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Nonlocal modeling of a randomly distributed and aligned long-fiber composite material

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ABSTRACT

The work under study is about the variational and stochastic modeling of randomly distributed and aligned long-fiber composites. Its objective is to derive a homogenized behavior that exhibits the nonlocal phenomenon of this type of material at the macroscopic scale. Several methods of applied mathematics are used in order to keep the maximum information about the nonlocal behavior after homogenization.

1. Introduction

The macroscopic behavior of randomly reinforced composite materials is studied herein. The material we have in mind is TexSol™ [1–3]. Contrary to the previous studies, in which the nonlocal aspect phenomenon is lost in the homogenization step, this paper aims to derive a homogeneous behavior with nonlocal effects.

We are interested in the macroscopic behavior of a random fiber structure whose reference configuration is the open subset of $\mathbb{R}^3$ $\mathcal{O} := \widehat{\mathcal{O}} \times (0, h)$, and base $\widehat{\mathcal{O}} := (0, l_1) \times (0, l_2) \subset \mathbb{R}^2$. We consider that there is no difference between $\mathbb{R}^3$ and the three-dimensional Euclidean physical spaces, equipped with an orthogonal basis denoted by (e_1, e_2, e_3). For all $x = (x_1, x_2, x_3)$ of $\mathbb{R}^3$, $\hat{x}$ stands for (x_1, x_2) and $\mathbf{M}^{3\times 3}$, $\mathbf{M}^{3\times 2}$ denotes the sets of 3×3 and 3×2 matrices, respectively. We denote by $\hat{Y}$ the unit cell $(0, 1)^2$ of $\mathbb{R}^2$ and by Y the unit cell $(0, 1)^3$ of $\mathbb{R}^3$.

More precisely, for $\varepsilon = \frac{1}{n}$ (with n the number of fibers), considering the fibers $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$ with $D(\omega) := \bigcup_{i\in\mathbb{N}} D(\omega_i)$. $D(.)$ are discs randomly distributed in $\mathbb{R}^2$ according to a stochastic process $\omega = (\omega_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^2$ associated with a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, defined in [4]. We are seeking to determine the limit equivalent model in a variational sense.

For the macroscopic behavior of the structure $(\mathcal{S})$ in Fig. 1, we study the behavior according to the variational energy functional H_ε of $\Omega \times L^p(\mathcal{O}, \mathbb{R}^3)$ with value in $\mathbb{R}^+ \cup \{+\infty\}$

$$H_\varepsilon(\omega, u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u)\,\mathrm{d}x + \int_{\mathcal{O}\cap T_\varepsilon} g(\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3) \\ +\infty & \text{else} \end{cases}$$

The space $W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3)$) is the admissibility kinetic space of functions u in $W^{1,p}(\mathcal{O}, \mathbb{R}^3)$ such that $u = 0$ in the trace sense on the bounded $\Gamma_0 := \hat{\mathcal{O}} \times \{0\}$.

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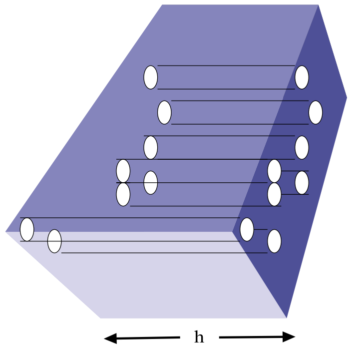

Fig. 1. Randomly fibered structure ($\mathcal{S}$).

The first integral, which we note $F_\varepsilon(\omega, u)$, is the internal elastic energy in the matrix $\mathcal{O} \setminus T_\varepsilon$, whose rigidity is of order ε^p. The second integral, denoted $G_\varepsilon(\omega, u)$, stands for the internal mechanical energy T_ε, of the set of parallel cylindrical fibers on $\mathcal{O}$. Thus, $H_\varepsilon(\omega, u)$ represents the total internal energy of the fibers and the matrix phase. We assume large deformations in the matrix and the fibers so that the strong and soft materials are hyperelastic (in this study, small deformations are assumed for fibers and large deformations for the matrix). Now, presuming that the two bodies are perfectly clamped and are subjected to an exterior loading $\mathcal{L}$, we derive the problem $(\mathcal{P}_{H_\varepsilon})$. We recall that the distribution of the cross-sections of the fibers is statistically homogeneous according to a stationary point process. The rigidity is very small order $\approx \varepsilon^p$ in the matrix $\mathcal{O} \setminus T_\varepsilon$, while the stiffness is of order 1 in the fibers. The function u represents the mechanical displacement of the structure subjected to a given loading L give and we consider the zero displacement on the basis of the complete structure $\Gamma_0 := \hat{\mathcal{O}} \times \{0\}$.

