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Joseph Boussinesq's legacy in fluid mechanics

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ABSTRACT

Joseph Boussinesq was the most prolific of all French contributors to nineteenth-century fluid mechanics. His scientific production included a novel theory of solitary waves, the KdV equation for finite deformations of the water surface in an open channel, a systematic study of open channel and pipe flow based on the concept of effective viscosity, pioneering derivations of boundary layers and entrance effects, new exact solutions of the Navier–Stokes equation under geometrically simple boundary conditions, and the 'Boussinesq approximation' for heat convection in a moving fluid under gravity. Although his extraordinary skills were quickly recognized and rewarded, other experts in the field were often unaware even of his most important results and they ended up rediscovering some of them. Boussinesq's unusual background and the resulting peculiarities of his style explain this problematic diffusion. They also account for the richness of his legacy.

1. Introduction

Joseph Valentin Boussinesq was born in 1842 in the small town of Saint-André-de-Sangonis in Southern France, to a family of humble cultivators who would have wished him to stay with them. Young Boussinesq enjoyed playing on the Hérault River, throwing stones into its waters, diving deep into it, or just watching its sometimes impetuous flows. Intensely curious about natural phenomena, he questioned the local school teacher about the water waves and whirls he knew how to create. At the age of nineteen, he got his bachelor degree in mathematics at Montpellier and went on to teach this subject in provincial colleges for a few years. During this period, he read the French classics of mathematical physics and tried his hand on difficult theoretical problems. His teaching did not go as well as his reading. He had a weak voice and little authority over classes of boisterous teenagers. In 1864, in the small college of Le Vigan, an inspector was surprised to find, on Boussinesq's desk, a copiously annotated volume of Laplace's *Mécanique céleste*. This incident won Boussinesq a promotion to a more important college in Gap in the Alps. There he had enough time to begin a dissertation on the theory of optics (see [1,2]).

Boussinesq's first publication, a brief note of 1865 in the *Comptes rendus* [B1], bore on elasticity theory applied to the optical ether and was read by the great elasticity theorist Gabriel Lamé. His colleague Émile Verdet, who then was the most competent French expert on optics and molecular theories, directed a dissertation in which Boussinesq sought a complete explanation of the interplay of ether and matter in various phenomena including optical dispersion, crystal optics, optical rotation, and the optics of moving bodies. The outcome was a remarkably powerful theory that anticipated the central features of Hendrik Lorentz's later electromagnetic theory (see [3], pp. 256–258). This theory appeared with some delay in 1868 and not as a doctoral thesis [B5, B6]. In place of it, Boussinesq successfully defended a thesis on the propagation

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of heat in anisotropic media at the end of 1867 [B3]. The reason for this change of topic was Verdet's untimely death in June 1866 and the Sorbonne's choice for examining the dissertation: Charles Briot. This disciple of Augustin Cauchy had recently developed an optical theory in which the molecules of matter modified the density of the surrounding ether, whereas Boussinesq assumed the interstitial ether to be completely identical with the ether in vacuum. In order to avoid confrontation, Boussinesq replaced his optical work with a memoir he had recently written on heat propagation [see B32, pp. 267n–268n].

In the same period, Boussinesq addressed two other questions: the general form of molecular forces [B2], and viscous flow in narrow tubes [B4]. He got the attention of the engineer, physicist, and freshly elected Academician Adhémar Barré de Saint-Venant, who became his friend, spiritual guide, and zealous supporter. One may wonder what brought Boussinesq to investigate, at nearly the same time, questions of optics, elasticity theory, molecular theory, and fluid mechanics. One may also wonder what brought Boussinesq and Saint-Venant so close together. The answers to these two questions are intimately related.

Saint-Venant regarded himself as the continuator of a tradition inaugurated by Laplace, Siméon Denis Poisson, Augustin Cauchy, Augustin Fresnel, and André Marie Ampère, in which physical bodies were regarded as collections of point-like molecules interacting through distance forces. The elasticity of solid bodies, the rigidity of the optical ether, the compressibility of gases, the viscosity of fluids, and thermal phenomena were all regarded as resulting from this molecular structure. In Saint-Venant's philosophy, the first task of mathematical physics was to define a clear molecular picture from which, by averaging over the many molecules of macroscopic elements, differential equations could be derived. The molecular foundation could then be largely ignored and approximation procedures were devised to solve concrete problems of practical interest. This molecular-based and step-wise physics, which Poisson called *mécanique physique*, largely dominated French mathematical physics in the first third of the nineteenth century; but it had much receded by 1850. One reason was the demise of two central concepts of Laplacian physics: the imponderable, molecular fluids of heat and light. Another reason was the empiricist promotion of theories more directly related to observation. A third and last reason was the preference of the mechanics section of the Paris Academy for the *mécanique rationnelle* of ideal rigid bodies, inextensible threads, and incompressible fluids. Physical mechanics nonetheless survived this hostile environment thanks to the few adepts of Cauchy's molecular optics and to the sustained efforts of Saint-Venant and Boussinesq, who both believed that molecular mechanics remained the best foundation for all pure and applied physics and both ignored domains of physics, such as electromagnetism, that did not fit in this theoretical frame (see [4]; [5], pp. 233–234; [6], chap. 5).

In his very first works on optics and elasticity theory, Boussinesq adopted a molecular foundation from which he promptly derived macroscopic partial differential equations. The skills required in solving these equations being nearly the same as those needed for the equations of fluid mechanics, he was naturally driven to fluid-mechanical problems. He excelled in finding new analytical solutions where others would not even try, for instance for the Poiseuille flow in a tube of triangular section [B7]. As we will later see, he also had a unique flair in finding approximate equations of practical interest. On the conceptual side, he sought a molecular understanding of the distinction between solid and liquid. This is why, in 1867 [B2], he propounded a general molecular force formula in which the force depended not only on the distance between the two interacting molecules (as Saint-Venant had assumed), but also on the density of the surrounding molecules. In general, Boussinesq believed in a molecular foundation of mechanics and physics, and he developed his own vibrational theory of heat in an Amperean tradition distinct from the British and German kinetic theory of gases (see [B25]; [6], chap. 5; [7,8]).

The affinities between Saint-Venant and Boussinesq should by now be evident: they both pursued a three-stage physics involving molecular foundations, partial differential equations, and clever approximation strategies. They both favored highly mathematical theories that did not lose sight of practical engineering interests, and they occasionally injected empirical input into the theory. It is difficult to decide to which extent Saint-Venant determined the preferences of his protégé. What is certain is that from 1868 he frequently reported on Boussinesq's memoirs for the Academy of Sciences, that he suggested improvements and new problems, and that both men exchanged many letters on problems, methods, philosophy, and religion.[1] Plausibly it was Saint-Venant who directed Boussinesq from the Poiseuille flow to problems of hydraulic interests.

Three years after his doctoral thesis, in 1873, Boussinesq won a physics chair at Lille University. In 1886 he obtained the more prestigious chair of mechanics at the Sorbonne. He entered the Academy of Sciences in the same year, a few days after his mentor and ardent supporter Saint-Venant passed away. In 1896 he succeeded Henri Poincaré on the chair of mathematical physics and probability at the Sorbonne. He retired in 1918 and died in 1929.

A large proportion of Boussinesq's works were devoted to fluid mechanics and hydraulics. According to one of his disciples, Auguste Boulanger, he wrote no less than 1800 pages on this topic. It is dubious that anyone read the totality of this corpus. So it remains possible that important results of Boussinesq's have been overlooked to this day. Until recently, it was for instance not known that he had obtained the KdV equation some twenty years before Korteweg and de Vries. The best one can do with a limited amount of time and patience is to identify a few major results and situate them in Boussinesq's scientific biography.

[1] The correspondence between Boussinesq and Saint-Venant is held at the library of the Institut de France.

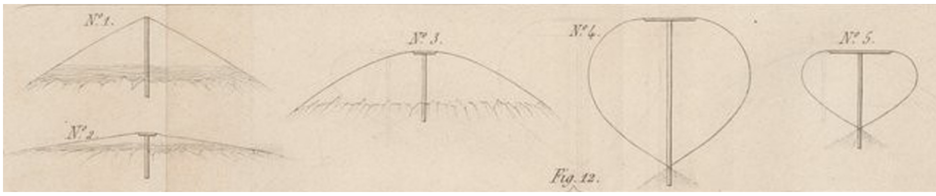

Fig. 1. Some of Savart's water bells. From [12], plate 5.

2. Regular flow

2.1. Poiseuille flow

The first fluid-mechanical problem that Boussinesq tackled was, as already mentioned, the Poiseuille flow in pipes of small section [B4, B7]. Being unaware of Stokes' pendulum memoirs, Boussinesq rediscovered the no-slip condition on the walls, with the same justification as Stokes: the dynamical equilibrium of a thin layer of fluid next to the wall requires the shear stress to be of the same order of magnitude on both sides of the layer, which seems intuitively impossible if the shear is infinitesimal on one side and finite on the other. With this boundary condition, Boussinesq computed the discharge in pipes of circular, elliptical, rectangular, and triangular sections. As became typical for him, he extended the problem to the limits of computability by considering non-stationary flows, and even flows in a curved channel or meander. In the latter case, he derived the helicoidal character of the flow and used it to explain the formation of meanders ([B7], pp. 378–379, 413–421), as James Thomson (Kelvin's brother) would do eight years later in a more qualitative manner [9] (see [10]).

