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A robotic model for Codman's paradox simulation and interpretation

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ABSTRACT

A robotic approach based on Denavit–Hartenberg parametrization is proposed for simulating and interpreting Codman's paradox. A 3-degree-of-freedom robot model of the glenohumeral joint, driving the arm reduced to its long humerus, is considered for simulating the two-step rotational sequence of Codman's paradox. We propose to use the classical distinction made in robotics between the joint space, i.e. the inner space of joint angles, and the operational space, i.e. the outer physical space, for interpreting this historical version of the paradox, as there is some kind of confusion between these two spaces to be considered for arm movement definition. In its extended form, developed by MacConnail, the three-step rotational sequence of Codman's paradox would highlight the motor redundancy of the shoulder joint, necessitating for its simulation, according to our robotic approach, a 4-axis model of the shoulder spheroid joint. Our model provides a general prediction of the conjunct rotation angle in full accordance with clinical observation for a two-step or three-step version of Codman's paradox. The relation of the paradox with a possible general law of motion is finally discussed.

1. Introduction

Codman's paradox is originally described by the Bostonian surgeon E.A. Codman [1] in his book *The Shoulder*, and it is still today intensively discussed either to claim that it is not a paradox [19,22,14], or to emphasize its importance for a general thought about shoulder movements [4,9,15,25]. However, the simple fact that an article dedicated to Codman's paradox can lead to a controversy in the journal where it has been published – see, for example, Paul [17] in response to Politti et al. [19], and Cheng [4] in response to Stepen and Otahal [22] – highlights the always actual difficulty to clearly interpret this powerful paradox. This fact is, according to us, in relation with the apparently purely kinematic character of Codman's paradox: multiple mathematical interpretations can then be proposed for explaining the result of the successive rotations at the shoulder joint considered by the paradox, in its initial form as in its renewed form developed in the 1950s by MacConaill [11,13]. Among the mathematical tools used for explaining Codman's paradox, rotation matrices applied to vectors [22,14] or points in homogeneous coordinates [19] appear to be efficient tools thanks to which it can be hoped that the paradox will be definitively solved. Things however do not appear so simple due to the assumption, made in many "solutions" of Codman's paradox, that abduction–adduction and flexion–extension axes are considered as fixed – this is clearly assumed in [19,22,14]. If, in the case of the look for parameters modeling the orientation of the

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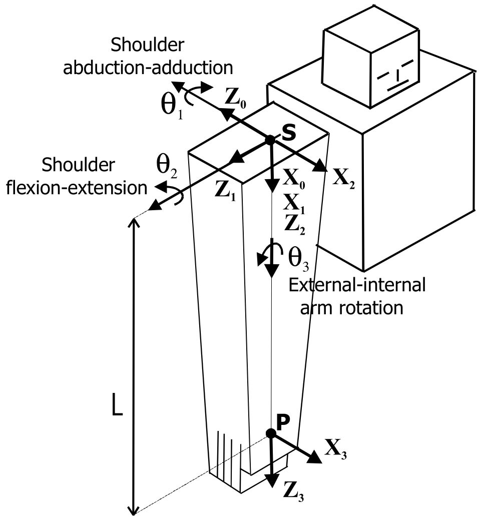


(a)

elementary movement	a_i	α_i	d_i	θ_i
abduction/ adduction	0	$-\pi/2$	0	θ_1
flexion/ extension	0	$+\pi/2$	0	$\theta_2-\pi/2$
int./ext. rotation	0	0	L	θ_3

(b)

Fig. 1. Three-degree-of-freedom robot model for Codman's paradox simulation. (a) Arm model and placement of frames according to R. Paul's notation. (b) Corresponding table with Denavit–Hartenberg parameters (a_i, α_i, d_i, θ_i), for $i = 1$ to 4 (see text for the definition of these parameters and also Fig. 2).

shoulder, such an assumption is perfectly relevant, as done for example in [7]; it is much more questionable for interpreting Codman's paradox. Such hypothesis indeed leads us to assume that the muscles actuating the shoulder complex have no influence on the location of movement axes, which is an opinion difficult to defend. Moreover, because the external–internal arm rotation axis is clearly a mobile axis to consider in Codman's paradox analysis, most of these mathematical models combine fixed abduction–adduction and flexion–extension axes with this last mobile axis; this is also a questionable point regarding the corresponding kinematic structure modeling the shoulder movements – i.e. the considered sequence of joint movements – because it then mixes fixed and mobile axes. To overcome such a difficulty, we propose in the framework of this article a robotic approach of Codman's paradox. Such an approach has, according to us, three advantages:

- it clearly defines a serial kinematic chain making possible the kinematic modeling of the whole arm structure considered as a non-jointed single unit arm-forearm-hand driven by the shoulder joint reduced to a ball-and-socket joint. The final hand position and orientation can then be deduced and compared with the predictions made by Codman's paradox;
- every rotation axis of the robotic model moves with respect to the previous one in the considered serial kinematic chain generating, as predicted by Roth's proof, an independence of the final hand location with the order of joint controls;
- the robotic model is based on the powerful and optimal Denavit–Hartenberg parametrization [6] of rotation axes in the affine space, leading to a synthetic computation of the composed rotations.

It is important to note that the proposed robotic approach, which we are going to develop by means of the so-called homogeneous matrices, makes it possible to deal with changes in both position and orientation, imposed by joint movements to the arm-tip in physical space. Although Euler angles can also be ordered in accordance with a chosen sequence of rotation axes, they are only devoted to specify the orientation of a body in space and not both its position and orientation. Finally, due to its analytic form, a robotic model would be able to cover all examples, which is actually an essential point in the appreciation of Codman's paradox simulation models as pointed out by Cheng [4]. Our look for this relevant robotic model adapted to Codman's paradox simulation will be done in two stages: the first one is limited to the examples considered by Codman himself for illustrating his paradox, the second one is dedicated to cover both this classic form and the renewed form of Codman's paradox after its rereading by MacConaill. We will so lead to propose a general definition of Codman's paradox and to question a possible 'law of motion' in relation to it.

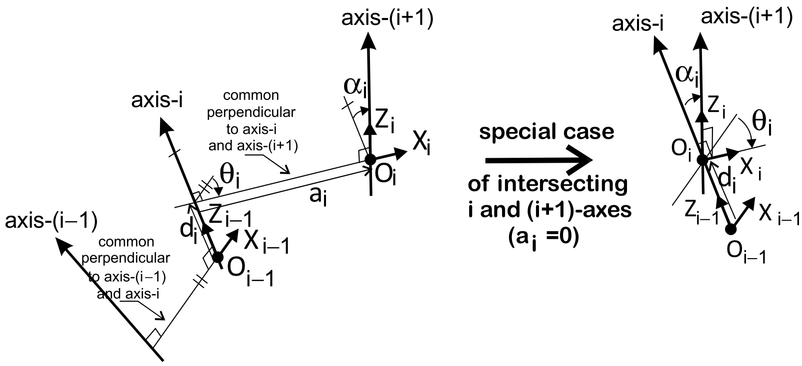


Fig. 2. General definition of Denavit–Hartenberg parameters according to R. Paul's notation and special case of intersecting joint axes with, as a consequence, a corresponding length a_i between axes-i and $i+1$ equal to zero (see text).