We assume large deformations in the fibers and the matrix (see for example [3]) so that the solid materials are hyperelastic. Energy densities f and g are two quasi-convex functions defined on the space $\mathbf{M}^{3\times 3}$ and that meet the standard growth condition of order $p > 1$: there exist two positive reals α, β, such that $\forall M, M' \in \mathbf{M}^{3\times 3}$

$$\alpha |M|^p \le f(M) \le \beta(1 + |M|^p) \tag{1}$$

idem for g. Note that f automatically satisfies the Lipschitz property

$$|f(M) - f(M')| \le \ell |M - M'|(1 + |M|^{p-1} + |M|^{p-1}) \tag{2}$$

with $\ell > 0$, idem pour g.

Further, we assume the existence of $\beta' > 0$, $0 < \gamma < p$ and of a positive homogeneous function of order p, written $f^{\infty,p}$, such that for all $M \in \mathbf{M}^{3\times 3}$

$$\left| f(M) - f^{\infty,p}(M) \right| \le \beta'(1 + |M|^{p-\gamma}) \tag{3}$$

From (3), (1) and (2), we deduce that $f^{\infty,p}$ verifies for all $M \in \mathbf{M}^{3\times 3}$

$$\alpha |M|^p \le f^{\infty,p}(M) \le \beta |M|^p \tag{4}$$

and

$$|f^{\infty,p}(M) - f^{\infty,p}(M')| \le \ell |M - M'|(|M|^{p-1} + |M|^{p-1}) \tag{5}$$

for all $(M, M') \in \mathbf{M}^{3\times 3} \times \mathbf{M}^{3\times 3}$.

In what follows, we assume that $f^{\infty,p}$ is a convex function. If we then assume that the two materials are perfectly bounded and subjected to a loading L, the displacement is a solution to problem $(\mathcal{P}_{H_\varepsilon})$

$$(\mathcal{P}_{H_\varepsilon}) \qquad \inf \left\{ H_\varepsilon(\omega, u) - \int_{\mathcal{O}} L \cdot u \, dx : u \in L^p(\mathcal{O}, \mathbb{R}^3) \right\}$$

with $L \in L^q(\mathcal{O}, \mathbb{R}^3)$, $q = \frac{p}{p-1}$.

Our objective is to study the variational problem $\mathcal{P}_{H_\varepsilon}$ when ε tends toward 0 to obtain a simplified model and the deterministic behavior of a TexSol™ type material [1–3]. We would like to find the limit of the total energy of this material (like in the Hashin–Shtrikman method). This terminal is both deterministic and above all will be a nonlocal model of our material. More precisely, we proposed that the functional Γ-limit H_ε be bounded by two nonlocal deterministic functionals. Indeed, we establish the following two estimates:

$$F_0^- {}_{\frac{+}{e}} G_0 \le \Gamma\text{-}\liminf H_\varepsilon(\omega, .) \le \Gamma \limsup H_\varepsilon(\omega, u) \le F_0^+ {}_{\frac{+}{e}} G_0$$

where $F_0^- \underset{e}{+} G_0$ and $F_0^+ \underset{e}{+} G_0$ are epigraphic sums defined in $L^p(\mathcal{O},\mathbb{R}^3)$ by

$$F_0^- \underset{e}{+} G_0\,(u) := \inf_{w \in L^p(\mathcal{O},\mathbb{R}^3)} \Big(F_0^-(u-w) + G_0(w) \Big)$$

and

$$F_0^+ \underset{e}{+} G_0\,(u) := \inf_{w \in L^p(\mathcal{O},\mathbb{R}^3)} \Big(F_0^+(u-w) + G_0(w) \Big)$$