Boussinesq then noted that in pipes or channels of hydraulic size, the laminar viscous flow implied absurdly high velocities for the central filets of water and proposed that in such cases even the tiniest asperities on the walls would induce eddies propagating through the fluid mass and enhancing the internal viscous damping by increasing the effective shearing surfaces within the fluid. He did not wait long to attack this complex hydraulic problem, which Saint-Venant then called a "hopeless enigma" [11, p. 774]. For a while, however, Boussinesq treated a few easier problems of the laminar kind: waves on water, water bells, and efflux.

2.2. Water bells

Let us begin with the water bells. In beautiful experiments published in 1833, Félix Savart [12,13] produced and studied axially-symmetric sheets of water by impact of a downward water jet on a disk of slightly larger diameter (see Fig. 1 and the recent study by Christophe Clanet [14]). Some forty years later, Boussinesq surmised that the shape and velocity of the sheet resulted from the combined action of capillarity forces and gravity on its particles [B9; B20, pp. 639–659].

Call z the vertical coordinate along the axis, r the distance from the axis, s the curvilinear abscissa along a generating curve of the sheet, θ the longitudinal angle, and e the thickness of the sheet. For the fluid particle within the elements ds and dθ, the product of the mass by the acceleration should be equal to the weight plus the capillary force:

$$\rho\, e\, \mathrm{d}s\, r\, \mathrm{d}\theta\, \ddot{\mathbf{r}} = \rho\, e\, \mathrm{d}s\, r\, \mathrm{d}\theta\, \mathbf{g} + 2\, C\, f\, \mathrm{d}s\, r\, \mathrm{d}\theta\, \mathbf{n} \tag{1}$$

wherein ρ denote the fluid density, C the mean curvature, f the capillarity constant, and $\mathbf{n}$ the normal unit vector. Taking into account the equation $Q = \rho 2\pi r e \upsilon$ for the conservation of the discharge, this gives:

$$\ddot{\mathbf{r}} = \mathbf{g} + kCr\upsilon\mathbf{n}, \quad \text{with } k = 4\pi f/Q \tag{2}$$

The radial projection of $\mathbf{n}$ being $-\mathrm{d}z/\mathrm{d}s$, its vertical projection $\mathrm{d}r/\mathrm{d}s$, and the velocity υ being equal to $\mathrm{d}s/\mathrm{d}t$, this is equivalent to the system

$$\ddot{r} = -2kCr\dot{z}, \qquad \ddot{z} = g + 2kCr\dot{r} \tag{3}$$

The mean curvature C is the average of the two principal curvatures:

$$\pm 2C = \frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}z}{\mathrm{d}s} + \frac{1}{r}\frac{\mathrm{d}z}{\mathrm{d}s}, \quad \text{with } \mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}z^2 \tag{4}$$

Boussinesq combined these equations to get the equations of motion of a fluid particle, and the equation of the water sheet by elimination of time. The latter equation reads:

$$(\upsilon - kr)\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}r}{\mathrm{d}z} + \left(1 + \frac{\mathrm{d}r^2}{\mathrm{d}z^2}\right)\left(k + \frac{g}{\upsilon}\frac{\mathrm{d}r}{\mathrm{d}z}\right) = 0, \quad \text{with } \upsilon = \sqrt{\upsilon_0^2 + 2gz} \tag{5}$$

In the case of negligible gravity, Boussinesq found the generating curve of the sheet to be a catenary. His adroit discussion of the true motion and the comparison with Savart's measurement are too complex to be summarized in a few lines. It included a derivation the stability condition $\upsilon - kr \geq 0$ for the stability of the sheet under small perturbations.

2.3. Efflux

Savart's water bells are a curiosity, whose main attraction for Boussinesq must have been their theoretical computability. In contrast, the efflux problem Boussinesq tackled in the same year was the most famous problem of the older hydrodynamics and had obvious hydraulic significance. This problem is to determine the quantity of water ejected through a small opening on a wall of a water container, the opening being at a considerable depth h from the water surface (and small compared to the radius of curvature of the wall). In his *Hydrodynamica* of 1738, Daniel Bernoulli derived the expression $\sqrt{2gh}$ of the water velocity at the free surface of the jet by what we would now interpret as an energy argument [15, pp. 258–260] (see [5], pp. 7–79). However, the discharge through the opening is not simply the product of $\sqrt{2gh}$ by the area of the opening, because the jet contracts when escaping from the container. Only after the contraction does the fluid velocity become uniform and perpendicular to the section of the jet, so that the discharge then is $\sqrt{2gh}$ times the section of the contracted vein.

Boussinesq did not try to determine this contraction.[2] Instead, he estimated the normal velocity profile in the plane of the opening [B10; B20, pp. 536–569]. Through potential theory applied to the velocity potential, he proved that the normal velocity should vanish on the edge of the opening. Through intuition and empirical observation he surmised that the velocity should also vanish in the center of the opening for a circular opening of radius R. Further assuming the profile $u(r)$ to be an even function of the distance r from this center, he tried

$$u(r) = ar^2\left(1 - r^2/R^2\right) \tag{6}$$

In order to determine the coefficient a in this formula, he proved that the velocity potential within the container was the electrostatic potential for an electric charge spread on the surface of the opening with the density $u(r)$, and he equated the resulting value of the radial velocity at the rim with $\sqrt{2gh}$. Lastly, he obtained the discharge Q by integrating $u(r)$ on the opening. The result,

$$Q = 0.6566\pi R^2\sqrt{2gh} \tag{7}$$

agreed reasonably well with the experimental value for not too small openings.

2.4. Solitary waves

The most important of Boussinesq's early works in hydrodynamics concerned wave propagation on the surface of a liquid. For plane sine waves of small amplitude on water of constant and finite depth, he rediscovered the laws earlier established in Britain by George Biddel Airy, George Green, Philip Kelland, and George Stokes [B8, B19] (see [5], chap. 2). He also studied spherical waves and their diffraction under the same conditions; he determined the behavior of finite waves to second order in the amplitude; and he showed that viscosity modified water motion in a thin layer next to the bottom and to the water surface only.

As Boussinesq knew, the most interesting and most accurate experimental studies of water waves belonged to John Scott Russell in Scotland and Henry Bazin in France (see [5], pp. 47–60, 232–235). Most strikingly, in 1834 Russell had accidentally discovered that "solitary waves" could be propagated over large distances along a channel without much deformation (see [16], p. 61). Joseph Louis Lagrange had long ago proven that small waves of any shape propagate without deformation with celerity $\sqrt{gh}$ on water of small depth h. But Russell's solitary waves were not small enough to be described by Lagrange's theory. Airy and Stokes had shown that waves of finite amplitude were deformed during propagation on shallow water. They also knew that periodic sine waves on water of depth comparable to the wavelength propagated with celerity values depending on the wavelength.[3] This dispersion implies the deformation of any wave packet formed by superposition of sine waves. In brief, both the non-linearity of the equations of motion and their dispersive character implied the deformation of any swell during its propagation (see [5], pp. 66–72). For this reason, Stokes repeatedly denied the existence of solitary waves and suggested that Russell had underestimated the deformation of his waves.

Being unfamiliar with the British theoretical literature, Boussinesq had no prejudice against Russell's solitary waves. On the contrary, he assumed the existence of waves propagating without deformation in an open channel of rectangular section, and derived an approximate equation for the profile of such waves from Euler's hydrodynamic equation. This equation turned out to have solutions that faithfully represented Russell's solitary waves [B12].

In a given section of the channel, call x and y the horizontal and vertical coordinates, u and v the corresponding velocity components, and φ the harmonic potential from which they derive (the flow being incompressible and irrotational). Following a procedure by Lagrange, Boussinesq developed the velocity potential in a power series of the distance y from the bottom. To second order, this gives

$$\varphi(x, y) = \beta(x) - \frac{1}{2}\beta''(x)y^2 \tag{8}$$

[2] In his *Eaux courantes* [B20, pp. 548–552], Boussinesq gave the potential equation and the boundary conditions that in principle determine the jet, and he proved their invariance by the rescaling $\varphi \to \alpha\sqrt{2gh}\varphi$, $\boldsymbol{r} \to \boldsymbol{r}/\alpha$, $t \to t\sqrt{2gh}/\alpha$, from which the proportionality of the discharge with $\sqrt{2gh}$ follows.

[3] The dispersion relation is $\omega^2 = gk\tan kh$ for waves of the form $a e^{i(\omega t - kx)}$.

and

$$u = \alpha(x) - \frac{1}{2}\alpha''(x)y^2, \upsilon = -\alpha'(x)y \quad \big(\text{with } \alpha = \beta'\big) \tag{9}$$

for the general form of a solution satisfying the boundary condition $\upsilon(x,0) = 0$ at the bottom. Call σ the elevation of the surface above its original height h, and c the celerity of the wave. The conservation of the flux in a reference system bound to the wave implies

$$\int_0^{h+\sigma} u\,\mathrm{d}y = c\sigma \tag{10}$$

The resulting constraint on the unknown function α is

$$\alpha(h+\sigma) - \frac{1}{6}\alpha''(h+\sigma)^3 = c\sigma \tag{11}$$

At the lowest order of approximation, the cubic term is dropped on the left-hand side, and σ is neglected with respect to h, so that $\alpha = c\sigma/h$. At the next order of approximation, the latter value of α is injected in the cubic term, and σ is neglected with respect to h in this term only. This gives:

$$\frac{\alpha}{c} = \frac{\sigma}{h+\sigma} + \frac{1}{6}\sigma'' h \tag{12}$$

and

$$\frac{u}{c} = \frac{\sigma}{h+\sigma} + \frac{1}{6}\frac{\sigma''}{h}\big(h^2 - 3y^2\big), \quad \frac{\upsilon}{c} = -\frac{\sigma'}{h}y \tag{13}$$

Boussinesq then obtained the equation of the surface by injecting these expressions in the condition of zero pressure on the open surface[4]:

$$u^2 + \upsilon^2 - 2\partial_t\varphi + 2g(y-h) = 0 \quad \text{for } y = h+\sigma \tag{14}$$

As the potential φ is a function of $x - ct$ only, $\partial_t\varphi$ is the same as $-cu$.