2. Codman's paradox first complexity level and its kinematic solving

For many students and readers in human joint physiology, Codman's paradox is often discovered in one of the translations of the famous Kapandji's treatise about human upper limb joint physiology [10]. Kapandji introduces Codman's paradox as follows: "Start from the reference position [. . .], with the upper limb hanging vertically alongside the trunk, the palm of the hand facing medially and the thumb facing anteriorly, and abduct the limb to 180° in the frontal plane and then extend the limb in the sagittal plane for −180°. The limb once more lies vertically alongside the body but with the palm facing laterally and the thumb facing posteriorly. The movements can also be reversed starting with 180° flexion followed by 180° adduction. The limb is laterally rotated through 180°. It is easy to note that the orientation of the palm has changed and that the limb has been axially rotated through 180° (p. 14)". Kapandji uses the term 'conjunct rotation' introduced after Codman by MacConaill [11] to designate "the mechanical and involuntary movement about the longitudinal axis of the upper limb [brought out] by this sequential movements" (idem, p. 14). If we adopt MacConaill's terminology, any analysis of Codman's paradox, limited for the moment to these two examples, must explain the origin of this 'involuntary' conjunct rotation and its experimental π-value. Because Codman's paradox seems to be limited to the interpretation of the glenohumeral joint – i.e. no translation of this joint induced by the shoulder complex is considered –, a simple kinematic interpretation of the paradox is based on a ball-and-socket joint model of this anatomical articulation. And since any ball-and-socket joint is equivalent to a sequence of three single degree of freedom revolute joints with intersecting axes, the simple robotic model shown in Fig. 1a can be considered: the arm and its hand define a single structure driven by the sequence of three revolute joints. We propose to adopt the following axis sequence: the first rotation axis is an abduction–adduction axis and the second rotation axis is a flexion–extension axis. In this way, for a π/2-abduction, the flexion–extension arm movements can be described as horizontal flexion–extension, in accordance, for example, with Kapandji's treatise. The third axis is then the rotation axis of the long humerus about itself. It is important to recall that, in this type of serial chain sequence, the first axis remains fixed and drives the second one, which drives the third one. Such a model is well adapted to study the movements of the long humerus induced by a combined sequence of abduction–adduction and flexion–extension movements. We are free to choose the direction of each axis of rotation. In the proposed model, the positive directions respectively correspond to abduction, flexion, and external rotation. The joint variables will be denoted by θ_1 for abduction–adduction, θ_2 for flexion–extension, and θ_3 for external–internal rotation. The end-point of the arm will be denoted by $\boldsymbol{P}$, situated at a distance L from the ball-and-socket joint center along the third rotation axis. Because hand position and hand orientation are both involved in Codman's paradox, we associate with the hand a frame whose origin is point $\boldsymbol{P}$ and whose axes are defined according to the classical Paul's notation [16], in relation with the Denavit–Hartenberg parameters, as illustrated in Fig. 2 in the case of intersecting axes, which is the only case to be considered here. Frame $\boldsymbol{R}_i$ is associated with axis i movement as follows: the origin $\boldsymbol{O}_i$ is located at the intersection between axis i and axis $i+1$, the $\boldsymbol{X}_i$-vector is put along the common perpendicular to both axes and the $\boldsymbol{Z}_i$-axis is put along the axis $i+1$ in its positive direction. Consequently, the variable $\boldsymbol{\theta}_i$ is the rotation angle from the $\boldsymbol{X}_{i-1}$-vector to $\boldsymbol{X}_i$-vector about $\boldsymbol{Z}_i$-axis. In order to express the homogeneous transformation from frame $\boldsymbol{R}_{i-1}$ to frame $\boldsymbol{R}_i$ four parameters are required whose one is the $\boldsymbol{\theta}_i$-joint variable and the three others are: the a_i-positive "distance" from axis i to axis $i+i'$ measured along the $\boldsymbol{X}_i$-vector, equal to zero in our case; the α_i-twist angle from axis i to axis $i+i'$ measured around the $\boldsymbol{X}_i$-vector; the d_i-distance from origin $\boldsymbol{O}_{i-1}$ to origin $\boldsymbol{O}_i$ measured along the $\boldsymbol{Z}_{i-1}$-vector, which can be freely chosen in the case of the first and last axes. This frame notation imposes a zero value to the joint angle, corresponding to the equality of vectors $\boldsymbol{X}_{i-1}$ and $\boldsymbol{X}_i$. It is however possible to introduce joint offsets in order to impose zero values to all joints of the robot model in a given reference configuration. It is easy to show that the offset is then simply equal to minus the corresponding angle from vector $\boldsymbol{X}_{i-1}$ to vector $\boldsymbol{X}_i$ in this reference configuration. In our case, the reference configuration will be the neutral initial

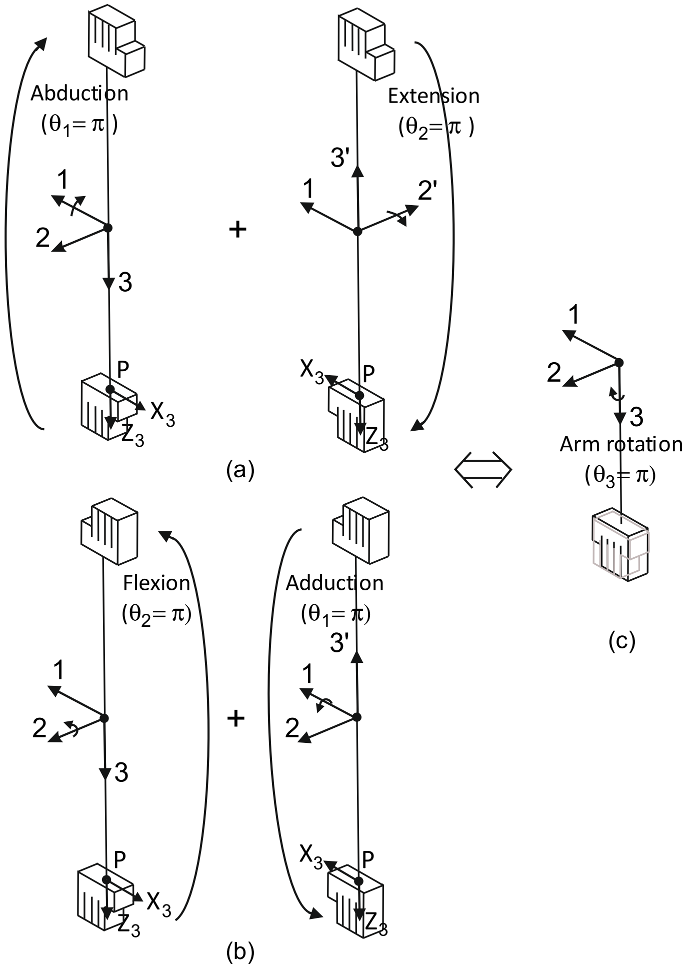


Fig. 3. Simulation of the two-step Codman's paradox with our 3 d.o.f. arm model. (a) Abduction is performed first, changing the initial axis sequence $(1,2,3)$ into $(1,2',3')$. (b) Flexion is performed first, changing the initial axis sequence $(1,2,3)$ into $(1,2,3')$. (c) Equivalent rotation performed about the arm rotation axis 3, showing that the final orientation of the hand (drawn in black while the initial one is drawn in light grey) is the same as the one resulting from the two-step rotational sequences shown in (a) and (b) – it is important to note that the labeled axes are the current axes and that the current movement is made around the axis marked with a rotation arrow; a similar approach will be used in Figs. 4 and 8.