The nonlocal effects are taken into account by the internal variable w, which is derived from the homogenized displacement fields of the fibers. Now we can define the functionals G_0, F_0^- and F_0^+:

$$G_0(u) = \begin{cases} \theta \displaystyle\int_{\mathcal{O}} (g^{\perp})^{**}(\frac{\partial u}{\partial x_3})\,\mathrm{d}x & \text{if } u \in V_0 \\ +\infty & \text{else} \end{cases}$$

$$V_0 := \left\{ u \in L^p(\mathcal{O},\mathbb{R}^3) : \frac{\partial u}{\partial x_3} \in L^p(\mathcal{O},\mathbb{R}^3),\ u(\hat{x},0) = 0 \text{ on } \widehat{\mathcal{O}} \right\}$$

The density $g^{\perp}$ is defined for all $\xi \in \mathbb{R}^3$ by:

$$g^{\perp}(\xi) := \inf_{m \in \mathbf{M}^{3\times 2}} g\,(m,\xi)$$

with $\mathbf{M}^{3\times 2}$ the set 3×2-matrix, so $(m,\xi) \in \mathbf{M}^{3\times 3}$ for all $\xi \in \mathbb{R}^3$. We note h^* the Legendre–Fenchel transform in $\mathbb{R}$ of the function h

$$h^*\,(s) := \sup\,\{s.x - h\,(x)\,;\ \ x \in \mathbb{R}\}$$

and h^{**} is the Legendre–Fenchel transform of the function h^*, and a classical property is that this function is convex.

The parameter $\theta \in (0,1)$ is an asymptotic area fraction of fibers $\theta := \int_{\Omega} |\hat{Y} \cap D(\omega)|\,\mathrm{d}\mathbf{P}(\omega)$, $\hat{Y} = (0,1)^2$.

We assume that:

$$F_0^-(u) = \int_{\mathcal{O}} f_0^-(u)\,\mathrm{d}x$$

where for all $\xi \in \mathbb{R}^3$,

$$f_0^-(\xi) = \sup_{n \in \mathbb{N}} \oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}(\cdot,\xi)$$

the operator $\oint$ is an inf-convolution as defined in [5], Remark 36, and Theorem VIII.40. Moreover, $A \mapsto \mathcal{S}^-_A(\omega,.)$ is a process such that its Legendre–Fenchel transform in $\mathbb{R}^3$ is a sub-additive process generated by $\hat{Y} \subset \mathbb{R}^2$.

2. Definition of energy densities f_0^- and f_0^+

A first step consists in defining a good space of probability; for this work; we can choose the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ defined in [4].

2.1. The density f_0^+

Note $\mathcal{I}$ the set (a,b) generated by $(0,1)^2$. For all $\hat{A}$ in $\mathcal{I}$ and $\xi \in \mathbb{R}^3$,

$$\mathcal{S}^+_{\hat{A}}(\omega,\xi) := \inf \left\{ \int_{\overset{\circ}{\hat{A}} \times (0,1) \setminus T(\omega)} f^{\infty,p}(\nabla w)\,\mathrm{d}\hat{x} : w \in \mathrm{Adm}^+_A(\omega,\xi) \right\}$$

$$\mathrm{adm}^+_{\hat{A}}(\omega,\xi) := \left\{ w \in W_0^{1,p}\big(\overset{\circ}{\hat{A}} \times (0,1) \setminus \overline{T}(\omega), \mathbb{R}^3\big) : \fint_{\hat{A} \times (0,1)} w\,\mathrm{d}\hat{x} = \xi \right\}$$

where $\overset{\circ}{\hat{A}}$ is the interior of $\hat{A}$, $\overline{T}$ is the adherence of T and $\fint_\Omega f\,\mathrm{d}x := \frac{1}{|\Omega|}\int_\Omega f\,\mathrm{d}x$ the average of f. We could take as $\mathcal{I}$ the set of all open intervals (a,b) generated by $\hat{Y}$ that we still denote by $\mathcal{I}$. In the following, the subadditivity condition becomes: for every $I \in \mathcal{I}$ such that there exists a finite family $(I_j)_{j\in J}$ disjoint interval $\mathcal{I}$ with $|I \setminus \bigcup_{j\in J} I_j| = 0$,