In order to clarify subsequent approximations, it is convenient to introduce the dimensionless variables $\varepsilon = \sigma/h$, $\varepsilon' = \sigma'$, $\varepsilon'' = h\sigma''$. Boussinesq assumed the wave to be small and gently sloped, and therefore treated ε, ε'/ε, and $\varepsilon''/\varepsilon$ as small quantities. He thus got the equation of the surface

$$c^2 = gh\left(1 + \frac{3}{2}\sigma/h + \frac{1}{3}h^2\sigma''/\sigma\right) \tag{15}$$

where terms in ε^2, $\varepsilon'^2/\varepsilon$, ε'' and all smaller terms are neglected.[5] This equation may be rewritten as

$$\varepsilon'' = 3K\varepsilon - \frac{9}{2}\varepsilon^2, \quad \text{with } K = c^2/gh - 1 \tag{16}$$

A first integration yields

$$\varepsilon'^2 = 3\varepsilon^2(K - \varepsilon) \tag{17}$$

The maximum $\varepsilon' = 0$ of the corresponding curve is reached when $\varepsilon = K$. Consequently, the celerity of the wave is related to the height σ_M of its summit through

$$c = \sqrt{g(h+\sigma_M)} \tag{18}$$

which is Russell's formula. Boussinesq then integrated a second time to reach

$$\frac{\sigma}{h} = \frac{2K}{1+\cosh[\sqrt{3K}(x-ct)/h]} \tag{19}$$

The shape and celerity of a solitary wave (see Fig. 2) are therefore completely determined by its height.

[4] The other boundary condition, that a particle of the surface should remain on the surface, is a consequence of Eq. (10).

[5] Boussinesq kept the $\varepsilon'^2/\varepsilon$ terms, but neglected them when he integrated the equations.

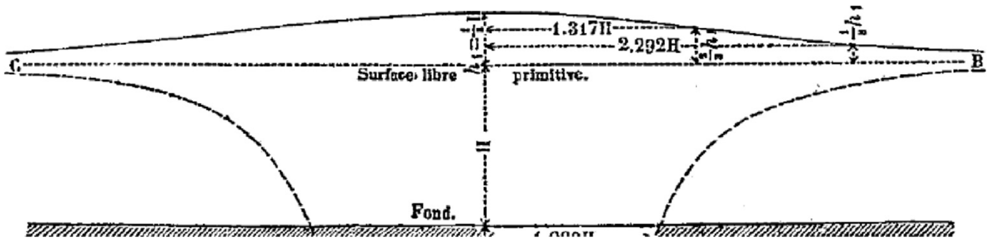

Fig. 2. Boussinesq's profile for a solitary wave whose elevation over the original free surface equals one third of the depth ($\sigma_M = h/3$ in my notation; $h_1 = H/3$ in Boussinesq's). From [B16], p. 70.

2.5. The Boussinesq equation

In order to derive the evolution of a wave of arbitrary shape, Boussinesq [B14, B15, B16] investigated the higher approximation, in which

$$\varphi = \beta - \frac{1}{2}\beta'' y^2 + \frac{1}{24}\beta'''' y^4 \tag{20}$$

The vanishing of pressure at the free surface gives

$$2g\sigma + 2\partial_t\varphi + (\nabla\varphi)^2 = 0 \quad \text{for } y = \sigma(x,t) \tag{21}$$

The condition that a particle originally on the surface should remain on the surface gives

$$\partial_y\varphi = \partial_t\sigma + \partial_x\varphi\partial_x\sigma \quad \text{for } y = \sigma(x,t) \tag{22}$$

At the lowest order of approximation, using dots for time derivatives and accents for derivatives with respect to x, these two conditions yield

$$\dot\beta = -g\sigma, \qquad \dot\sigma = -\beta'' h \tag{23}$$

The elimination of β gives Lagrange's wave equation

$$\ddot\sigma = gh\sigma'' \tag{24}$$

Consequently, at this order σ is the sum of a function of $x - c_0 t$ and a function of $x + c_0 t$, with $c_0 = \sqrt{gh}$. Boussinesq retained only the first component, which represents a perturbation traveling at the constant speed c_0 in the direction of increasing x.

At the next order of approximation, the two conditions give

$$\dot\sigma = -\beta'' h - \beta''\sigma - \beta'\sigma' + \frac{1}{6}\beta'''' h^3, \qquad \dot\beta = -g\sigma + \frac{1}{2}\dot\beta'' h^2 - \frac{1}{2}\beta'^2 \tag{25}$$

where $h + \sigma$ has been replaced with h in terms that have a derivative of third order or higher in factor. In order to eliminate β, Boussinesq derived the first equation with respect to time and the second twice with respect to x. This gives

$$\ddot\sigma = -\dot\beta'' h - \big(\dot\beta'\sigma\big)' - \big(\beta'\dot\sigma\big)' + \frac{1}{6}\dot\beta'''' h^3, \qquad \dot\beta'' = -g\sigma'' + \frac{1}{2}\dot\beta'''' h^2 - \frac{1}{2}\big(\beta'^2\big)'' \tag{26}$$

In the terms that follow the first, dominant term in each one of these equations, $\dot\sigma$ and $\dot\beta$ can be replaced by their first approximation (23), and the operators ∂_t and $-c_0\partial_x$ are interchangeable. This gives

$$\ddot\sigma = -\dot\beta'' h + g\big(\sigma^{2'}\big)'' - \frac{1}{6}gh^3\sigma'''', \qquad \dot\beta'' = -g\sigma'' - \frac{1}{2}gh^2\sigma'''' - \frac{1}{2}gh^{-1}\big(\sigma^2\big)'' \tag{27}$$

Hence follows the Boussinesq equation for the evolution of the perturbation:

$$\ddot\sigma = gh\sigma'' + \frac{3}{2}g\big(\sigma^2\big)'' + \frac{1}{3}gh^3\sigma'''' \tag{28}$$

In order to ease the integration of this equation, Boussinesq imagined a series of fictitious vertical planes moving in such a manner that the volume of liquid between two consecutive planes remains constant. The velocity w of these planes is easily seen to depend on their abscissa x in such a way that

$$\dot\sigma = -(\sigma w)' \tag{29}$$

With the notation

$$\gamma = \frac{3}{2} g\sigma^2 + \frac{1}{3} gh^3\sigma'' \tag{30}$$

Eq. (28) leads to

$$\partial_t(\sigma w) + c_0^2\sigma' + \gamma' = 0 \tag{31}$$

In terms of the auxiliary quantity

$$\chi = \sigma(w - c_0) - \gamma/2c_0 \tag{32}$$

this equation can be rewritten as

$$\dot{\chi} = c_0\chi' \tag{33}$$

granted that the operator ∂_t can be replaced with $-c_0\partial_x$ when applied to the small quantity γ. This means that χ is a function of $x + c_0 t$ only. As it also is a combination of quantities that are functions of $x - c_0 t$ vanishing at infinity, it must vanish. This implies

$$w = c_0 + \gamma/2c_0\sigma \tag{34}$$

and, approximately,

$$w^2 = gh\left(1 + \frac{3}{2}\sigma/h + \frac{1}{3}h^2\sigma''/\sigma\right) \tag{35}$$

Boussinesq then injected his expression of w into Eq. (29) to get the convective variation of the height of the fluid slices as

$$\dot{\sigma} + w\sigma' = -c_0\left(\frac{3}{2}\sigma/h + \frac{1}{3}h^2\sigma''/\sigma\right)' \tag{36}$$

Boussinesq verified that the volume, momentum, and energy of a swell evolving according to this equation were invariable.

Remembering that w is the velocity of constant-volume slices of the swell, the shape of a swell is permanent if and only if w is a constant c which represents the celerity of the wave:

$$c^2 = gh\left(1 + \frac{3}{2}\sigma/h + \frac{1}{3}h^2\sigma''/\sigma\right) \tag{37}$$

This condition is identical to Eq. (15), earlier reached by Boussinesq by a more direct method. For anyone familiar with the calculus of variations, this equation derives from the condition that the integral

$$M = \int_{-\infty}^{+\infty} (\sigma'^2 - 3\sigma^3/h^3)\mathrm{d}x \tag{38}$$

be a minimum for a fixed value of the energy integral $\int_{-\infty}^{+\infty} \rho g\sigma^2 \mathrm{d}x$. This remarkable property of the quantity M prompted Boussinesq to examine its evolution for arbitrary swells. He found it to be a constant of motion. From this property, he inferred that M measured the departure of a swell from a solitary wave, or the speed at which its shape varied in time. This remark justified the name *moment d'instabilité*. It also explained the ease with which Russell and Bazin had produced solitary waves [B16, p. 100]:

> If the moment of instability of a wave slightly exceeds the minimum value, the shape of the swell will oscillate about that of a solitary wave with the same energy, without ever differing much from the latter wave: indeed a notable difference would imply an increase of the moment of instability, which is impossible, since this moment does not vary in time; or, rather, a solitary wave will soon be formed; because frictional forces, which we have neglected so far, damp the oscillations of the effective form of the swell about its limiting form [...] And we may even conceive, in the absence of any stable form about which a wave might oscillate, that any swell susceptible, by its positive and moderate volume, to form a solitary wave with a height small enough not to break, should assume this form after a certain time. Thus is explained the ease with which solitary waves are produced.