attitude of the arm to be considered in Codman's paradox. According to these rules, the frames $\boldsymbol{R}_0$ to $\boldsymbol{R}_3$ are located as shown in Fig. 1a, and we give in Fig. 1b the corresponding Denavit–Hartenberg table, gathering the $(a_i, \alpha_i, d_i, \theta_i)$ parameters for each joint i, $i = 1$ to 4. Let us check that such a robotic model is well adapted to the simulation of Codman's paradox, as illustrated in Fig. 3: if an abduction of π is made first, the initial rotation is made around axis-1 transforming axes 2, 3 into 2′, 3′ – axis 1 is unchanged – and then an extension of π can be made by rotation of the arm around axis 2′ (Fig. 3a); conversely, if a flexion of π is made first around axis 2, adduction can then be made around the unchanged axis 1 (Fig. 3b); in both cases, the final hand position is equal to the initial one, with a conjunct rotation angle equal to π. This value of the conjunct rotation can be determined from the so-called direct kinematic model of our robot model. Always according to Paul's notations associated with the Denavit–Hartenberg parameters this direct kinematic model can be written as the homogeneous matrix denoted by $\boldsymbol{T}_{\mathrm{arm}}(\theta_1, \theta_2, \theta_3)$ expressing the frame $\boldsymbol{R}_3$ components with respect to frame $\boldsymbol{R}_0$, resulting from the following equation:

$$\boldsymbol{T}_{\mathrm{arm}}(\theta_1, \theta_2, \theta_3) = \boldsymbol{A}_1(\theta_1)\boldsymbol{A}_2(\theta_2)\boldsymbol{A}_3(\theta_3) \tag{1}$$

where

$$\boldsymbol{A}_i(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the so-called Denavit–Hartenberg matrix, as rewritten by Paul [16]; while the first three columns respectively express, in homogeneous coordinates, the vectors $\boldsymbol{X}_i$, $\boldsymbol{Y}_i$, $\boldsymbol{Z}_i$, the last one expresses the point $\boldsymbol{O}_i$, with respect to the previous $\boldsymbol{R}_{i-1}$ frame. From the table given in Fig. 1b, we get:

$$\boldsymbol{T}_{\mathrm{arm}}(\theta_1,\theta_2,\theta_3)=\begin{bmatrix} -C_1S_2C_3-S_1S_3 & C_1S_2S_3-S_1C_3 & C_1C_2 & LC_1C_2 \\ -S_1S_2C_3+C_1S_3 & S_1S_2S_3+C_1C_3 & S_1C_2 & LS_1C_2 \\ C_2C_3 & -C_2S_3 & S_2 & LS_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where C_i and S_i are respectively for $\cos(\theta_i)$ and $\sin(\theta_i)$. One fundamental and, according to us, too less emphasized property of any serial chain robotic model results from the so-called Roth proof: in a general study about screws, Roth [20,21] considers a case of special motions resulting from the fact that two screws axes are assumed to be rigidly linked, i.e. the distance and the inclination are fixed between them. In the case of pure rotations about two axes ϕ_{A} and ϕ_{B}, "the order of the transformations about ϕ_{A} and ϕ_{B} is immaterial" – in [21], p. 600 – due to the fact that the transformation matrices about the second and mobile axis in the first sequence ϕ_{A} then ϕ_{B} and second sequence ϕ_{B} and ϕ_{A} are similar. If indeed, transformation matrices of the two rotations are respectively denoted by $\boldsymbol{A}$ and $\boldsymbol{B}$ with respect to a reference frame, the homogeneous matrix of the rotation about moved axis ϕ_{B} into ϕ_{B2} is given by $\boldsymbol{B}_2=\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1}$ which is a typical similarity transformation – see for example [18] – leading to $\boldsymbol{B}_2=(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{-1})\boldsymbol{A}=\boldsymbol{A}\boldsymbol{B}$. The term of Roth's proof was introduced by some biomechanicians in their studies of coordinates systems applied to motor limb physiology [23,2,8] – see also [3], without however an explicit reference to Roth's proof. We elsewhere showed that Roth's proof can also be applied to any robot defined as a serial chain whose bodies are rigidly linked by revolute or prismatic joints in order to show that the order of joint controls has no effect on the resulting location of the robot flange [24]. In this way, robot joint motions can be called "similarity-transformation motions" in Roth's meaning, and also it follows that joint space is a vector space. Because the joint control order is immaterial, there is no ambiguity in any so-called direct kinematic model of a robot in the form $\boldsymbol{T}$ $(q_1,\ldots,q_n)$, where $\boldsymbol{T}$ is the homogeneous transformation from robot base frame to robot hand frame and $q_1,\ldots,q_n$ are the joint variables. Consequently, the look for the conjunct rotation predicted by Codman's paradox can be considered here as the look for the θ_{3c} value solving the matrix equation:

$$\boldsymbol{T}_{\mathrm{arm}}(\theta_1,\theta_2,0)=\boldsymbol{T}_{\mathrm{arm}}(0,0,\theta_{3c})\Rightarrow\begin{bmatrix} -C_1S_2 & S_1 & C_1C_2 & LC_1C_2 \\ -S_1S_2 & C_1 & S_1C_2 & LS_1C_2 \\ C_2 & 0 & S_2 & LS_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 & L \\ S_{3c} & C_{3c} & 0 & 0 \\ C_{3c} & -S_{3c} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

We easily deduce that, for non-zero θ_1 and θ_2 angles, this equality is true if and only if $\theta_1=\theta_2=\pi$ and, in this case, $\theta_{3c}=\pi$, which is in full accordance with both considered examples for illustrating Codman's paradox. As a conclusion, such a purely kinematic approach inspired by robotics appears to be one way to "solve" Codman's paradox, in its first two-step version. But we think that a robotic model can bring some supplementary material in the understanding of Codman's paradox: robotics distinguishes the joint space, which is the inner space of joint angles, from the operational or task space, which is the outer physical space of positions and orientations reached by the chosen end-point of the robot. This operational space can also be defined by the final frame, $\boldsymbol{R}_3$, in the case of this first robot model. According to a robotic point of view, the final location of the hand after the movement sequence of Codman's paradox is limited, since the final position is the same as the initial one, to a change of orientation in the operational space. This is a phenomenon well known in robotics: when we move the robot arm, without moving its wrist, both position and orientation of the hand change. Always according to a purely robotic point of view, the notion of conjunct rotation has no meaning and the fact that the two joint vectors $[\pi,\pi,0]^{\mathsf{T}}$ and $[0,0,\pi]^{\mathsf{T}}$ generate a same final hand frame is a consequence, especially in the case of this location, of a double joint solution to reach it. But, as well underlined by Codman himself, what is truly paradoxical in this movement sequence is that, in its final attitude, the arm, which was initially in a neutral rotation around its long humerus, is now in extreme external rotation. No robotic model can explain this phenomenon because joint 3 was not controlled during the two-step Codman's movement sequence. According to us, this simple fact highlights that Codman's paradox is not a purely kinematic paradox easy to solve by some combination of rotations of a kinematic model of the shoulder joint, but both a motor and kinematic paradox. This appears more clearly through the second type of Codman's paradox movement sequence that we consider now.