$$\mathcal{S}_I^+(\cdot) \le \sum_{j\in J} \mathcal{S}_{I_j}^+(\cdot)$$

The set $\Omega \times L^p(\mathcal{O},\mathbb{R}^3)$ is equipped with product σ-algebra $\mathcal{A}\otimes\mathcal{B}$, where $\mathcal{B}$ is a σ-algebra associated with the $L^p(\mathcal{O},\mathbb{R}^3)$ norm. So it is easy to see that, for all $\hat{A}$ fixed of $\mathcal{I}$ and ξ fixed in $\mathbb{R}^3$, $\omega \mapsto \mathcal{S}_{\hat{A}}^+(\omega,\xi)$ is measurable.

We can define the following density. For all $\xi \in \mathbb{R}^3$ fixed,

$$\begin{aligned} \mathcal{S}^+(.,\xi): \mathcal{I} &\longrightarrow L^1(\Omega,\mathcal{A},\mathbf{P}) \\ \hat{A} &\longmapsto \mathcal{S}_{\hat{A}}^+(.,\xi) \end{aligned}$$

and for all $\xi \in \mathbb{R}^3$, $\hat{A} \in \mathcal{I}$ and all $\delta > 0$ very small,

$$\mathcal{S}_{\hat{A}}^+(\omega,\xi) \le \frac{C(p)}{\delta^p \left|(\hat{Y} \setminus D(\bar{\omega}))_{2\delta}\right|} |\xi|^p |\hat{A}| \tag{6}$$

where $C(p)$ is a positive constant depending on p.

Thus the limit $\lim\limits_{n\to\infty} \dfrac{\mathcal{S}_{I_n}^+(\omega,\xi)}{|I_n|}$ exists $\mathbf{P}$-almost surely and

$$\lim_{n\to\infty} \frac{\mathcal{S}_{I_n}^+(\xi,\omega)}{|I_n|} = \lim_{n\to\infty} \frac{\mathcal{S}_{[0,n[^2}^+(.,\xi)}{n^2} = \inf_{m\in\mathbb{N}^*} \left\{ \mathbf{E}\frac{\mathcal{S}_{[0,m[^2}^+(.,\xi)}{m^2} \right\}$$

where $\mathbf{E}$ is the mathematical expectancy. This limit will be called $f_0^+(\xi)$.

To simplify the proof of the upper estimate of the Γ-limit, it will be convenient to introduce a new subadditive process $A \mapsto \tilde{\mathcal{S}}_A$ converging toward the same limit $F_0^+(\xi)$, where A varies in $\mathbb{R}^3$. Specifically, we note again $\mathcal{I}$ the set of open intervals (a,b) generated by $Y=(0,1)^3$, and we apply the Ackoglu–Krengel Theorem (see [6]) with $N=3$ to the defined process for any $A \in \mathcal{I}$ and all $\xi \in \mathbb{R}^3$ by

$$\tilde{\mathcal{S}}_A(\omega,\xi) := \inf\left\{ \int_{A\setminus T(\omega)} f^{\infty,p}(\nabla w)\,\mathrm{d}x : w \in \mathrm{adm}_A(\omega,\xi) \right\}$$

$$\mathrm{adm}_A(\omega,\xi) := \left\{ w \in W_0^{1,p}\big(\overset{\circ}{A} \setminus \overline{T}(\omega), \mathbb{R}^3\big) : \fint_A w\,\mathrm{d}x = \xi \right\}$$

Theorem 2.1. *Let $\xi \in \mathbb{R}^3$ fixed, the function*

$$\begin{aligned} \tilde{\mathcal{S}}(.,\xi): \mathcal{I} &\longrightarrow L^1(\Omega,\mathcal{A},\mathbf{P}) \\ A &\longmapsto \tilde{\mathcal{S}}_A(.,\xi) \end{aligned}$$

is a subadditive process defined by $(\tau_z)_{z\in\mathbf{Z}^3}$, such that $\tau_z(\omega) = \omega - \hat{z}$, where $z = (\hat{z}, z_3)$. So for all $(I_n)_{n\in\mathbb{N}}$ in $\mathcal{I}$ the limit $\lim\limits_{n\to\infty} \dfrac{\tilde{\mathcal{S}}_{I_n}(\omega,\xi)}{|I_n|}$ exists almost surely for all $\omega\in\Omega$ and $\lim\limits_{n\to\infty} \dfrac{\tilde{\mathcal{S}}_{I_n}(\xi,\omega)}{|I_n|} = f_0^+(\xi)$.

