energy
Pseudo-parabolic equation

ABSTRACT

This paper deals with a pseudo-parabolic equation involving variable exponents under Dirichlet boundary value condition. The author proves that the solution is not global in time when the initial energy is positive. This result extends and improves a recent result obtained by Di et al. (2017) [1].



## 1. Introduction

In this paper, we study the following $m(x)$-Laplacian equation:

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) = |u|^{p(x)-2}u, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$. The exponents $m(x)$ and $p(x)$ are two continuous functions satisfying

$$2 < m^- \leqslant m(x) \leqslant m^+ < p^- \leqslant p(x) \leqslant p^+ < \infty \tag{1.2}$$

and both satisfy the following condition:

$$|p(x) - p(y)| \leqslant \frac{A}{\log\left(\frac{1}{|x-y|}\right)}, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta \tag{1.3}$$

where $A > 0$ and $0 < \delta < 1$. The upper bound for the blow-up time was obtained when the initial energy is nonpositive [1]. However, it is natural to pose an interesting problem: does the blow-up phenomena occur when the initial energy is positive? To the best of our knowledge, there exist only rare results about such problems.

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https://doi.org/10.1016/j.crme.2019.09.003
1631-0721/

Pseudo-parabolic equations describe many important physical processes, such as the seepage of homogeneous fluids through a fissured rock, the aggregation of populations, etc. The interested readers may refer to [2,3] and the references therein. For the study of pseudo-parabolic equations, many results have been obtained [4–6]. Especially, Xu and Su [4] studied a class of semilinear pseudo-parabolic equations (where $m(x) = 2$, and the source term $|u|^{p(x)-2}$ is replaced by $u^p$). They proved global existence, non-existence, and asymptotic behavior of solutions with initial energy by introducing a family of potential wells and obtained finite-time blow-up with high initial energy by the comparison principle. Since there exist variable exponents, we cannot obtain the blow-up result with high initial energy by trying to consider the potential-well method for Problem (1.1). Variable exponents and the term $\Delta u_t$ cause some difficulties, thus we have to develop other techniques to overcome these difficulties. Before stating our result, let us recall some results on the existence and blow-up of the solution to Problem (1.1) with nonpositive initial energy.

**Theorem 1.1.** *[1] Assume that (1.2) and (1.3) hold, then for any $u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$, there exists a $T_0 \in (0, T]$ such that the Problem (1.1) has a unique local solution $u \in L^\infty([0, T_0], W^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega))$ with $u_t \in L^2([0, T_0], W^{1,2}(\Omega))$ satisfying*

$$(u_t, v) + (\nabla u_t, \nabla v) + (|\nabla u|^{m(x)-2}\nabla u, \nabla v) = (|u|^{p(x)-2}u, v), \quad \text{for } v \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega) \tag{1.4}$$

Define the energy functional by

$$E(t) = \int_\Omega \Big(\frac{1}{m(x)}|\nabla u|^{m(x)} - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\Big)\,\mathrm{d}x \tag{1.5}$$

It obviously follows that

$$E'(t) = -\|u_t\|^2_{H_0^1} \leqslant 0 \tag{1.6}$$

which implies

$$E(t) = E(0) - \int_0^t \|u_s\|^2_{H_0^1}\,\mathrm{d}s \tag{1.7}$$

**Theorem 1.2.** *[1] Assume that (1.2) and (1.3) hold. Let $u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ such that $\|u_0\|_{H_0^1(\Omega)} > 0$ and $E(0) \leqslant 0$. Then the solution $u(x,t)$ to Problem (1.1) blows up in a finite time $T^*$. Moreover, an upper bound for the blow-up time is given by*

$$T^* \leqslant \frac{2(\|u_0\|^2_{H_0^1})^{1-\frac{m_-}{2}}}{(m_- - 2)\beta}$$

*where $\beta > 0$ is a constant.*

Our main result is as follows:

**Theorem 1.3.** *Assume that (1.2) and (1.3) hold. Let $u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ be non-zero. If the following condition holds:*

$$0 < E(0) < C \tag{1.8}$$

*then the solution $u(x,t)$ to Problem (1.1) is non-global, where*

$$C = \frac{p^- - m^+}{m^+}\frac{1}{p^-}\min\Big\{\frac{2}{m^+}\Big(\frac{m^-|\Omega|}{2}\Big)^{-\frac{m^+-2}{2}}\Big(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\Big)^{\frac{m^+}{2}}\|u_0\|^{m^+}_{H_0^1},$$
$$\frac{2}{m^-}\Big(\frac{m^-|\Omega|}{2}\Big)^{-\frac{m^--2}{2}}\Big(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\Big)^{\frac{m^-}{2}}\Big(\frac{m^+-2}{m^+}\frac{m^-}{m^--2}\Big)^{-\frac{m^--2}{2}}\|u_0\|^{m^-}_{H_0^1}\Big\}$$

*Here $\lambda_1$ is the first eigenvalue of the problem*

$$\begin{cases} \Delta\omega + \lambda\omega = 0, & \text{in } \Omega, \\ \omega = 0, & \text{on } \partial\Omega \end{cases}$$

In comparison with Theorem 1.2, we add something substantial to $E(0)$, in the sense that we obtain the non-global existence of solutions when the initial energy is positive. From this point of view, this improvement is in the spirit of the one performed by Merle and Zaag [7] for some parabolic equation derived from the semilinear heat equation; the interested readers can refer to Proposition 2.1 in [7].

## 2. Proof of Theorem 1.3

In order to prove our main result, we argue by contradiction. We assume that the solution is global in time. Next, we will divide this proof into two steps.

**Step 1.** It directly follows from [1] that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H_0^1}^2 &= 2\int_\Omega |u|^{p(x)}\,\mathrm{d}x - 2\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x\\ &\geqslant 2p^-\int_\Omega \frac{|u|^{p(x)}}{p(x)}\,\mathrm{d}x - 2\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x\\ &= 2p^-\int_\Omega \frac{|u|^{p(x)}}{p(x)}\,\mathrm{d}x - 2p^-\int_\Omega \frac{|\nabla u|^{m(x)}}{m(x)}\,\mathrm{d}x + 2p^-\int_\Omega \frac{|\nabla u|^{m(x)}}{m(x)}\,\mathrm{d}x - 2\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x\\ &\geqslant -2p^-E(t) + \frac{2(p^- - m^+)}{m^+}\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x\end{aligned} \tag{2.1}$$

**Step 2.** In order to complete the proof, we consider the following two cases.

**Case 1.** We first consider the case $E(t) \geqslant 0$, for all $t > 0$.

Firstly, **we give an estimate of a lower bound for the solution in $H_0^1$ norm**. In fact, by virtue of (1.8), we choose an $\alpha$ satisfying the following inequality

$$1 < \alpha < \frac{C}{E(0)} \tag{2.2}$$

Combining (1.7) with (2.1) (2.2) and $E(t) \geqslant 0$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H_0^1}^2 &\geqslant -2p^-E(t) + \frac{2(p^- - m^+)}{m^+}\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x\\ &= 2p^-(\alpha-1)E(t) - 2p^-\alpha E(t) + \frac{2(p^- - m^+)}{m^+}\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x\\ &\geqslant -2p^-\alpha E(0) + 2p^-\alpha\int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s + \frac{2(p^- - m^+)}{m^+}\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x\end{aligned} \tag{2.3}$$

Young's inequality implies that

$$\|\nabla u\|_2^2 \leqslant \frac{2\sigma}{m^-}\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x + \frac{m^+-2}{m^+}|\Omega|\max\left\{\sigma^{\frac{-2}{m^+-2}}, \sigma^{\frac{-2}{m^--2}}\right\} \tag{2.4}$$

for all $\sigma > 0$. It follows from Poincaré's inequality that $\|\nabla u\|_2^2 \geqslant \lambda_1\|u\|_2^2$. Therefore, we get

$$\|\nabla u\|_2^2 = \frac{\lambda_1}{1+\lambda_1}\|\nabla u\|_2^2 + \frac{1}{1+\lambda_1}\|\nabla u\|_2^2 \geqslant \frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2 \tag{2.5}$$