3. Second complexity level of Codman's paradox and the question of motor redundancy

3.1. An over-actuated robotic model for simulating Codman's paradox

MacConaill [11] reconsiders Codman's paradox by substituting to the two-step closed-loop movement a three-step closed-loop movement defined, for example, as follows: from same neutral initial position as considered earlier, a first

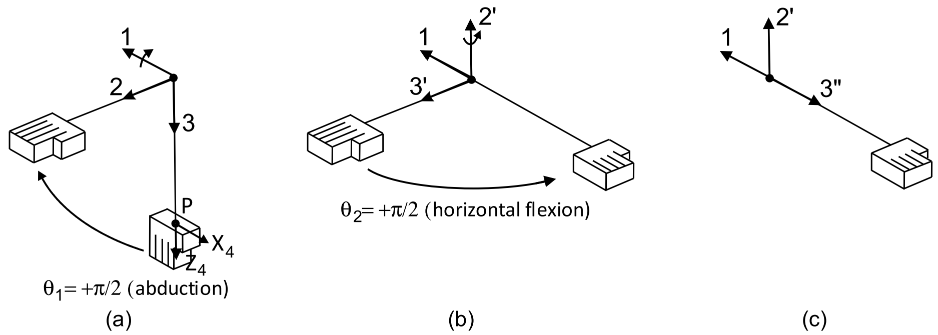
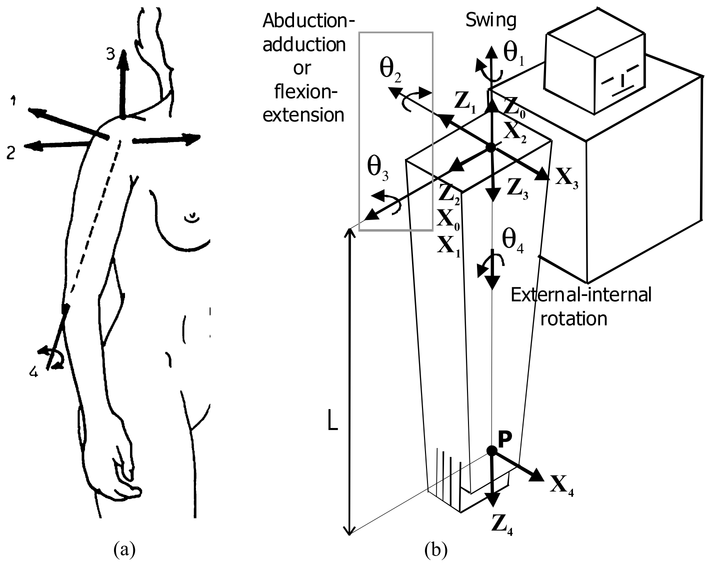


Fig. 4. Impossibility to realize with a 3 d.o.f. robot model the sequence of three elementary joint movements imposed by MacConaill's version of Codman's paradox. (a) First abduction movement changing the initial axis sequence $(1, 2, 3)$ into $(1, 2', 3')$. (b) Second swing movement changing the axis sequence $(1, 2', 3')$ into $(1, 2', 3'')$. (c) Resulting singular configuration due to the confusion between axis 1 and axis $3''$ making impossible the final extension back movement.



elementary movement	a_i	α_i	d_i	θ_i
swing	0	$+\pi/2$	0	θ_1
abd./add. or flex./ext.	0	$+\pi/2$	0	$\theta_2+\pi/2$
abd./add. or flex./ext.	0	$+\pi/2$	0	$\theta_3-\pi/2$
int./ext. rotation	0	0	L	θ_4

(c)

Fig. 5. Over-actuated model of the arm driven by a sequence of four revolute joints with intersecting axes. (a) Kapandji's scheme combining three shoulder fixed axes with the mobile axis of the long humerus about itself – in [10], page 3. (b) Proposed serial chain 4-axis robot model with the ordered sequence of joints 1 to 4: while joint 1 realizes a vertical swing around a fixed axis, and joint 4 corresponds to the external–internal rotation of the arm similar to this of Kapandji's scheme, joint 2 and joint 3 are now responsible for abduction–adduction or flexion–extension movements according to the spatial location of their corresponding axis – see text. (c) Corresponding Denavit–Hartenberg table.

abduction movement of $\pi/2$ is performed, then a second swing or horizontal flexion movement of $\pi/2$ is considered before a final vertical flexion, to bring back the arm at its initial position. Let us see how this sequence of three joint movements can be simulated with our robotic model: abduction is performed by a rotation around axis 1, then swing by a rotation around axis $2'$ but neither axis $2'$ or axis $3''$ is then able to realize the final back movement in extension, as illustrated in Fig. 4, due to the singularity configuration resulting from the confusion between axis 1 and axis $3''$. The so-called gimbal-lock phenomenon is well known in theory of mechanisms: the 3 d.o.f. ball-and-socket joint becomes singular when two of its axes come to be the same. But, in daily life experience, it is clear that no such gimbal-lock phenomenon occurs in human shoulder joint. The great originality of MacConaill's version of Codman's paradox would then be, according to us, to highlight the possibility for the shoulder complex to avoid this singularity. One way to model such a kinematic possibility,

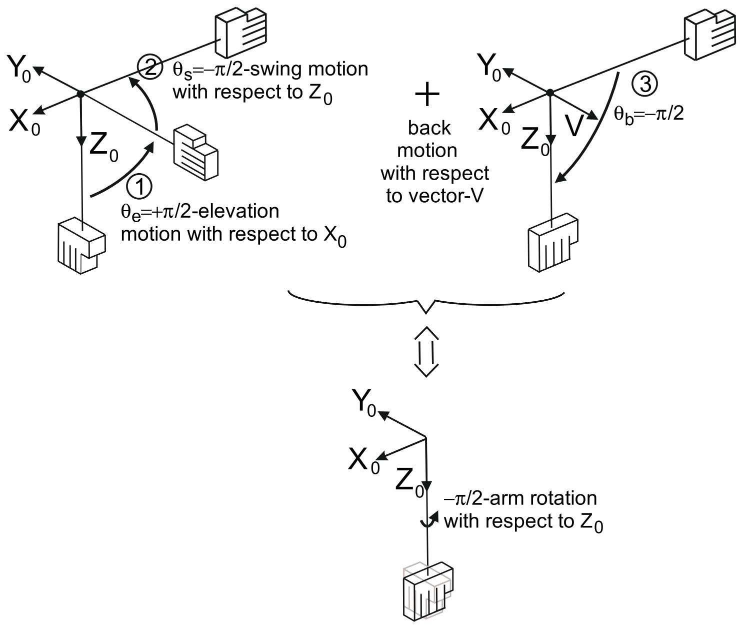


Fig. 6. Cheng's law illustrated in the case of a three-step closed-loop motion: $+\pi/2$ elevation, $-\pi/2$ swing and back motion: the $-\pi/2$ angle corresponding to the resulting conjunct rotation is indeed equal to the swing angle (the initial hand location is drawn in light grey and the final one in black).