Proof. See in [7] the extension to the 3D case. □

In the periodic case, we have the following corollary.

Corollary 2.1. *Suppose a periodic fiber distribution (that of a chessboard with a random position), then for all $\xi \in \mathbb{R}^3$,*

$$f_0^+(\xi) = \inf_{n\in\mathbb{N}^*} \frac{\tilde{\mathcal{S}}_{(0,n)^2}(\xi)}{n^2}$$

with

$$\tilde{\mathcal{S}}_A(\xi) := \inf\left\{\int_A f^{\infty,p}(\nabla w)\,\mathrm{d}x : w \in \mathrm{adm}_A(\xi)\right\}$$

$$\mathrm{adm}_A(\xi) := \left\{w \in W_0^{1,p}\big(A \setminus T, \mathbb{R}^3\big) : \fint_A w\,\mathrm{d}x = \xi\right\}$$

and

$$f_0^+(\xi) = \inf\left\{\int_Y (f^{\infty,p})(\nabla w)\,\mathrm{d}\hat{y} : w \in W_\#^{1,p}(Y, \mathbb{R}^3),\ \int_Y w\,\mathrm{d}\hat{y} = \xi,\ w = 0 \text{ on } D\right\}$$

where $W_\#^{1,p}(Y, \mathbb{R}^3)$ is a Y-periodic function of $W^{1,p}(Y, \mathbb{R}^3)$.

Proof. See [7]. □

Remark 2.1. We note here that in the periodic case, the energy density limit $f_0^+(\xi)$ equals that obtained by Licht & Michaille [8].

The following result is due to estimation (6). The function f_0^+ is a convex and p-order positively homogeneous function, satisfying the growth condition (4) with the same constant α, another constant $\beta > 0$, and the Lipschitz condition (5) with a constant $L > 0$ possibly different. From a mechanical point of view, in the elastic case, this energy density allows us to obtain the upper bound of the homogenized parameters.

2.2. The density definition f_0^-

To obtain the "best" lower bound of Γ-$\liminf H_\varepsilon$, we will build an upper process $\mathcal{S}^-$ lower than $\mathcal{S}^+$, such that the Fenchel–Moreau method works. For all $\hat{A} \in \mathcal{I}$ and all $\xi \in \mathbb{R}^3$, we define

$$\mathcal{S}_{\hat{A}}^-(\omega, \xi) = \inf\left\{\int_{\hat{A}\setminus D(\omega)} f^{\infty,p}(\nabla w, 0)\,\mathrm{d}x : w \in \mathrm{adm}_{\hat{A}}^-(\xi)\right\}$$

with

$$\mathrm{adm}_{\hat{A}}^-(\omega, \xi) := \left\{w \in W^{1,p}(\hat{A}, \mathbb{R}^3),\ w = 0 \text{ on } D(\omega),\ \fint_{\hat{A}} w\,\mathrm{d}x = \xi\right\}$$

Note that this process $\mathcal{S}^-$ is noted a subadditive process, which is due to the lack of a bound condition on $\hat{A}$. However, the Legendre–Fenchel transform is a subadditive process and verifies all conditions of the ergodic subadditive theorem [8]. From a mechanical point of view, in the elastic case, this energy density allows us to obtain the lower bound of the homogenized parameters.

Lemma 2.1. *The Legendre–Fenchel transform of $\xi \mapsto \frac{\mathcal{S}_{\hat{A}}^-(\omega,\xi)}{|\hat{A}|}(.)$ is defined for all ξ^* in $\mathbb{R}^3$ by*

$$\left(\frac{\mathcal{S}_{\hat{A}}^-}{|\hat{A}|}\right)^*(\xi^*) = \inf\left\{\frac{1}{|\hat{A}|}\int_{\hat{A}\setminus T} (f^{\infty,p})^*(\sigma, 0)\,\mathrm{d}x : \sigma \in \mathrm{adm}_{\hat{A}}^*(\xi^*)\right\}$$

with

$$\mathrm{adm}_{\hat{A}}^*(\xi^*) := \left\{\sigma \in L^q(\hat{A} \setminus D, \mathbf{M}^{3\times 2}) : -\mathrm{div}\sigma = \xi^* \text{ in } \hat{A} \setminus D,\ \sigma.\nu = 0 \text{ on } \partial\hat{A}\right\}$$

and ν is a unit normal vector of $\partial\hat{A}$.