Combining (2.4) with (2.5), and replacing $\frac{m^-}{2\sigma}\frac{\lambda_1}{1+\lambda_1}$ by $\kappa$, then

$$\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x \geqslant \kappa\|u\|_{H_0^1}^2 - \frac{m^+-2}{m^+}\frac{m^-}{2}|\Omega|C_1(\kappa) \tag{2.6}$$

where $C_1(\kappa) = \max\left\{\left(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\right)^{\frac{-m^+}{m^+-2}}\kappa^{\frac{m^+}{m^+-2}}, \left(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\right)^{\frac{-m^-}{m^--2}}\kappa^{\frac{m^-}{m^--2}}\right\}$. Inserting (2.6) into (2.3), it directly follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H_0^1}^2 \geqslant -2p^-\alpha E(0) + 2p^-\alpha\int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s + \frac{2(p^- - m^+)\kappa}{m^+}\|u\|_{H_0^1}^2 - \frac{2(p^- - m^+)}{m^+}\frac{m^+ - 2}{m^+}\frac{m^-}{2}|\Omega|C_1(\kappa) \tag{2.7}$$

Removing the second term on the right-hand side of the above inequality, and then solving the ordinary differential inequality, we obtain

$$\|u\|_{H_0^1}^2 \geqslant \frac{m^+}{2(p^- - m^+)\kappa}\Big(2p^-\alpha E(0) + \frac{2(p^- - m^+)}{m^+}\frac{m^+ - 2}{m^+}\frac{m^-}{2}|\Omega|C_1(\kappa)\Big) \times \Big(1 - \mathrm{e}^{\frac{2(p^- - m^+)\kappa t}{m^+}}\Big) + \|u_0\|_{H_0^1}^2\mathrm{e}^{\frac{2(p^- - m^+)\kappa t}{m^+}} \tag{2.8}$$

Secondly, **we prove that the solution is not global in time**. Define $y(t) = \int_0^t \|u\|_{H_0^1}^2\,\mathrm{d}s$, then $y'(t) = \|u\|_{H_0^1}^2$, $y''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H_0^1}^2$. By inserting (2.8) into (2.7), we have

$$y''(t) \geqslant 2p^-\alpha\int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s + \Big(\frac{2(p^- - m^+)\kappa}{m^+}\|u_0\|_{H_0^1}^2 - \frac{2(p^- - m^+)}{m^+}\frac{m^+ - 2}{m^+}\frac{m^-}{2}|\Omega|C_1(\kappa) - 2p^-\alpha E(0)\Big)\mathrm{e}^{\frac{2(p^- - m^+)\kappa t}{m^+}} \tag{2.9}$$

Due to (2.2), we can take $\varepsilon > 0$ small enough that

$$\varepsilon < \frac{2(C - \alpha E(0))}{\alpha\|u_0\|_{H_0^1}^2} \tag{2.10}$$

And then we may choose sufficiently large $c > 0$ satisfying

$$c > \frac{1}{4}\varepsilon^{-2}\|u_0\|_{H_0^1}^4 \tag{2.11}$$

Denoting $\varphi(t) = y^2(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2 y(t) + c$, we get

$$\varphi'(t) = \Big(2y(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2\Big)y'(t) \tag{2.12}$$

$$\varphi''(t) = \Big(2y(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2\Big)y''(t) + 2(y'(t))^2 \tag{2.13}$$

Set $\delta = 4c - \varepsilon^{-2}\|u_0\|_{H_0^1}^4$, then $\delta > 0$ with the help of (2.11). From (2.12), we obtain that

$$(\varphi'(t))^2 = (4\varphi(t) - \delta)(y'(t))^2$$

which implies

$$4\varphi(t)(y'(t))^2 = (\varphi'(t))^2 + \delta(y'(t))^2 \tag{2.14}$$

(2.13) and (2.14) imply that

$$4\varphi(t)\varphi''(t) = 4\Big(2y(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2\Big)y''(t)\varphi(t) + 2\Big((\varphi'(t))^2 + \delta(y'(t))^2\Big) \tag{2.15}$$