without abandoning the reference to elementary shoulder movements, consists in considering an over-actuated model of the shoulder ball-and-socket joint. If the number of degrees of freedom of this model is always equal to 3, its mobility – i.e. the number of independent joint control parameters – is now equal to 4, according to the following four axes sequence: the first one is a vertical axis realizing swing motions, the second and third ones are perpendicular axes realizing according to the context abduction–adduction or flexion–extension movements, and the fourth one is the external–internal rotation axis with which the conjunct rotation is associated. It is worthy to note that, while the first and second axes in our previous 3 d.o.f. robot model respectively corresponded to shoulder abduction–adduction and shoulder flexion–extension, it is no longer the case of this model, due to the presence of the first swing axis, which can lead to interpret joint 2 and joint 3 movements either like shoulder abduction–adduction or shoulder flexion–extension. It is also interesting to remark that our model can be considered as some robotic interpretation of the amazing scheme proposed by Kapandji [10] and reproduced in Fig. 5a: this model combines three fixed axes ('transverse axis', 'antero-posterior axis', vertical axis', respectively numbered 1, 2, 3 in the figure) with the mobile fourth axis of arm rotation about the long axis of the humerus. It is worthy to note that, in our robotic model shown in Fig. 5b, only the first axis is fixed. By using the Denavit–Hartenberg table shown in Fig. 5c, we derive, as done in the previous section, the following homogeneous transformation from frame $\boldsymbol{R}_0$ to frame $\boldsymbol{R}_4$ associated with the hand, denoted by $\boldsymbol{T}_{\text{arm}+}$:

$$\boldsymbol{T}_{\text{arm}+}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{bmatrix} -(C_1S_2S_3+S_1C_3)S_4+C_1C_2S_4 & (C_1S_2S_3+S_1C_3)S_4+C_1C_2C_4 & C_1S_2C_3-S_1S_3 & L(C_1S_2C_3-S_1S_3) \\ -(S_1S_2S_3-C_1C_3)C_4+S_1C_2S_4 & (S_1S_2S_3+C_1C_3)S_4+S_1C_2C_4 & S_1S_2C_3+C_1S_3 & L(S_1S_2C_3+C_1S_3) \\ C_2S_3C_4+S_2S_4 & -C_2S_3S_4+S_2C_4 & -C_2C_3 & -LC_2C_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Let us now show the relevance of this model for Codman's paradox simulation.

3.2. *Towards a general definition of Codman's paradox*

As particularly well highlighted by Cheng [4], "[. . .] there is no study explaining why there are different examples [. . .] when referring to Codman's paradox, and what is the common law that governs the motion in the different examples" (p. 1201). Consequently, Cheng proposed the following original definition of Codman's paradox (idem, pp. 1202–1203): "1. Codman's paradox involves a closed-loop motion that consists of three sequential rotations of the long axis of the arm in reference coordinate system. 2. The first rotation is about an axis that is perpendicular to the initial position of the long-axis of the arm. The second rotation is about an axis that coincides with the initial position of the long-axis. The third rotation is to move the long-axis back to its initial position. 3. The questioned axial rotation angle is defined as a rotation about the long-axis." And, in relation to this general definition, he proposes the following law of motion: '*The equivalent axial rotation angle is equal to the angle of swing*' (idem, p. 1203), where "swing" in Cheng's terms corresponds to the second long-axis rotation. Let us see first how we could demonstrate Cheng's law of motion.[1] Let us call Z_0 the initial fixed long-axis arm and X_0 an arbitrary axis perpendicular to Z_0, as illustrated in Fig. 6.[2] The result of an elevation of angle θ_e around a fixed X_0 axis followed by a "swing" of angle θ_s around a fixed Z_0 axis and finally the back motion considered in Cheng's law around some unknown vector $\boldsymbol{V}$ of an unknown angle θ_b can be expressed by a sequence of three successive rotations, performed in the fixed frame $\boldsymbol{R}_0$. Cheng's law can now be expressed according to the following mathematical formulation – we would remind that rotation matrices must be left multiplied because all rotations are defined in the same fixed frame, while our previous robotic models imposed a right multiplication of homogeneous matrices due to the definition of each rotation in a current frame:

$$\forall\theta_e \in \boldsymbol{R},\ \forall\theta_s \in \boldsymbol{R},\ \exists\boldsymbol{V} \in \boldsymbol{R}^3, \exists\theta_b \in \boldsymbol{R} \text{ such as: } \mathbf{Rot}(\boldsymbol{V}, \theta_b)\mathbf{Rot}(\boldsymbol{Z}_0, \theta_s)\mathbf{Rot}(\boldsymbol{X}_0, \theta_e) = \mathbf{Rot}(\boldsymbol{Z}_0, \theta_s) \tag{5}$$

where **Rot** designates the vector rotation matrix-operator whose first argument is a rotation vector and the second argument is a rotation angle. We deduce from (5) the following matrix equation:

$$\mathbf{Rot}(\boldsymbol{V}, \theta_b) = \mathbf{Rot}(\boldsymbol{Z}, \theta_s)\mathbf{Rot}(\boldsymbol{X}_0, -\theta_e)\mathbf{Rot}(\boldsymbol{Z}, -\theta_s) = \begin{bmatrix} C_e + C_s^2(1-C_e) & C_sS_s(1-C_e) & -S_sC_e \\ C_sS_s(1-C_e) & 1-C_s^2(1-C_e) & C_sS_e \\ S_sS_e & -C_sS_e & C_e \end{bmatrix} \tag{6}$$

where C_e, S_e, C_s, S_s are respectively for $\cos(\theta_e)$, $\sin(\theta_e)$, $\cos(\theta_s)$ and $\sin(\theta_s)$. Let us now consider the well-known formula:

$$\mathbf{Rot}(\boldsymbol{V}, \theta_b) = \cos(\theta_b)\boldsymbol{I}_3 + \big(1-\cos(\theta_b)\big)\boldsymbol{n}\boldsymbol{n}^\mathsf{T} + \sin(\theta_b)[\tilde{\boldsymbol{n}}] \tag{7}$$

where $\boldsymbol{V} = [l_1\ l_2\ l_3]^\mathsf{T}$ and:

$$\boldsymbol{n}\boldsymbol{n}^\mathsf{T} = \begin{bmatrix} l_1^2 & l_1l_2 & l_1l_3 \\ l_1l_2 & l_2^2 & l_2l_3 \\ l_1l_3 & l_2l_3 & l_3^2 \end{bmatrix} \quad \text{and} \quad [\tilde{\boldsymbol{n}}] = \begin{bmatrix} 0 & -l_3 & +l_2 \\ +l_3 & 0 & -l_1 \\ -l_2 & +l_1 & 0 \end{bmatrix} \tag{8}$$

which leads to:

$$\mathbf{Rot}(\boldsymbol{V}, \theta_b) = \begin{bmatrix} C\theta_b + (1-C\theta_b)l_1^2 & l_1l_2(1-C\theta_b) - l_3S\theta_b & l_1l_3(1-C\theta_b) + l_2S\theta_b \\ l_1l_2(1-C\theta_b) + l_3S\theta_b & C\theta_b + (1-C\theta_b)l_2^2 & l_2l_3(l-C\theta_b) - l_1S\theta_b \\ l_1l_3(1-C\theta_b) - l_2S\theta_b & l_2l_3(1-C\theta_b) - l_1S\theta_b & C\theta_b + (1-C\theta_b)l_3^2 \end{bmatrix} \tag{9}$$