2.6. The KdV equation

Boussinesq's success in this difficult problem depended on his familiarity with Lagrangian methods, on great flair in judging which terms should be retained in successive approximations, and on the intuitive artifact of fluid slices. Unlike modern writers on this topic, he did not rely on dimensionless constants and equations. He nevertheless understood that dispersive terms and non-linear terms both contributed to the evolution of finite swells in a channel of finite depth, and that the deformations caused by the two kinds of terms could compensate each other for properly shaped waves. The introduction of fluid slices of constant volume enabled him to replace the Boussinesq equation, which is of second order with respect to time derivation, with a first-order equation and to intuitively grasp the evolution of a general swell. His Eqs. (29) and (34) together yield

$$\dot{\sigma} = -\sqrt{gh}\left(\sigma + \frac{3}{4}\sigma^2/h + \frac{1}{6}h^2\sigma''\right)' \tag{39}$$

which is the KdV equation. Boussinesq gave this equation as well as a more direct derivation in footnote of his *Eaux courantes* of 1877 [B20, p. 360n]. The Dutch theorist Diederik Korteweg and his doctoral student Gustav de Vries rediscovered it in 1895 [17]. In the meantime, Lord Rayleigh had rediscovered the Boussinesq profile of a stationary wave in 1876 [18] and acknowledged Boussinesq's priority in this regard.

3. Tumultuous flow

In Boussinesq's agenda, the theory of water waves only was a first step in a general investigation of flow in open channels. Since the beginning of the century there had been many experimental and theoretical attempts to understand aspects of flow in rivers, torrents, and canals. Boussinesq, with Saint-Venant's probable guidance, was well aware of these mostly French studies. In order to appreciate his own contribution, it will be useful to recall the contents of a few of these older hydraulic studies.

3.1. Older theories of backwaters, hydraulic jumps, torrents, and tidal bores

A first question of interest for canal builders is the relation between the slope i of the canal (the sine of its inclination) and the discharge Q (the total quantity of water crossing one of its section in a unit time). For a long time, the answer to this question was based on Pierre du Buat's idea of a balance between the weight of a fluid slice and the friction of this slice on the wetted surface of the canal. The velocity U was regarded as roughly uniform within the slice, with a slide at the bottom of the canal. Calling ρ the density of water, g the acceleration of gravity, S the surface of a slice, χ the wetted perimeter, and F_U the retarding force per unit surface, du Buat's condition reads

$$\rho g S i = \chi F_U \tag{40}$$

Combined with an assumption for the form of F_U, usually taken to be the sum of a quadratic term and of a smaller linear term, this law gave the discharge $Q = SU$ as a function of the slope i. This was the basic theory of uniform permanent flow, for which the section of the channel is uniform, and its slope is constant and equal to the slope of the water surface (see [5], pp. 221–233).

Another problem of hydraulic interest was the formation of backwaters (*remous*), namely, the curving of the water surface when the flow is about to encounter a weir. It was a common practice to improve the navigability of rivers through a succession of weirs, chosen so that the depth of the backwater of the nth weir at the foot of the $(n-1)$th weir should exceed the minimal depth required for navigation. A lock on the side of each weir permitted the passage of the boats. Jean-Baptiste Bélanger [19], a Polytechnique-trained engineer of roads and bridges, gave the first theory of backwater in 1828, in answer to a prize question of the Metz Academy. In this problem, the water surface is no longer parallel to the surface of the channel, the pressure P of the water varies along the channel, and so does too the section S and the velocity u of the fluid slices (still assumed to be uniform within a slice). By Newton's second law, the sum of the pressure, gravity, and friction forces acting on a fluid particle must be equal to the product of its mass and its acceleration.[6] For a fluid particle at the bottom of the channel, this gives:

$$-\frac{\partial P}{\partial s} + \rho g \sin\gamma - \frac{\chi}{S}F_u = \rho\frac{\mathrm{d}u}{\mathrm{d}t} \tag{41}$$

wherein s is the curvilinear abscissa along the bottom of the channel, and γ the inclination of the bottom of the channel (see Fig. 3).

The pressure P at the bottom is given hydrostatically by $\rho g h \cos\gamma$, wherein h is the perpendicular depth; the velocity u is inversely proportional to the section S; the acceleration $\mathrm{d}u/\mathrm{d}t$ reduces to the convective acceleration $u\mathrm{d}u/\mathrm{d}s$; the section S is a function of the depth h and the abscissa s; and the friction at the bottom is taken to be $F_u = bu^2$. Taking all these conditions into account, Eq. (41) can be used to express the derivative $\mathrm{d}h/\mathrm{d}s$ as a function of s. In the simplest case for which the channel has a constant slope and a uniform, rectangular section, the resulting equation of backwaters is

$$\frac{\mathrm{d}h}{\mathrm{d}s} = \frac{h^3 - h_0^3}{h^3 - h_c^3}\operatorname{tg}\gamma, \quad \text{with } h_0 = \left(bq^2/\rho g \sin\gamma\right)^{1/3} \text{ and } h_c = \left(q^2/g\right)^{1/3} \tag{42}$$

[6] Bélanger assumed the frictional force to be uniformly distributed in the fluid section S, as should indeed be the case for it to be balanced by gravity in the uniform permanent case. Hence comes the form $\chi F_u/S$ of the frictional term in Eq. (41).

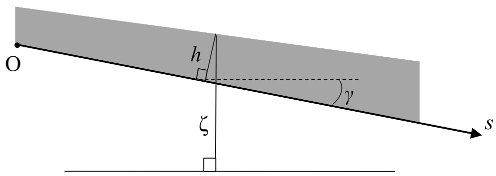

Fig. 3. Geometrical parameters for gradually varied channel flow.

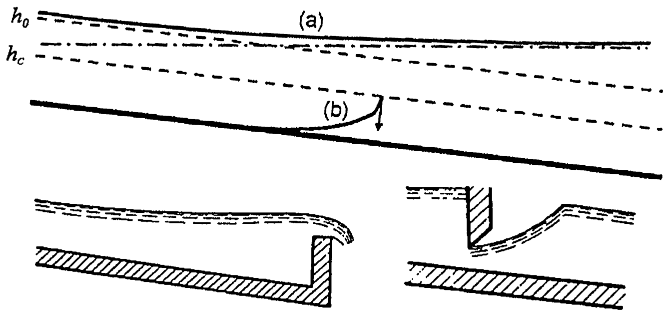

Fig. 4. The backwater curves in an inclined channel (from [5], p. 225), and their concrete realization according to Forchheimer [51, p. 181].

The letter q here denotes the discharge per unit breadth of the channel ($q = uh$), h_0 the depth that the flow would have in the uniform case, and h_c the critical depth for which the slope of the backwater curve becomes infinite.

In the frequent case of a swell ($h > h_0$) on a small-slope bed ($h_0 > h_c$), the curve $h(s)$ is concave and has an upstream asymptote parallel to the bed and a horizontal downstream asymptote (Fig. 4a). This means that the flow is asymptotically uniform in the upstream direction and then swells owing to a downstream cause, which could be a weir or the merging into a lake. In the case $h < h_c < h_0$, which would occur when water is forced through a sluice gate into a small-slope channel or when a high-slope channel turns into a small-slope one, the depth increases in the downstream direction until it reaches the critical value h_c for which the slope $\mathrm{d}h/\mathrm{d}s$ becomes infinite (Fig. 4b). The part of the curve close to this critical point cannot be trusted, for it contradicts the approximation of parallel-slice flow. Bélanger surmised that in this case the water level would suddenly increase to a value higher than critical and then again vary smoothly according to Eq. (42). He identified this behavior with the "hydraulic jumps" that the Italian hydraulician Giorgio Bidone [20] had studied in the 1820s (see [21]).

Another way to derive Bélanger's backwater equation is to equate the variation of the kinetic energy of a fluid slice during the time $\mathrm{d}t$ with the work of gravitational and pressure forces on this slice. The result of this balance is

$$\rho g \mathrm{d}\zeta + (\chi/S)F_u \mathrm{d}s + \mathrm{d}(\rho u^2/2) = 0 \quad \text{or } I \equiv -\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \frac{\chi F_u}{S\rho g} + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{u^2}{2g}\right) \tag{43}$$

wherein ζ denotes the height of the water surface and I its slope (see Fig. 3). Jean-Victor Poncelet and Claude-Louis Navier both obtained this equation, which connects to Bélanger's through the relations $\mathrm{d}\zeta = -\sin\gamma \mathrm{d}s + \cos\gamma \mathrm{d}h$ and $\mathrm{d}u/\mathrm{d}t = \mathrm{d}(\upsilon^2/2)/\mathrm{d}s$. In 1836, Gaspard Coriolis [22] improved the reasoning by taking into account the non-uniformity of the velocity of the fluid across a section of the channel, although he still assumed a frictional sliding of fluid slices at the average velocity U with respect to the walls. The result,

$$I \equiv -\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \frac{\chi F_U}{S\rho g} + \alpha\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{U^2}{2g}\right), \quad \text{with } \alpha = S^{-1}\iint (\upsilon/U)^3 \mathrm{d}S \tag{44}$$

differs from Poncelet's through the coefficient α, which depends on the velocity profile in a fluid section.