Proof. By definition of the Legendre–Fenchel transform,

$$\left(\frac{\mathcal{S}_{\hat{A}}}{|\hat{A}|}\right)^* (\xi^*) = \sup_{\xi\in\mathbb{R}} \left\{ \xi^*\cdot\xi - \inf\left\{ \frac{1}{|\hat{A}|} \int_{\hat{A}\setminus D(\omega)} f^{p,\infty}(\nabla u)\,\mathrm{d}x,\ u\in \mathrm{adm}^-_{\hat{A}}(\omega,\xi) \right\}\right\}$$
$$= \sup_{(\xi,u)\in\mathbb{R}^3\times W^{1,p}\left(\hat{A},\mathbb{R}^3\right)} \left\{ \xi^*\cdot\xi - \left\{ \frac{1}{|\hat{A}|} \int_{\hat{A}\setminus D(\omega)} f^{p,\infty}(\nabla u)\,\mathrm{d}x + I(\xi,u) \right\}\right\} \tag{7}$$

where

$$I(a,u) = \begin{cases} 0 & \text{if } u\in \mathrm{adm}^-_{\hat{A}}(\omega,\xi) \\ +\infty & \text{else} \end{cases}$$

Let the variable $\zeta \in L^p\left(\hat{A}\setminus T, \mathbb{R}^3\right)$, we have:

$$\left(\frac{\mathcal{S}_{\hat{A}}}{|\hat{A}|}\right)^* (\xi^*) = \sup_{(\xi,\zeta)\in\mathbb{R}^3\times L^p\left(\hat{A}\setminus D(\omega),\mathbb{R}^3\right)} \left\{ \xi^*\cdot\xi + \int_{\hat{A}\setminus T} 0:\zeta\,\mathrm{d}x - \left\{ \frac{1}{|\hat{A}|} \int_{\hat{A}\setminus D(\omega)} f^{p,\infty}(\zeta)\,\mathrm{d}x + \widetilde{I}(\xi,\zeta) \right\}\right\}$$

where

$$\widetilde{I}(\xi,\zeta) = \begin{cases} 0 & \text{if } \exists u\in \mathrm{adm}^-_{\hat{A}}(\omega,\xi),\ \zeta=\nabla u \ \text{ in } Y\setminus D(\omega) \\ +\infty & \text{else} \end{cases}$$

So, $\left(\frac{\mathcal{S}_{\hat{A}}}{|\hat{A}|}\right)^* (\xi^*) = \left(J+\widetilde{I}\right)^*(\xi^*,0)$, where J is a functional of $\mathbb{R}^3\times L^p\left(\hat{A}\setminus D(\omega),\mathbb{R}^3\right)$ with

$$J(\xi,\zeta) = \frac{1}{|\hat{A}|} \int_{\hat{A}\setminus D(\omega)} f^{p,\infty}(\zeta)\,\mathrm{d}x$$

Consequently (see Proposition 9.4.1 in [9]), $\left(\frac{\mathcal{S}_{\hat{A}}}{|\hat{A}|}\right)^*$ is the following epigraphic sum:

$$\left(\frac{\mathcal{S}_{\hat{A}}}{|\hat{A}|}\right)^* (\xi^*) = \left(J^* \underset{c}{+} \widetilde{I}^*\right)(\xi^*,0)$$
$$= \inf_{(b^*,z^*)\in\mathbb{R}^3\times L^q\left(\hat{A}\setminus D(\omega),\mathbb{R}^3\right)} J^*\left(\xi^*-b^*,-z^*\right) + \widetilde{I}^*\left(b^*,z^*\right) \tag{8}$$

Now, we will explain the functions I^* and J^*.