Let us assume that $\boldsymbol{V}$ is a unit-vector. We get from Eqs. (6) and (9):

$$\mathrm{Trace}\big(\mathbf{Rot}(\boldsymbol{V}, \theta_b)\big) = 1 + 2C\theta_b = 1 + 2C\theta_e \tag{10}$$

from which we deduce: $\theta_b = \pm\theta_e$ and then from equality of terms $(3, 3)$ in both matrices of Eq. (6) and Eq. (9): $l_3 = 0$; finally, we can check that two solutions solve our problem:

$$\boldsymbol{V} = [C\theta_s, S\theta_s, 0]^\mathsf{T} \quad \text{with } \theta_b = -\theta_e \quad \text{or } \boldsymbol{V} = [-C\theta_s, -S\theta_s, 0]^\mathsf{T} \quad \text{with } \theta_b = +\theta_e \tag{11}$$

We will consider the first one. Let us illustrate it in the example shown in Fig. 6: the arm long axis is initially set vertically, defining the $\boldsymbol{Z}_0$ direction; a $\theta_e = +\pi/2$ elevation is first performed around the fixed $\boldsymbol{X}_0$-axis and then a $\theta_s = -\pi/2$ swing is performed around the fixed $\boldsymbol{Z}_0$-axis. From Eq. (11) we get $\boldsymbol{V} = [0, -1, 0]^\mathsf{T}$ with $\theta_b = -\pi/2$, and we can check that a third rotation of $-\pi/2$ around fixed $\boldsymbol{Y}_0$-axis bring back the arm long axis in its initial position, with a $-\pi/2$ conjunct rotation.

Our demonstration of Cheng's law only used general properties of spatial vector rotations, without considering any joint specificity. This is due, according to us, to the fact that all rotations are defined in a fixed frame. In particular, the swing rotation axis and the long-arm rotation axis are the same, although, as noted by Cheng [4] himself, "[...] swing is generally used to describe a rotation around a vertical axis" (p. 1203). Our robotic model precisely proposed to distinguish a fixed vertical swing axis from a mobile rotation arm axis, as justified by our daily experience (see Fig. 7). Moreover, because Cheng's definition refers to a three-step sequence, it appears to be badly adapted for two-step Codman's paradox sequences

[1] In a reply to the letter to the editor written by Stepan and Otahal [22], Cheng [4] proposes a mathematical demonstration of his law; we decided to propose an alternative demonstration of Cheng's law, because it will help us to better justify our own mathematical approach.

[2] In his 2006-paper, Cheng considers a reference frame in which is defined the initial arm long axis by its longitude and latitude. Because elevation and swing in Cheng's law are both defined with respect to the initial long-axis positioning, we decided in this study to choose a reference frame $\boldsymbol{R}_0$ directly linked to the initial long-axis arm positioning. Our demonstration is consequently independent of this initial long-axis position.

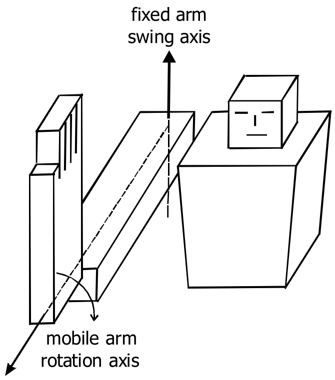


Fig. 7. Justification of the vertical fixed swing axis: when, for example, the arm is placed in the horizontal plane, it can both horizontally swing and rotate its forearm around its long axis.

which involve no swing. This is why, inspired by Cheng's work, we propose the following revised definition of Codman's paradox:

Assuming the arm initially set in a neutral attitude, i.e. with zero joint angles, Codman's paradox involves a closed-loop, eventually repeated, motion of the arm's long axis that consists of a sequence of abduction–adduction and flexion–extension relative movements, mixed with eventual swings about a fixed vertical axis – relative means here that successive movements are performed with respect to the current axis.

The third point of Cheng's definition questioning the resulting axial rotation angle keeps the same. It is important to note that, by comparison with Cheng's redefinition of Codman paradox, we force the arm to be initially placed in a rest position along the body. In his 2006-paper, Cheng reports examples with non-zero initial arm positions. Our robotic model is able to deal with such initial situations; however, a closed-loop movement of the arm in full extension, starting from a non-neutral attitude, corresponds to what is called, in joint physiology, 'arm circumduction'. Arm circumduction is generally defined as a movement of the shoulder in a circular motion so that, if the elbow and fingers are fully extended, the subject draws a circle in the physical space. We consider that the determination of a possible conjunct rotation during any arm circumduction is a next step, beyond Codman's paradox analysis, in the understanding of shoulder movements, which we do not want to deal with in the framework of this article. Also, for limiting the complexity of our presentation, we will consider the initial arm situation imposed by Codman and MacConnail analyses.

Let us apply our relative 4R kinematic structure to this revised definition of Codman's paradox. Due to the independence of joint movements inside the considered (θ_1, θ_2, θ_3, θ_4)-sequence, it is worthy to note that the resulting axial rotation is independent of the order in which are performed swing, abduction–adduction, or flexion–extension movements. The resulting axial rotation angle – or conjunct rotation angle – we note θ_{4c} then results from the following matrix equation to be solved:

$$\begin{aligned}
&\boldsymbol{T}_{\text{arm}+}(\theta_{1S},\theta_{2S},\theta_{3S},0)=\boldsymbol{T}_{\text{arm}+}(0,0,0,\theta_{4c})\\
&\Rightarrow\begin{bmatrix}-C_{1S}S_{2S}S_{3S}-S_{1S}C_{3S} & C_{1S}C_{2S} & C_{1S}S_{2S}C_{3S}-S_{1S}S_{3S} & L(C_{1S}S_{2S}C_{3S}-S_{1S}S_{3S})\\ -S_{1S}S_{2S}S_{3S}+C_{1S}C_{3S} & S_{1S}C_{2S} & S_{1S}S_{2S}C_{3S}+C_{1S}S_{3S} & L(S_{1S}S_{2S}C_{3S}+C_{1S}S_{3S})\\ C_{2S}S_{3S} & S_{2S} & -C_{2S}C_{3S} & -LC_{2S}C_{3S}\\ 0 & 0 & 0 & 1\end{bmatrix}\\
&=\begin{bmatrix}S_{4c} & C_{4c} & 0 & 0\\ C_{4c} & -S_{4c} & 0 & 0\\ 0 & 0 & -1 & -L\\ 0 & 0 & 0 & 1\end{bmatrix}
\end{aligned}\tag{12}$$

where θ_{1S}, θ_{2S} and θ_{3S} are, respectively, the sum of angles θ_1, θ_2 and θ_3 performed during the closed-loop sequential movement, and C, S are respectively for the cosine and sinus of the considered angles. Let us still emphasize that due to Roth's proof and its application to serial chain robots, the order in joint movement sequence has no effect on the final hand location, making it possible to gather the different values of a given joint variable into their sum. It clearly appears, first, from the comparison between the last columns of the two matrices of Eq. (12), that the closed-loop movement condition of Codman's paradox is realized if and only if $C_{2S}C_{3S}=1$, which corresponds to only two choices for the variables θ_{2S} and θ_{3S}: $(\theta_{2S},\theta_{3S})=(0,0)$ and $(\theta_{2S},\theta_{3S})=(\pi,\pi)$. In the first case, it also appears, by a simple comparison of other matrix terms that $\theta_{4c}=-\theta_{1S}$, and, in the second case, that $\theta_{4c}=\pi-\theta_{1S}$. Let us check this result on MacConaill's examples. In the first one, the lateral abduction is performed by $\theta_2=+\pi/2$, followed by a swing corresponding to a joint-1 control