Noticing that

$$\int_0^t\int_\Omega (uu_s + \nabla u\nabla u_s)\,\mathrm{d}x\mathrm{d}s \leqslant \Big(\int_0^t\int_\Omega (u^2 + |\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^t\int_\Omega (u_s^2 + |\nabla u_s|^2)\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{1}{2}}$$

it is easy to verify that

$$
\begin{aligned}
(y'(t))^2 = \|u\|_{H_0^1}^4 &= \Big(\|u_0\|_{H_0^1}^2 + 2\int_0^t \int_\Omega (uu_s + \nabla u \nabla u_s)\,\mathrm{d}x\,\mathrm{d}s\Big)^2 \\
&\leqslant \|u_0\|_{H_0^1}^4 + 4\|u_0\|_{H_0^1}^2 (y(t))^{\frac{1}{2}} \Big(\int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s\Big)^{\frac{1}{2}} + 4y(t)\int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s \\
&\leqslant \|u_0\|_{H_0^1}^4 + 2\varepsilon\|u_0\|_{H_0^1}^2 y(t) + 2\varepsilon^{-1}\|u_0\|_{H_0^1}^2 \int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s + 4y(t)\int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s
\end{aligned} \tag{2.16}
$$

for all $\varepsilon > 0$. Therefore, combining (2.16) with (2.15) and (2.9), we have

$$
\begin{aligned}
&4\varphi''(t)\varphi(t) - (2+\alpha p^-)(\varphi'(t))^2 \\
&= 4\Big(2y(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2\Big) y''(t)\varphi(t) + 2\Big((\varphi'(t))^2 + \delta(y'(t))^2\Big) - (2+\alpha p^-)(\varphi'(t))^2 \\
&= 4\Big(2y(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2\Big) y''(t)\varphi(t) + 2\delta(y'(t))^2 - \alpha p^-(\varphi'(t))^2 \\
&= 4\Big(2y(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2\Big) y''(t)\varphi(t) + 2\delta(y'(t))^2 - \alpha p^-(4\varphi(t) - \delta)(y'(t))^2 \\
&= 4\Big(2y(t) + \varepsilon^{-1}\|u_0\|_{H_0^1}^2\Big) y''(t)\varphi(t) + (2+\alpha p^-)\delta(y'(t))^2 - 4\alpha p^-\varphi(t)(y'(t))^2 \\
&\geqslant 4\left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right) y''(t)\varphi(t) - 4\alpha p^-\varphi(t)(y'(t))^2 \\
&\geqslant 4\left(2y(t) + \varepsilon^{-1}\|u_0\|_2^2\right)\Big(2p^-\alpha \int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s + G(\kappa)\mathrm{e}^{\frac{2(p^- - m^+)\kappa t}{m^+}}\Big)\varphi(t) - 4\alpha p^- \\
&\quad \times \varphi(t)\Big(\|u_0\|_{H_0^1}^4 + 2\varepsilon\|u_0\|_{H_0^1}^2 y(t) + 2\varepsilon^{-1}\|u_0\|_{H_0^1}^2 \int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s + 4y(t)\int_0^t \|u_s\|_{H_0^1}^2\,\mathrm{d}s\Big)
\end{aligned} \tag{2.17}
$$

where

$$
G(\kappa) = \frac{2(p^- - m^+)\kappa}{m^+}\|u_0\|_{H_0^1}^2 - \frac{2(p^- - m^+)}{m^+}\frac{m^+ - 2}{m^+}\frac{m^-}{2}|\Omega| C_1(\kappa) - 2p^-\alpha E(0)
$$

Next we prove the non-negativity for the right-hand side of (2.17). Solving the equation $G'(\kappa) = 0$, then we obtain one of $\kappa_1$ and $\kappa_2$, respectively

$$
\kappa_1 = \|u_0\|_{H_0^1}^{m^+ - 2}\Big(\frac{m^-}{2}|\Omega|\Big)^{-\frac{m^+ - 2}{2}}\Big(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\Big)^{\frac{m^+}{2}}
$$

$$
\kappa_2 = \|u_0\|_{H_0^1}^{m^- - 2}\Big(\frac{m^-}{2}|\Omega|\Big)^{-\frac{m^- - 2}{2}}\Big(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\Big)^{\frac{m^-}{2}}\Big(\frac{m^+ - 2}{m^+}\frac{m^-}{m^- - 2}\Big)^{-\frac{m^- - 2}{2}}
$$