There is more to say about the critical depth h_c. In 1851, Saint-Venant noted that the form of the backwater curves below and above the critical depth corresponded to tumultuous and tranquil flows respectively [23, p. 319]. The case $h < h_c$ defined "*torrents*, the various parts of which seem to flow independently of each other and whose acquired velocity allows them to flow over small obstacles." The case $h > h_c$ defined "*rivers* or quiet streams whose successive slices press on each other and move along together, so that they can only get over obstacles by means of the weight of the elevated water and so that every elevation in one part is felt in the upstream direction to a finite distance."

For a rectangular canal of small slope, the condition $h \approx h_c$ for the possibility of a jump is equivalent to $u \approx \sqrt{gh}$. As Saint-Venant noted in 1870 [24], this means that the velocity of the water is the same as the propagation velocity of a small swell as given by Lagrange. In a hydrostatic canal closed by two distant gates, with a rise of the water level obtained by constantly feeding water at one of the gates, the higher level propagates as a step along the canal with the Lagrangian celerity $\sqrt{gh}$, as indicated in Fig. 5. A small hydraulic jump can be obtained by superposing to this motion a constant flow at the velocity $-\sqrt{gh}$. Saint-Venant used this remarkable connection between jumps and waves to confirm his distinction

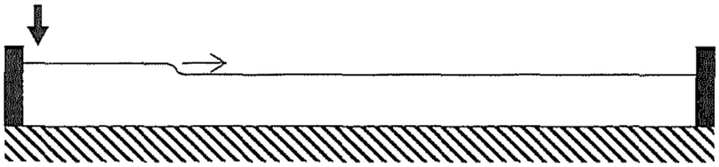

Fig. 5. Progressive swell caused by continuous injection of water at one end of a channel (from [5], p. 228).

between river and torrents. In a stream slower than $\sqrt{gh}$, the swells created by an obstacle must propagate in the upstream direction, so that water accumulates before it can pass the obstacle. This is the case of a river. In a stream faster than $\sqrt{gh}$, the water can pass obstacles without previous accumulation. This is the case of a torrent.

The backwater theories concerned gradually varied permanent flow only. In order to account for tides in the lower course of rivers, equations had to be found for covering the non-permanent case. Airy did so in 1845 in the context of his general theory of tides, by applying momentum balance to a fluid slice in a third-order approximation in the amplitude of the tidal waves [25] (see [5], pp. 66–68). This implies the variant

$$c = \sqrt{g(h + 3\sigma/2)} \tag{45}$$

of Lagrange's celerity formula (σ being the elevation of the summits of the waves), as well as a gradual deformation of the wave during its upstream travel: the front of the wave becomes steeper than its rear, which explains why the rise of water takes more time than its descent at a station far from the mouth. In some cases, the front becomes so steep that a *mascaret* or tidal bore is formed. In 1871, the *Ponts et Chaussées* engineer Henri Partiot gave a simple, intuitive explanation of this phenomenon, by picturing the tidal influx as a succession of step-shaped water laminas [25]. The successive laminas encounter higher and higher levels of water, and therefore propagate at higher and higher velocities, so that the later laminas of water catch up with the earlier ones. The front of the tidal wave thus becomes steeper and steeper. For strong tides or quickly narrowing beds, it can reach the vertical slope for which breaking occurs. Saint-Venant [26] soon gave a general equation for gradually varied non-permanent show, and thus retrieved Airy's and Partiot's results.

All these theories of open channel flow relied on simple idealizations with some empirical input. Most of them assumed a uniform velocity profile with a finite slide on a bottom and an empirical friction formula. Coriolis admitted a non-uniform profile in his energy balance, but sill appealed to the sliding slice picture in his estimate of the work of frictional forces. In every case, the open-channel theorists ignored the palpable irregularities of the flow and they tacitly assumed that the average, macroscopic motion measured by standard gauging methods was the only one to be considered in mechanical reasoning.

3.2. Saint-Venant's effective viscosity

Yet there were reasons to believe that the tumultuous, eddying character of the fluid motion played an essential role in the retardation process. In his famous memoir of 1822 including the Navier–Stokes equation, Navier had already remarked that this equation could apply only to "linear motion" and not to the "more complex motions" observed in hydraulic cases [27]. In 1843, his former student and admirer Saint-Venant offered a new derivation of the Navier–Stokes equation that included "the case when partial irregularities of the fluid motion force us to take faces of a certain extension so as to have regularly varying averages" [28, p. 1242n]. That is to say, Saint-Venant applied Newton's second law of a small cube with faces large enough to average out the pressure fluctuation.

Call τ_{ij} the ith component of the pressure exerted by the fluid on a face normal to the jth axis. The irregularities of motion being isotropic, this stress system must share the symmetries of the average rate of deformation $\partial_i \upsilon_j + \partial_j \upsilon_i$ of the fluid around the cube. This condition yields

$$\tau_{ij} = \varepsilon(\partial_i \upsilon_j + \partial_j \upsilon_i) + \varpi \delta_{ij} \tag{46}$$

wherein the coefficients ε and ϖ may depend on the local irregularities of motion and may therefore vary from place to place, according to the global conditions of the flow. The Navier–Stokes equation obtains for a constant value of the ε coefficient that determines the friction between successive layers of the fluid. In hydraulic cases, Saint-Venant expected this coefficient to vary with the size of the pipes and channels and with the distance from the walls. As he put it in 1851 [23, p. 229]:

> Newton's hypothesis, as reproduced by MM. Navier and Poisson, consists in making internal friction proportional to the relative velocity of the filaments sliding on one another; if it can be approximately applied to the various points of the same fluid section, every known fact indicates that the proportionality *coefficient* must increase with the dimensions of transverse sections; which may be to some extent explained by noticing that the filaments do not proceed in parallel directions with a regular gradation of velocity, and that the *ruptures*, the whirls and other complex and oblique motions that must considerably influence the intensity of friction develop better and faster in large sections.

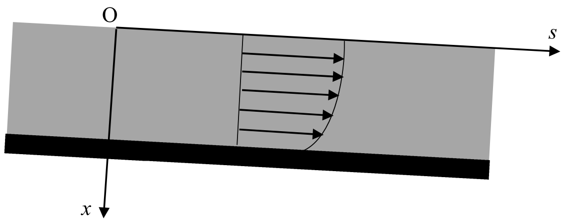

Fig. 6. Boussinesq's velocity profile for permanent flow in a wide rectangular channel.

Saint-Venant did not attempt to quantify this insight. In his own studies of open channel flow, he kept relying on a global retardation formula for the fluid slices.

On the experimental slide, the knowledge of pipe flow and open channel flow greatly progressed in the 1850s and 1860s thanks to the efforts of the *Ponts et chaussées* engineer Henry Darcy and his disciple Henri Bazin (see [21], pp. 169–173; [5], pp. 232–233). Through many systematic and precise measurements, Darcy corrected old laws of retardation in pipes, demonstrated the effect of the roughness of the walls and of the diameter of the pipes, and determined the velocity profile. He and Bazin did the same for open channel flow, using the Burgundy canal as a laboratory. They found a quadratic velocity profile in rectangular sections, and a cubic profile in semi-circular sections (with sliding on the wall). They also studied non-permanent flow, including solitary waves and tidal bores. The wealth of new regularities and phenomena defied theory, Bazin thought [29]: "Maybe such a delicate part of science must long remain in the realm of experiment."

3.3. Boussinesq's *Eaux courantes*

In a series of notes and memoirs written between 1870 and 1872, Boussinesq did his best to refute Bazin's pessimist forecast [B11, B13, B17, B18, B20]. The three keys to his success were Saint-Venant's concept of enhanced viscosity,[7] simple assumptions on the value of this viscosity, and an uncanny ability to integrate the resulting equations in concrete cases of open channel flow. For a canal of rectangular section, he took the effective viscosity ε to be a constant proportional to the depth h of the water and to the sliding velocity υ_0 at the bottom. In his intuition, the asperities on the bottom wall create whirling motion at a rate proportional to the sliding velocity, and these whirls then fill the entire mass of the water homogeneously, thus enormously increasing the viscous damping. For a circular canal, Boussinesq took the viscosity to be proportional to the radius of the channel and to increase linearly with the distance from the axis of the channel.