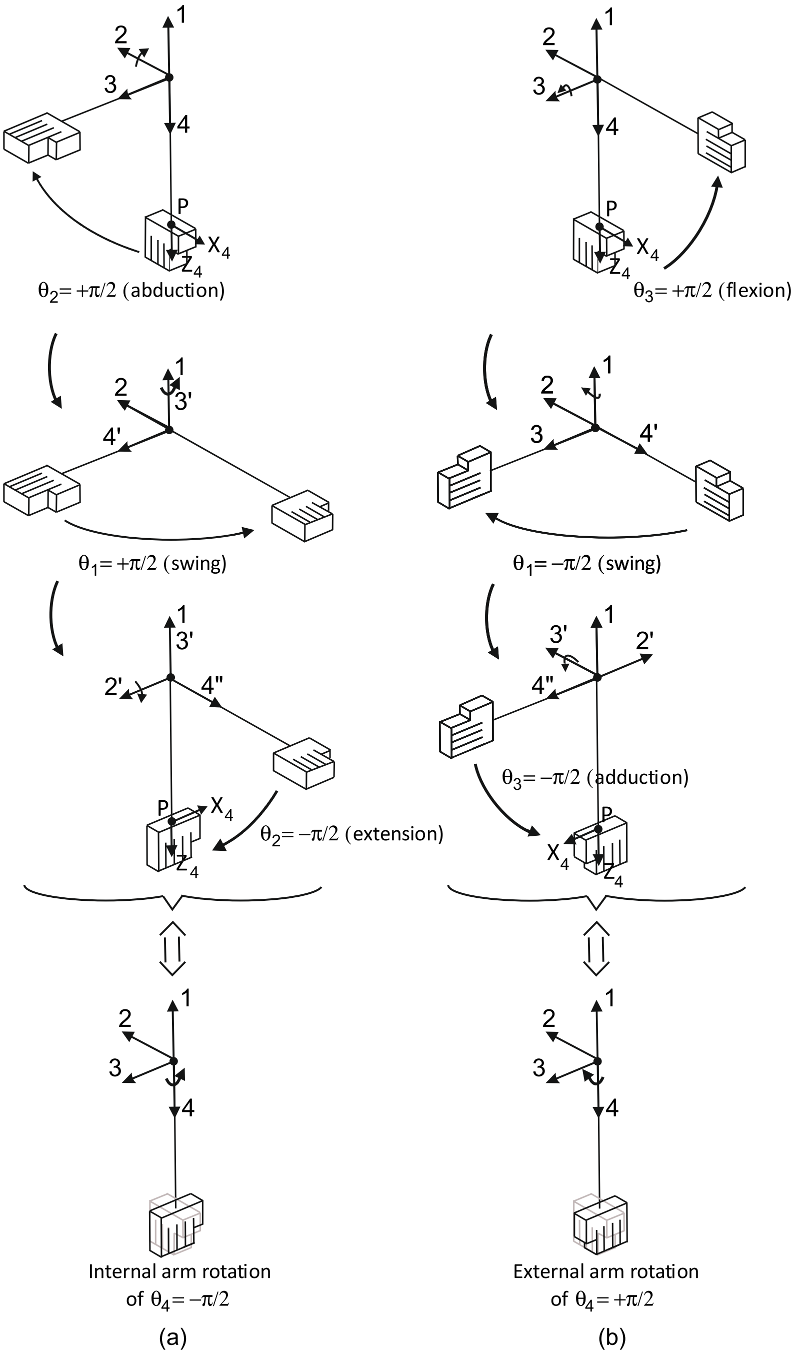


Fig. 8. Simulation of our over-actuated robot model applied to the three-step MacConaill's version of the Codman's paradox: (a) the final hand orientation resulting from a first θ_2 angle equal to $+\pi/2$ (abduction), followed by a θ_1 angle equal to $+\pi/2$ (swing), and finally a θ_2 angle equal to $-\pi/2$ (extension) is equivalent to the one resulting from a θ_4 angle equal to $-\pi/2$ (internal rotation). (b) The final hand orientation resulting from a first θ_3 angle equal to $+\pi/2$ (flexion), followed by a θ_1 angle equal to $-\pi/2$ (swing) and finally a θ_3 angle equal to $-\pi/2$ (adduction) is equivalent to the one resulting from a θ_4 angle equal to $+\pi/2$ (external rotation).

$\theta_1 = +\pi/2$ while the final movement is made by a $\theta_2 = -\pi/2$, as illustrated in Fig. 8a. Consequently, we are in the case $\theta_{2S} = +\pi/2 - \pi/2 = 0$, $\theta_{3S} = 0$, and we effectively check that the conjunct rotation angle measured around axis 4 is equal to: $\theta_{4c} = -\theta_{1S} = -\pi/2$. In the alternative example, as described for example in Cheng [5], the first movement would be a right angle flexion ($\theta_3 = +\pi/2$) followed by and horizontal swing of $\theta_1 = -\pi/2$, and a final adduction movement ($\theta_3 = -\pi/2$); we are still in the case $(\theta_{2S}, \theta_{3S}) = (0, 0)$, and we can check that the resulting conjunct rotation angle is well given by formula: $\theta_{4c} = -\theta_{1S}$, i.e. $+\pi/2$, as illustrated in Fig. 8b. Let us also remark that the two examples cited in Section 2 are