Noticing that

$$
G(\kappa_1) = \frac{2(p^- - m^+)}{m^+}\frac{2}{m^+}\Big(\frac{m^-}{2}|\Omega|\Big)^{-\frac{m^+ - 2}{2}}\Big(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\Big)^{\frac{m^+}{2}}\|u_0\|_{H_0^1}^{m^+} - 2p^-\alpha E(0)
$$

$$
\begin{aligned}
G(\kappa_2) = &\frac{2(p^- - m^+)}{m^+}\frac{2}{m^-}\Big(\frac{m^-}{2}|\Omega|\Big)^{-\frac{m^- - 2}{2}}\Big(\frac{m^-}{2}\frac{\lambda_1}{1+\lambda_1}\Big)^{\frac{m^-}{2}}\Big(\frac{m^+ - 2}{m^+}\frac{m^-}{m^- - 2}\Big)^{-\frac{m^- - 2}{2}}\|u_0\|_{H_0^1}^{m^-} \\
&- 2p^-\alpha E(0)
\end{aligned}
$$

we apply (2.10) to obtain

$$
G(\kappa_{\max}) \geqslant \min\{G(\kappa_1), G(\kappa_2)\} = 2p^-(C - \alpha E(0)) > \varepsilon p^-\alpha\|u_0\|_{H_0^1}^2
$$

$\kappa_{\max} \in \{\kappa_1, \kappa_2\}$ is the maximum point of the function $G$. Utilizing the fact that $\mathrm{e}^{\frac{2(p^- - m^+)\kappa t}{m^+}} \geqslant 1$ and (2.10), (2.17), we obtain

$$
\begin{aligned}
&4\varphi''(t)\varphi(t) - (2+\alpha p^-)(\varphi'(t))^2 \\
&\geqslant 4\left(2y(t) + \varepsilon^{-1}\|u_0\|^2_{H_0^1}\right)\left(2p^-\alpha\int_0^t \|u_s\|^2_{H_0^1} ds + \varepsilon p^-\alpha\|u_0\|^2_{H_0^1}\right)\varphi(t) \\
&\quad - 4\alpha p^-\varphi(t)\left(\|u_0\|^4_{H_0^1} + 2\varepsilon\|u_0\|^2_{H_0^1} y(t) + 2\varepsilon^{-1}\|u_0\|^2_{H_0^1}\int_0^t \|u_s\|^2_{H_0^1}\, ds + 4y(t)\int_0^t \|u_s\|^2_{H_0^1}\, ds\right) \\
&= 0
\end{aligned}
$$

which indicates

$$
\varphi''(t)\varphi(t) - \frac{2+\alpha p^-}{4}(\varphi'(t))^2 \geqslant 0
$$

Applying $\varphi(0) > 0$, $\varphi'(0) > 0$, and the concavity method [8–10], we get the maximal existence time of the solution $T_{\max} = \frac{4\varphi(0)}{(\alpha p^- - 2)\varphi'(0)}$, which shows that the solution $u(x,t)$ is non-global. This is a contradiction with the assumption.

**Case 2.** There exists some $\tilde{t} > 0$ such that $E(\tilde{t}) < 0$. Because the energy functional $E(t)$ is a continuous function and $E(0) > 0$, we can assume that there exists a first time $t_0 > 0$ such that $E(t_0) = 0$ and $E(\tilde{t}) < 0$ for some $\tilde{t} > t_0$. Integrating (2.1) from $t_0$ to $\tilde{t}$, then $\|u(x,\tilde{t})\|_{H_0^1(\Omega)} > 0$. Now we choose $u(x,\tilde{t})$ as a new initial datum, then it is clear from Theorem 1.2 that the solution is non-global. This is a contradiction with the assumption.

This completes the proof of Theorem 1.3.

## Acknowledgements

The author wishes to express her gratitude to the China Scholarship Council for the scholarship and Professor Wenjie Gao for his support and constant encouragement.

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