For uniform permanent flow in a rectangular canal of constant inclination i, Boussinesq balanced the weight of a short portion of a fluid filament with the stresses acting on its lower and upper sides to get

$$-\varepsilon\frac{\mathrm{d}^2\upsilon}{\mathrm{d}x^2}=\rho g i \tag{47}$$

where x is the distance of the fluid filament from the surface [B20, pp. 72–73]. Taking $\varepsilon=\rho g A h\upsilon_0$ for the viscosity and $F=\rho g B\upsilon_0^2$ for the friction on the bottom, this second-order equation and the boundary conditions $\upsilon'(0)=0$, $-\varepsilon\upsilon'(h)=F$ lead to the formula

$$\frac{\upsilon}{\upsilon_0}=1+\frac{B}{2A}\left(1-\frac{x^2}{h^2}\right) \tag{48}$$

as shown in Fig. 6. Consequently, the average velocity U is proportional to the velocity υ_0, which is itself related to the slope i by the balance $\rho g h i=F=\rho g B\upsilon_0^2$ between the weight of a normal slice of fluid and the external friction F. In conclusion, Chézy's proportionality between slope and squared discharge still holds, despite the transverse variation of velocity.

In the case of varied permanent flow, the fluid particles are accelerated along the fluid filaments owing to their variable section in a first approximation, and also owing to their curvature in a second approximation. In the notation earlier used for backwaters, Boussinesq's resulting equation for the slope I of the water surface and the average velocity U in a wide rectangular channel reads

$$I=\frac{\chi F_U}{S\rho g}+(1+\eta+\beta)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{U^2}{2g}\right)-h^2\left[\frac{1}{3}\frac{\mathrm{d}^3}{\mathrm{d}s^3}\left(\frac{U^2}{2g}\right)+\frac{1}{2}\frac{U^2}{gh}\frac{\mathrm{d}^2 i}{\mathrm{d}s^2}\right] \tag{49}$$

[7] Like Saint-Venant, Boussinesq traced the effective viscosity to the enhanced internal friction associated with turbulent fluctuation or "ruptures". In this view, turbulence implies intensive sliding of fluid over fluid at the turbulence scale, hence a fast dissipation of the energy of the large-scale motion. Boussinesq did not perceive the conceptual connection between shear stress and transverse momentum transfer (I have unfortunately suggested the contrary in [3, pp. 235, 297]). According to Reynolds' later theory [45], the medium-scale average of the term $\rho\mathbf{v}\cdot\nabla\mathbf{v}=\partial_i(\rho\upsilon_i\upsilon_j)$ in the Navier–Stokes equation gives $\rho\bar{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}+\partial_i\sigma_{ij}$, with $\sigma_{ij}=\rho\overline{(\upsilon_i-\bar{\upsilon}_i)(\upsilon_j-\bar{\upsilon}_j)}$ for the stresses created by turbulent momentum transfer. On the contrary, Boussinesq identified the time-average of $\mathbf{v}\cdot\nabla\mathbf{v}$ with $\bar{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}$ [B20, pp. 28–31] (see Schmitt [35, pp. 619–620]). This error and the different intuition of effective viscosity may be traced to Saint-Venant's and Boussinesq's persistent belief in a quasi-static molecular model of fluids (regarded as temporary solids), whereas Reynolds adopted the kinetic-molecular theory and its fundamental relation between shear stress and momentum transfer.

wherein η and β are constants depending on the velocity profile. The three terms represent the effects of friction, variation of section, and curvature respectively. Except for the different value and meaning of the coefficient of the second term, the two first terms agree with Coriolis' earlier equation and therefore lead to similar consequences. The third term removes the singularity in the backwater equation. Boussinesq used it to deduce the profile of a hydraulic jump and to confirm Bazin's empirical distinction between two sorts of jumps with or without long-range oscillation. He also refined Saint-Venant's distinction between torrents and rivers, now introducing an intermediate category of "moderate torrents" for which jumps can occur, but only long and wavy ones. Lastly, he obtained equations for non-permanent flow, and used them to discuss all sorts of waves, river tides, tidal bores, and floods. It is impossible, in just a few lines, to do justice to the numerous, refined, and largely verified predictions of this theory.

The sum of Boussinesq's works on open channels appeared in 1877 in his 774-page-long *Essai sur la théorie des eaux courantes* [B20], with additions covering his earlier works on water bells, efflux, flow in curved channels. In 1897 he offered a more compact account of the case of rectangular channels [B28]. His disciples Alfred Flamant [30] (in 1891) and Auguste Boulanger [31] (in 1909) did their best to extract the essential from this voluminous work. Most readable is Saint-Venant's account of 1887 [32], for it replaces Boussinesq's difficult integrations of the generalized Navier–Stokes equation with simpler considerations of momentum balance on fluid slices. The academic report Saint-Venant wrote on the *Eaux courantes* was highly praiseful (in [B20], p. XXI):

> These numerous results of a high analysis, founded on a detailed discussion and on judicious comparisons of quantities of various orders of smallness, sometimes to be kept, sometimes to be neglected or abstracted, and their constant conformity with the results obtained by the most careful experimenters and observers, appear most remarkable to me.

Yet, the sheer abundance of results and the stylistic lengthiness must have hampered the diffusion of Boussinesq's works. In the case of highest hydraulic interest, which is gradually varied permanent flow, Boussinesq arrived in a complicated manner to an equation that Coriolis had already obtained in a much simpler, though fundamentally flawed manner. The considerations on rapidly varying and non-permanent flaw produced far more original results, including hydraulic jumps, solitary waves and the deformation of arbitrary swells. But the physicists who could have learned from them read only the shorter notes in the *Comptes rendus*. We will return to this problematic reception in a moment.

4. Viscous boundary layers

In his *Eaux courantes* Boussinesq usually assumed the velocity profile to have reached the steady value it takes long after the flow has been established and far from the entrance of the pipes or channels. In later years he studied the profiles that occur before the steady uniform profile has been reached (in the laminar case).

4.1. The diffusion of vorticity

Boussinesq gave a first simple example of such transitory profiles in 1880 [21], in reaction to a note by Jacques Bresse. This hydraulic engineer had erroneously extended to viscous fluids a famous theorem by Lagrange, according to which a velocity potential exists for any incompressible fluid motion started from rest. Boussinesq corrected him by solving the following simple ideal case.

A constant, uniform, and horizontal accelerating force $k\rho$ is applied at time zero and onward to the entire mass of a viscous fluid resting over a horizontal plane wall situated at $z = 0$. The resulting flow is obviously parallel to the plane, and its velocity u vanishes on the plane at any time. The Navier–Stokes equation for kinematic viscosity ν gives

$$\partial_t u = k + \nu \partial_z^2 u \tag{50}$$

in which Boussinesq recognized Fourier's equation for the diffusion of heat. The relevant solution is

$$u = kt\left[1 - \frac{2}{\sqrt{\pi}} \int_{\alpha}^{\infty} \left(1 - \alpha^2/\beta^2\right) e^{-\beta^2} d\beta\right] \tag{51}$$

with $\alpha = z/2\sqrt{\nu t}$. Consequently, the retarding effect of the wall is sensible only in a layer of thickness comparable to $\sqrt{\nu t}$.[8]

In a second note, Boussinesq insisted that wall stress played an essential role in determining the nature of the motion in any hydraulic problem and that his simple calculation revealed the general mechanism through which rotational motion began in any fluid [B22, p. 967]:

> The retarding influence of a wall will first only be sensible in the vicinity of this wall. Hence some time will elapse before the similar influences of the other walls reach this region, and it will therefore be permitted to evaluate the

[8] Stokes had already treated a similar problem in his pendulum memoir of 1850 [33].

velocity variation at the beginning of motion as if [...] the wall under consideration has infinite breadth and the fluid mass has infinite thickness [...] Hence, [my previous calculation] *most simply expresses what happens at the beginning of any flow, and demonstrates the general mechanism, abstracted from accessory complications*.

4.2. Entrance effects

Ten years later, Boussinesq examined a similar question in an attempt to correct for entrance effects in some of Poiseuille's experiments: how does the velocity profile of a viscous fluid entering a capillary tube evolve toward the uniform, parabolic profile? [B26, B27]. Assuming a rectangular profile at the entrance of the tube, he understood that an annular layer of retarded fluid grew from the walls until it reached the central part of the tube. From the Navier–Stokes equation with the usual boundary condition of the tube walls, he derived an approximate equation for the evolution of the velocity profile along the tube. This equation having the same form as the equation for the propagation of heat in one dimension (the abscissa along the tube here playing the role of time) with a variable heat capacity, he was able to solve it by successive approximation. In the end he found that the departure from the steady profile varied as $\mathrm{e}^{-16\nu x/UR^2}$, wherein x is the distance from the entrance, R the radius of the tube, U the average velocity, and ν the kinematic viscosity. The retarded layer reaches the thickness R for a distance of the order $x = UR^2/\nu$, which means that the thickness grows with x as $\sqrt{\nu x/U}$. The reader here recognizes Ludwig Prandtl's law for the growth of a laminar boundary layer. The context was however different: Boussinesq dealt with the retardation of pipe flow, Prandtl with the resistance problem at low viscosity (see [5], pp. 283–286).