$$\widetilde{I}^*\left(b^*,z^*\right) = \sup_{(b,z)\in\mathbb{R}^3\times L^p\left(\hat{A}\setminus D(\omega),\mathbb{R}^3\right)} \left\{ b^*\cdot b + \int_{\hat{A}\setminus D(\omega)} z:z^*\,\mathrm{d}x - \widetilde{I}(b,z) \right\}$$
$$= \sup_{(b,u)\in\mathbb{R}^3\times \mathrm{adm}^-_{\hat{A}}(\omega,b)} \left\{ b^*\cdot b + \int_{\hat{A}\setminus D(\omega)} \nabla u : z^*\,\mathrm{d}x \right\}$$
$$= \sup_{(b,u)\in\mathbb{R}^3\times \mathrm{adm}^-_{\hat{A}}(\omega,b)} \left\{ \int_{\hat{A}\setminus D(\omega)} u\cdot\frac{b^*}{|\hat{A}|}\,\mathrm{d}x - \int_{\hat{A}\setminus D(\omega)} u\cdot \mathrm{div}\left(z^*\right)\mathrm{d}x \right\}$$
$$= K\left(b^*,z^*\right) \tag{9}$$

where

$$K\left(b^*,z^*\right) = \begin{cases} 0 & \text{if } \mathrm{div}(z^*) = b^* \text{ in } \hat{A}\setminus D(\omega), \\ +\infty & \text{else} \end{cases}$$

Moreover, it is easy to see that

$$J^*\left(c^*, z^*\right) = \sup_{(c,z)\in\mathbb{R}^3\times L^p\left(\hat{A}\setminus D(\omega),\mathbb{R}^3\right)} \left\{ c^*\cdot c + \int\limits_{\hat{A}\setminus D(\omega)} z^* : z\,\mathrm{d}x - \frac{1}{|\hat{A}|}\int\limits_{\hat{A}\setminus D(\omega)} f^{p,\infty}(z)\,\mathrm{d}x \right\}$$

so

$$J^*\left(c^*, z^*\right) = \begin{cases} \dfrac{1}{|\hat{A}|}\displaystyle\int\limits_{\hat{A}\setminus D(\omega)} (f^{p,\infty})^*\left(|\hat{A}|z^*\right)\,\mathrm{d}x & \text{if } c^* = 0 \\ +\infty & \text{else} \end{cases} \tag{10}$$

We can conclude with (8), (9) and (10). □

Theorem 2.2. *The process $\hat{A} \mapsto \inf\left\{\int_{\hat{A}\setminus T} f^{\infty,p}(\sigma,0)\,\mathrm{d}x : \sigma \in \mathrm{adm}^*_{\hat{A}}(\xi^*)\right\}$ is a sub-additive process defined by $(\tau_z)_{z\in\mathbf{Z}^2}$. Therefore, for any regular family $(I_n)_{n\in\mathbb{N}}$ of $\mathcal{I}$, almost surely and for all $\xi^*\in\mathbb{R}^3$, we have:*

$$\lim_{n\to+\infty}\left(\frac{\mathcal{S}^-_{I_n}}{|I_n|}\right)^*(\omega,\xi^*) = \inf_{n\in\mathbb{N}^*}\int\limits_{\Omega}\left(\frac{\mathcal{S}^-_{(0,n)^2}}{n^2}\right)^*(\omega,\xi^*)\,\mathrm{d}\mathbf{P}(\omega)$$

Proof. One easily verifies that all hypotheses of the subadditive Theorem [8] are satisfied. □

Corollary 2.2 *(The definition of density f_0^-). Almost surely and for all $\xi\in\mathbb{R}^3$, we have for all regular family $(I_n)_{n\in\mathbb{N}}$ of $\mathcal{I}$*

$$\lim_{n\to+\infty}\frac{\mathcal{S}^-_{I_n}(\omega,\xi)}{|I_n|} = \sup_{n\in\mathbb{N}^*}\left(\oint\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\right)(\xi) \tag{11}$$

with

$$\left(\oint\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\right)(\xi) = \inf\left\{\int\limits_{\Omega}\frac{\mathcal{S}^-_{n\hat{Y}}(\omega,X(\omega))}{n^2}\,\mathrm{d}P(\omega) : X\in L^1_P(\Omega),\ \int\limits_{\Omega}X(\omega)\,\mathrm{d}\mathbf{P}(\omega) = \xi\right\}$$

we will note $f_0^-(\xi)$ this limit. For all $\xi\in\mathbb{R}^3$

$$\lim_{n\to+\infty}\left(\oint\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\right)(\xi) = f_0^-(\xi) \tag{12}$$