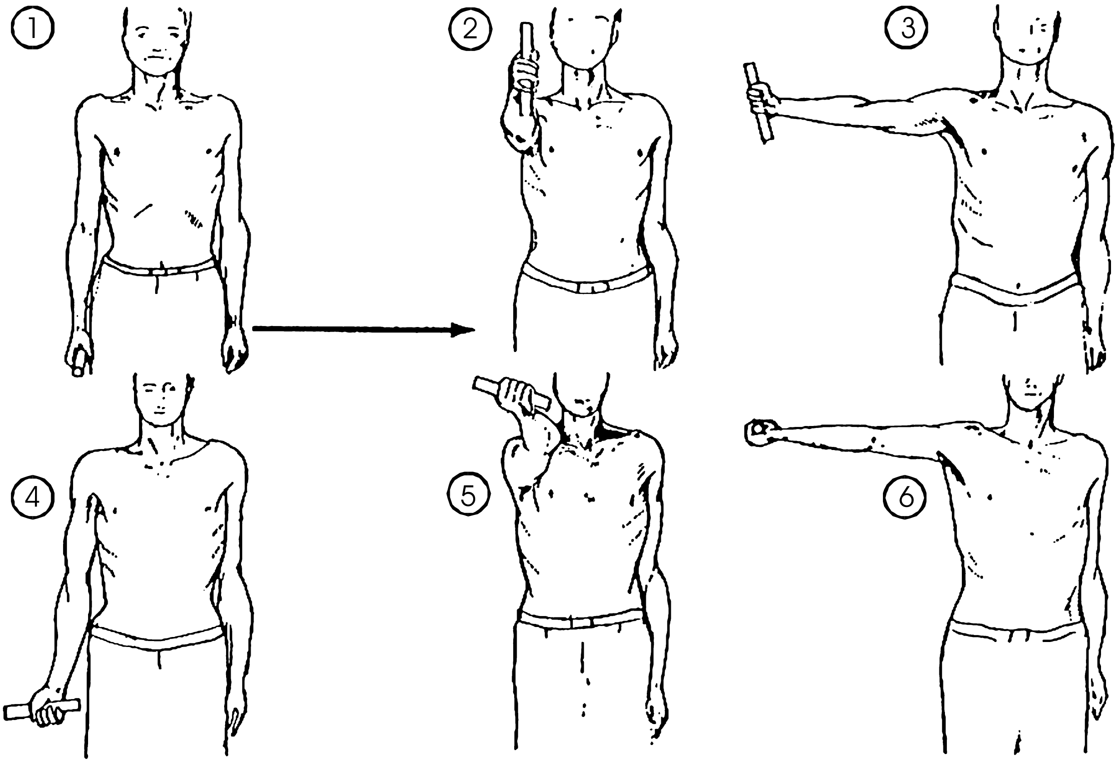


Fig. 9. Repetition of the three-step cycle considered in Fig. 8b and the resulting additive conjunct rotation – from MacConaill [13]. p. 76: from the initial attitude of image ①, the arm performs a first complete cycle, leading the arm back to its initial position with a resulting external arm rotation of $+\pi/2$ shown in image ④, before repeating the two first steps of the cycle (images ⑤ and ⑥), at the end of which it would try to perform an external arm rotation of π.

also included in this new robot model, since they correspond to the case $(\theta_{2S}, \theta_{3S}) = (\theta_2, \theta_3) = (\pi, \pi)$, with $\theta_1 = 0$ and so $\theta_{4c} = \pi - \theta_1 = \pi$, in accordance with the historical form of Codman's paradox. Furthermore, our model can also consider what MacConaill calls successive cycles in the performance of Codman's paradox movements, as illustrated in Fig. 9: images ① to ④ correspond to the realization of the cycle considered in Fig. 8b, while images ⑤ and ⑥ correspond to the repetition of this cycle, which would lead to an external arm-rotation of π, not shown in Fig. 9. According to our model, at the end of repetition of the two considered cycles, we always get: $(\theta_{2S}, \theta_{3S}) = (0, 0)$ but now $\theta_{1S} = -\pi/2 - \pi/2$ and so $\theta_{4c} = +\pi$, i.e. the arm is in full external rotation without permitting no supplementary rotation around the long humerus. It is worthy to remark that our model can simulate examples with joint angles different from $-\pi/2$, 0, $+\pi/2$, π but such examples are not fully, according to us, within the spirit of Codman's paradox, which is, essentially, a paradox to be tested and read on his/her own body.

To conclude, in accordance with our robotic model, Codman's paradox only occurs in two cases: both the sum of θ_2-angles and the sum of θ_3-angles are either equal to 0, or equal to π. In the first case, the resulting conjunct rotation is equal to minus the sum of swing angles, in the second case to π minus the same sum. This result is slightly different from the general law of motion proposed by Cheng although, as in Cheng's law, it appears that *the conjunct rotation, which does not result from a two-step closed-loop motion, only depends on the sum of swing angles*. If the two-step version of Codman's paradox is considered, the θ_2-angle corresponds to an abduction or adduction movement of value equal to π and the θ_3-angle corresponds to a flexion or extension movement of value equal to π, while no swing – corresponding to a zero θ_1-angle – is performed; if the three-step version of Codman's paradox is considered, the θ_2 and θ_3 angles correspond either to abduction–adduction movements or flexion–extension movements according to the situation in the physical space of their corresponding axis.

At his time, MacConaill [12] also attempts to derive, from Codman's paradox, a general law that he expressed as follows: 'It can be shown, both theoretically and experimentally, that the amount of conjunct rotation is directly proportional to the amount of backward swing' (p. 362) – 'backward swing' is here understood by MacConaill as a swing around a vertical axis, as we did it. No mathematical proof of this law is, however, given in the mentioned article as, at our knowledge, in other MacConaill's articles. MacConaill even goes further by claiming, always in the same mentioned paper, what he calls the 'Law of the Conservation of Axial Rotation', which he presents in the following way: 'the effect of any axial rotation is conserved unless means be taken either to prevent this or to undo it' (idem, p. 362). According to him, this law – for which he clearly says that he has not been able to prove it – 'plays the same part in mechanics of the joints as the law of the conservation of energy does in mechanics generally' (idem, pp. 362–363). How our analysis of Codman's paradox can help us to understand this law? On the one hand, the idea of a conservation of joint motions can be related to the fact that, according to Roth's proof, joint motions are independent on the order in which they are performed. Consequently, we can say that the external–internal rotation movements inside a complex limb motion add theirs effects, i.e. the performed angles. But the same thing could be said with abduction–adduction or flexion–extension movements. On the other hand, if we accept – although obviously excessive – the relevance of a comparison between a so-called internal rotation conservation principle and the energy conservation principle, vertical swing, as previously defined, would play the role of a "potential" internal rotation whose conjunct rotation would be the effect to be summed up with any voluntary internal rotation. This last property would justify *a posteriori* our choice for an over-actuated robotic model combining a vertical swing axis responsible for involuntary arm rotation with the own arm rotation axis responsible for external–internal rotation. Beyond Cheng's law expressed in a fixed frame and our robotic model of Codman's paradox, this is indeed this surprising ability of the upper limb to sum up voluntary and involuntary arm rotations that reveals Codman's paradox.

4. Conclusion

In his book, Codman [1] introduces his famous "pivotal" paradox as follows (p. 43): "And now we come to a curious paradox which I have only recently observed, although I have studied the motions of the shoulder for years. *You can prove that the completely elevated arm is in either extreme external rotation or in extreme internal rotation*." What will be called later "conjunct rotation" appears to be initially defined by its bounds. In the robotic perspective of our paper, this historical definition of Codman's paradox could be interpreted as some kind of confusion between the arm joint space (whose elements are here the joint vectors composed of the two angles performed from the rest position in abduction and flexion) and the operational space (whose elements are the hand position and orientation): when the arm is elevated due to a given joint vector, the position and also orientation of its hand change in a given reference frame, and if a closed-loop motion is performed, the position of the hand is finally the same, but not its orientation, giving the illusion of some involuntary, to use Kapandji's term, long-humerus rotation. At this stage, Codman's paradox could also be read as the shoulder musculature adaptation to large changes in the orientation of the hand, resulting from the combination of a full abduction and of a full flexion.

At a further level, introduced by MacConaill with its three-step closed-loop motion, Codman's paradox would introduce, according to us, another fundamental aspect of shoulder adaptation: its motor redundancy, which is notably evidenced by the absence of a gimbal-lock-like kinematic singularity. We proposed to represent this motor redundancy by a 4-axis robotic model whose sequence is made of a swing around a first fixed axis in series with three mobile axes for abduction–adduction, flexion–extension, and internal–external rotation. The conjunct-rotation angle deduced from this model is in good accordance with what is predicted by the different versions of Codman's paradox. Moreover, this original approach led us to propose a general definition of Codman's paradox in relation with Cheng's recent proposed definition. However, an important question remains, which we think may be deeply associated with Codman's paradox and its multiple commentaries: what is the best way for expressing the shoulder complex mobility and its apparent absence of kinematic singularities?

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