4.3. Viscous flow induced by a moving sphere

Boussinesq's approximations sometimes led to equations whose solution was already known, for instance Fourier's equation of propagation in boundary-layer problems. In other cases, Boussinesq invented new ways of solving the differential equations he had reached. A favorite trick of his was to introduce a generalized potential through which a symmetry of the problem could be exploited. A stunning example is his treatment of viscous incompressible flow around a moving sphere in the linear approximation for which the quadratic terms are neglected in the Navier–Stokes equation [B24; B32, pp. 224–242]. The equations of the problem are

$$\nabla\cdot\mathbf{v}=0,\qquad \rho\partial_t\mathbf{v}=-\nabla P+\mu\Delta\mathbf{v} \tag{52}$$

Boussinesq introduces the potential ψ for which

$$\mathbf{v}=\mathbf{i}\Delta\psi-\partial_x\nabla\psi \tag{53}$$

($\mathbf{i}$ being the unit vector on the x axis). The incompressibility condition is then automatically satisfied, and the linearized Navier–Stokes equation is satisfied if and only if the potential verifies

$$\Delta(\rho\partial_t\psi-\mu\Delta\psi)=0 \tag{54}$$

and the pressure is adjusted to

$$P=\partial_x(\rho\partial_t\psi-\mu\Delta\psi) \tag{55}$$

Boussinesq seeks solutions for which ψ depends only on the distance r from the origin of the moving sphere. Using $\Delta\psi = r^{-1}\partial_r^2(r\psi)$ and integrating twice (neglecting the motion of the center of the sphere in the computation of $\partial_t\psi$), Eq. (54) yields

$$\big(\rho\partial_t-\mu\partial_r^2\big)(r\psi)=\alpha(t),\quad \text{or } \big(\rho\partial_t-\mu\partial_r^2\big)\big[r\psi(r,t)+\beta(t)\big]=0 \tag{56}$$

which has the same form as the equation of propagation of heat in one spatial dimension.

Taking $\psi=0$ on the surface $r=R$ of the sphere, the quantity $\theta(t)=r\psi+\beta$ is formally the same as the temperature of a half-infinite bar whose extremity $r=R$ had the temperature $\beta(\tau)$ at any anterior time τ. It is therefore given by

$$\theta(r,t)=\sqrt{2/\pi}\int_0^\infty \beta\big[t-\big(\rho/\mu\lambda^2\big)(r-R)^2\big]\mathrm{e}^{-\lambda^2/2}\mathrm{d}\lambda \tag{57}$$

Now suppose the sphere to be moving at the velocity $\mathbf{V}(t)$ on the x axis. As Boussinesq shows, the boundary conditions $\mathbf{v}=\mathbf{V}$ on the surface $r=R$ of the sphere and $\mathbf{v}=\mathbf{0}$ at infinity are satisfied if we take

$$\beta=(3\mu/2\rho)RX(t),\quad \text{with } X(t)=\int_{-\infty}^{t}V(\tau)\mathrm{d}\tau \tag{58}$$

The net force $-F$ exerted by the fluid on the sphere is then given by the flux $\oint \tau_{ij} \mathrm{d}S_j$of the total stress $\tau_{ij} = -P\delta_{ij} + \mu(\partial_i \upsilon_j + \partial_j \upsilon_i)$ across the surface of the sphere, the pressure P being given by Eq. (55) and the velocity $\mathbf{v}$ by Eq. (53) for the choice $\psi = r^{-1}(\theta - \beta)$ of the potential. The result is

$$-F = 6\pi\mu RV + \frac{m}{2}\frac{\mathrm{d}V}{\mathrm{d}t} + 6\sqrt{\pi\rho\mu}R^2 \int_{-\infty}^{t} \frac{V'(\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau \tag{59}$$

The first term is the one Stokes obtained in 1850 in his pendulum studies [33]. The second term is the mass-correction first derived by Poisson in 1831 [34]. The last term, which depends on the previous history of the motion of the sphere, was essentially original, although Stokes already knew its value in the special case in which the motion of the sphere is sinusoidal.

5. Thermal convection

Early in the twentieth century, Boussinesq published two thick volumes of lectures on optics and heat propagation [B31, B32]. Toward the end of the second volume, he addressed the difficult question of the propagation of heat in a moving fluid [B32, lectures 34–35].

5.1. The fundamental equations

As Boussinesq explained in his introduction [B32, pp. vi–viii], Fourier and Poisson had long ago given the relevant system of equations, but no solution was known in any case. On the fluid-mechanical side, the relevant equations are the continuity equation

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0 \tag{60}$$

and Euler's equation

$$\rho\dot{\mathbf{v}} = -\nabla P - \rho\mathbf{g} \tag{61}$$

wherein the dot stands for the convective derivative $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{v}\cdot\nabla$. On the thermal side, we have the thermodynamic equation

$$\dot{Q} = \dot{u} - P\dot{\rho}/\rho^2 \tag{62}$$

which gives the heat gained by a convected mass element of the fluid per unit time as a function of its energy variation $\dot{u}$ and its density variation $\dot{\rho}$; and we also have the equation

$$\dot{Q} = \nabla\cdot(K\nabla\theta) \tag{63}$$

for the divergence of the heat flux, if θ denotes the temperature and K the thermal conductivity. The quantities K, P, and u are given as functions of ρ and θ. After eliminating $\dot{Q}$, we therefore have five (scalar) equations and five unknown functions $(\upsilon_x, \upsilon_y, \upsilon_z, \rho, \theta)$ of time and location. In principle, this is all we need to know to solve the problem of heat convection under given boundary conditions. In practice, the integration is hopeless if only because the conductivity K in the Fourier equation contains one of the unknown functions ρ.

5.2. The Boussinesq approximation

Having in mind the case of the convection caused by a heated body immersed in a liquid, Boussinesq argued [B29, p. 1382]:

> In order to reach maximum simplicity without losing the essence of the phenomenon, we will assume that the thermal dilatability of the liquid is small enough and the gravity g is strong enough, so that the reduction of *weight* of the unit volume of the liquid caused by the heating at θ is sensible, whereas the relative change of the liquid *volumes* is insensible in the terms in which it is not multiplied by g.

Boussinesq later extended this approximation to gases, further assuming that the heat change $\dot{Q}$ was approximately isobaric and that the relevant heat capacity was a constant C [B32, pp. 154–160]. This leads to the equation

$$C(\partial_t\theta + \mathbf{v}\cdot\nabla\theta) = K\Delta\theta \tag{64}$$

which differs from Fourier's equation by the convective term only. The other equations are

$$\nabla \cdot \mathbf{v} = 0 \quad \text{and} \quad \rho_0(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla P - (\rho_0 - \alpha\theta)\mathbf{g} \tag{65}$$

Subtracting from the pressure P the hydrostatic component $\rho_0 gz$, we may rewrite the latter equation as

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\rho_0^{-1} \nabla P + \gamma\theta\,\mathbf{k} \quad (\text{with } \gamma = \alpha g) \tag{66}$$

$\mathbf{k}$ being the vertical, downward unit vector.

Boussinesq focused on the permanent state for which $\partial_t \theta = 0$ and $\partial_t \mathbf{v} = 0$. The boundary conditions on the surface of the heated body are $\theta = \theta_0$ and $\mathbf{v} \cdot \mathbf{n} = 0$ ($\mathbf{n}$ being the normal unit vector). Far from the body, Boussinesq took $\theta = 0$ and $P = 0$, and he considered two cases: 1) the fluid is asymptotically at rest [B29], 2) the fluid moves asymptotically at the constant velocity $\mathbf{V}$ [B30].

5.3. Static convection, non-dimensionalized

In case (1), Boussinesq judged the equations to be still too complex to permit integration. This is why he relied on proper rescaling of the relevant quantities in order to restrict the form of the heat flux as a function of the temperature excess θ_0 of the body. The various constants of the problem enabled him to form the length $L = (\gamma\theta_0 C^2/K^2)^{1/3}$, the velocity K/LC and the pressure $\rho_0 K^2/L^2C^2$, and to write a system of dimensionless equations in place of the former equations, with proper rescaling of the boundary conditions. Although scaling arguments had been used since Newton's times, Boussinesq may have been first to introduce dimensionless equations in this manner. The solution to the dimensionless system being unique, the heat flux per unit surface, $-K\nabla\theta$, must be proportional to $L^{-1}\theta_0$ or $\theta_0^{-2/3}$, and the net flux to $L\theta_0$ or $\theta_0^{4/3}$. Boussinesq compared this law with the $\theta_0^{1.23}$ law empirically established by Pierre Louis Dulong and Alexis Thérèse Petit (for gases) and suggested that the 10% discrepancy in the exponents might be explained by the convexity of the body.[9]

5.4. Newton's law of cooling

In Boussinesq's second case, the asymptotic flow has the constant velocity $\mathbf{V}$. If the fluid glides fast enough on the surface of the heated body, its temperature and its density do not change enough to produce a significant value of the $\gamma\theta$ term in Eq. (66), so that gravity does not play any role. In this case, scaling analysis gives fluid velocities everywhere proportional to V, pressures proportional to $\rho_0 V^2$, and a heat flux proportional to the temperature excess θ_0, in conformity with Newton's law of cooling of bodies under a uniform air stream. For more precise results, Boussinesq considered the subcase in which the velocity $\mathbf{V}$ is parallel to the Oz axis and the heated body is a thin plate in the half plane $x = 0$, $z \geq 0$, with temperature $\theta_0(z)$. Under these assumptions, the flow is undisturbed by the plate and we have $\mathbf{v} = \mathbf{V}$ everywhere, except in the plate. In addition, Boussinesq neglected the second derivative $\partial_z^2\theta$ for large enough V. The resulting temperature equation reads

$$CV\partial_z\theta = K\partial_x^2\theta \tag{67}$$

Its solution for $z \geq 0$ under the conditions $\theta(x,z) = 0$ for $z \leq 0$ and $\theta(0,z) = \theta_0(z)$ for $z \geq 0$ reads

$$\theta = \sqrt{2/\pi}\int_0^\infty \theta_0\big(z - CVx^2/2K\lambda^2\big)\mathrm{e}^{-\lambda^2/2}\mathrm{d}\lambda \tag{68}$$

whence follows a simple expression for the heat flux per unit surface $-K\partial_x\theta$ on one side of the plate.

5.5. Reception

Nowadays, Boussinesq's name is attached to a number of theories and formulas. In hydrodynamics and hydraulics only, he is remembered as the first theorist who solved the problem of solitary waves, as the inventor of eddy viscosity (see [35]), and, in a minor genre, as the first theorist of water bells; there is a "Boussinesq equation" for swells propagating in an open channel, a "Boussinesq approximation" for thermal convection,[10] a "Boussinesq coefficient" for the momentum variation in open channel flow, and a "Boussinesq–Basset" force for the motion of a sphere in a viscous liquid (see [2,37]).[11]

Yet much of Boussinesq's production has gone unnoticed, forgotten, or undervalued. Most strikingly, the KdV equation is named after Korteweg and de Vries, whereas Boussinesq had it some twenty years earlier, implicitly in 1871 as an immediate consequence of two other equations, and explicitly in 1877 in a footnote to the *Eaux courantes*. His pioneering work on viscous boundary layers also seems to have remained unnoticed, save for his work on entrance effects which

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[9] Boussinesq obtained more precise results for the ascending heat current on the sides of a vertical flat plate.

[10] Lord Rayleigh exploited and publicized this approximation in his famous article of 1916 on Rayleigh–Bénard convection [36].

[11] Also well-remembered is Boussinesq's work on the old problem of earth pressure (*poussée des terres*), which is not addressed here since it does not belong to fluid mechanics *stricto sensu*. On this work, see [B23, pp. 42–50] and [38–40].

Prandtl praised in his lectures [41, pp. 24–29]. Much of the substance of the monumental *Eaux courantes* is forgotten, although Prandtl once remarked that Boussinesq's velocity profiles were a tolerable approximation of the truer logarithmic profile [42, p. 833n].

A sure sign that Boussinesq's writings were not widely known is the frequent occurrence of rediscoveries of his results. Rayleigh rediscovered the equation of a solitary wave in 1876 [18]. As was just mentioned, Korteweg and de Vries rediscovered the KdV equation in 1895 [17]. They also rediscovered the form Boussinesq had given in his *Eaux courantes* to finite periodic waves on shallow water [B20, pp. 390–396], in the same approximation as for the solitary waves.[12] The only Boussinesq reference in their memoir is to his first note on solitary waves in the *Comptes rendus* (with an error in the volume number). They presumably did not read the later note in which the KdV equation is nearly contained as a consequence of the Boussinesq equation, or the relevant note of the *Eaux courantes*. This is not to say that their study was redundant or worthless: on the contrary, they brought new insights and methods (see [43]). But it remains true that Boussinesq's extraordinarily astute discussion of the evolution of arbitrary swells went unnoticed.[13] Turbulence-induced shear stresses were reintroduced by Osborne Reynolds in 1895 [45], with no reference to Boussinesq. To be fair, the notion truly belongs to Saint-Venant, and Reynolds understanding of eddy-viscosity, being based on momentum exchange in analogy with the kinetic theory of gases, is deeper than Saint-Venant's or Boussinesq's and closer to the modern understanding.[14] The Cambridge mathematician Alfred Barnard Basset [46]; [47, chap. 22] rediscovered the Boussinesq force for a sphere moving in a viscous fluid in 1887, two years after Boussinesq's note in the *Comptes rendus*. His method of calculation differs from Boussinesq's, since he relied on Stokes' stream function for incompressible axial-symmetric flows.[15] In 1904, Prandtl [48] famously described the growth of a viscous boundary layer from the edge of a plate immersed in a parallel flow, with a kind of approximation and a law already known to Boussinesq. Prandtl's concern and strategy widely differed from Boussinesq's: Prandtl wanted to compute fluid resistance at high Reynolds number, a problem in which Boussinesq had little interest, and he crucially coupled the boundary layer with the Eulerian part of the flow beyond the layer (see [5], pp. 283–286).

Altogether, Boussinesq's works were less appreciated and read than they deserved. A first, evident explanation of this relative neglect is found in his lengthy, meticulous, and at times tedious style. Being mostly an autodidact in physics, Boussinesq never learned how to conciliate completeness and concision. His notes in the *Comptes rendus* were too elliptic to convey his full reasoning, and his synthetic memoirs and books so long and so full of details of variable interest that the reader could easily get discouraged. Although later hydraulicians often praised his *Eaux courantes*, few of them are likely to have read more than a few pages.

The relative opacity of Boussinesq's writings could have been a lesser handicap if he had been able to create his own school of physics from his Sorbonne chair. But he probably was no great communicator. His weak voice and his excessive thoroughness must have been serious handicaps, although his published lectures display a high level of clarity, order, and consistency. He wanted to convey everything he knew, and he knew too much. That said, he had a few colleagues, friends, and students who generously contributed to the diffusion of his theories, mainly Saint-Venant, Flamant, and Boulanger. In the early twentieth century, French hydraulicians could easily get acquainted with Boussinesq's main results through Boulanger's encyclopedia article [31]. From Boussinesq's correspondence, one gathers that hydraulic engineers often consulted him on technical matters.

As for mathematicians and mathematical physicists, they are not likely to have paid much attention to Boussinesq's works, despite his holding the most prestigious Sorbonne chair in mathematical physics. Mathematicians despised his intuitive, pre-Cauchy mathematics, although they recognized his problem-solving prowess (see [1], pp. xxx, xxxii). As Saint-Venant long lamented, the *Classe de mécanique* of the French Academy had little or zero interest in the *mécanique physique* based on molecules, averages, and approximations and meant to satisfy the needs of the engineer. French contributions to fluid mechanics, chiefly Poincaré's and Marcel Brillouin's ones, furnished rigorous mathematical theorems in ideal cases of little or no practical interest, for instance regarding vortices and discontinuity surfaces in perfect liquids. As documented by an exchange he had in with a budding expert on discontinuity surfaces, Henri Villat, Boussinesq regarded this strictly mathematical approach as completely misconceived. Even at high Reynolds number, he told Villat, the production of eddies largely determined fluid resistance in most cases (see [49]).[16] Reciprocally, the younger generation of French mathematical physicists regarded him as an admirable figure of the past.

There were other excuses for prematurely burying Boussinesq. In his lectures, he rejected any new or recent theory that did not fit his own kind of mechanical explanation. He condemned Maxwell's electromagnetic theory of light for replacing *obscurum per obscurius* [B31, p. 15], and he brushed away relativity and quantum theory as speculative aberrations

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[12] This waves being expressible by means of elliptic functions, Korteweg and de Vries called them "cnoidal waves" (after the cn notation for the elliptic cosine). See [5], p. 85.

[13] The first exception may have been the historical paper by John Miles [44].

[14] Boussinesq's conception has the advantage of specifying the relation between the stress tensor and the tensor of deformations. Yet it cannot quite be regarded as a "closure" of the system of large-scale equations, since these equations do not tell us how the effective viscosity ε depends on the global circumstances of the flow.

[15] Stokes's function ψ gives $\upsilon_z = -r^{-1}\partial\psi/\partial z$ and $\upsilon_r = r^{-1}\partial\psi/\partial r$ for the velocity component in cylindrical coordinates, so that the divergence $\nabla\cdot\mathbf{v} = r^{-1}\partial_r(r\upsilon_r) + \partial_z\upsilon_z$ automatically vanishes.

[16] As we may retrospectively judge from Prandtl's successes, Boussinesq underestimated the heuristic importance of Eulerian flow with discontinuity surfaces and circulation.

[B33, p. 93]. Although he never departed from his gentle, modest manner, he grew very lonely toward the end of his life. Philosophy, faith, and memory of his academic achievements were his consolation to the infelicities of his personal life. He married three times, had no children, and lost his three wives. Here is an anecdote by Laurent Schwarz [50, p. 156]:

> The mathematician–mechanician Boussinesq, great scientist, once lost his wife. The burial, which had begun on a very clear day, ended under pouring rain. Everyone got soaked. He remarried and became a widower again. The same meteorological phenomenon occurred during the funeral. When his third spouse died as well, the funeral went under a persistently blue sky, but all the attending academics had brought their umbrella. Émile Borel turned to Polya[17] (then in France) and said:
>
> – Listen, Polya, isn't it lamentable? We are academics, we are not superstitious, I am a probability theorist, I surely know that there cannot be any relation between the rain and the burial of M^me Boussinesq, and yet I brought my umbrella!
> – Not at all, Polya answered, we work with scientific facts; and it is a scientific fact that there often rains at the burial of Boussinesq's wifes.

The laws of probabilities may indeed have been cruel to Boussinesq. He lost three wives and theoretical physics took a turn that was not to his liking. Yet, in the twilight of old age he had the satisfaction of having written some of what he called the *roman de la physique* (the novel of physics) [B33, p. 395], an attempt to capture and control the physical world through geometrical fictions. He never doubted that his endeavors would have lasting value, as today's physicists do not hesitate to confirm.

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The following abbreviations are used: *CR*, Académie des sciences, *Comptes rendus hebdomadaires des séances*; *JMPA*, *Journal de mathématiques pures et appliquées*; *MSE*, Académie des sciences de l'Institut de France, *Mémoires présentés par divers savants*.

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