Proof of (11). The almost sure limit (11) is due to Theorem 2.2. Indeed, with this theorem and [10] we have for all $\xi^*\in\mathbb{R}^3$

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{\mathcal{S}^-_{I_n}}{|I_n|}\right)^*(\omega,\xi^*) &= \inf_{n\in\mathbb{N}^*}\int\limits_{\Omega}\left(\frac{\mathcal{S}^-_{(0,n)^2}}{n^2}\right)^*(\omega,\xi^*)\,\mathrm{d}\mathbf{P}(\omega)\\ &= \inf_{n\in\mathbb{N}^*}\left(\oint\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\right)^*(\xi^*)\\ &= \left(\sup_{n\in\mathbb{N}^*}\oint\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\right)^*(\xi^*)\end{aligned}$$

Using the Mosco-convergence that gives the equivalence of a simple convergence of Legendre–Fenchel of processed $h_n^*\to h^*$ and simple convergence $h_n\to h$, the proposal is essential for viewing, also demonstrating the limit (12)

Let $\xi^*\in\mathbb{R}^3$, by definition of the limit f_0^- and of the Mosco-convergence, we have for all $\omega\in\Omega$ the almost sure limit,

$$\lim_{n\to+\infty}\left(\frac{S_{n\hat{Y}}}{n^2}\right)(\omega,\xi^* = \left(f_0^-\right)^*(\xi^*)$$

We now take the mathematical expectancy of this equality; using the dominated convergence theorem of Lebesgue, we get:

$$\lim_{n\to+\infty}\mathbf{E}\left[\left(\frac{S_{n\hat{Y}}}{n^2}\right)(\omega,\xi^*)\right] = \mathbf{E}\left[\left(f_0^-\right)^*(\xi^*)\right] = \left(f_0^-\right)^*(\xi^*) \tag{13}$$

Then by the theorem of M. Valadier [10], for all $n \in \mathbb{N}$ and for all $\xi^* \in \mathbb{R}^3$,

$$\mathbf{E}\Big[\Big(\frac{S_{n\hat{Y}}}{n^2}\Big)(\omega,\xi^*)\Big] = \Big(\oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^*(\xi^*)$$

the equality (13) becomes

$$\lim_{n\to+\infty}\Big(\oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^*(\xi^*) = \Big(f_0^-\Big)^*(\xi^*)$$

We can conclude with the Mosco-convergence argument. □

The energy density f_0^- is a convex and p-order positively homogeneous function, satisfying the growth condition (4) with the same constant α, another constant $\beta > 0$, and the Lipschitz condition (5) with a constant $\ell > 0$, possibly different.

3. Variational bounds of the energy associated with structure ($\mathcal{S}$)

Remember that we would like to establish the next frame of the Γ-Limit functional H_ε. We need a compactness result giving us weak convergence in $L^p(\mathcal{O},\mathbb{R}^3)$ suites of finite energy. We recall that this convergence sets the topology for which the functional terminal is H_ε.

Proposition 3.1 *(Compactness result). Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of $W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R})$ such that $\sup_{\varepsilon>0}\Big(H_\varepsilon(\omega,u_\varepsilon) - \int_{\mathcal{O}} L\cdot u_\varepsilon\,\mathrm{d}x\Big) < +\infty$ for almost every $\omega \in \Omega$. Then, for almost every $\omega \in \Omega$, there exists a sub-sequence, and $(u,v) \in L^p(\mathcal{O},\mathbb{R}^3) \times V_0$ such that*

(i) $u_\varepsilon \rightharpoonup u$ *in* $L^p(\mathcal{O},\mathbb{R}^3)$;
(ii)

$$a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon \rightharpoonup v \text{ in } L^p(\mathcal{O},\mathbb{R}) \tag{14}$$

$$a(\omega,\frac{\cdot}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial v}{\partial x_3} \text{ in } L^p(\mathcal{O},\mathbb{R}) \tag{15}$$

This result is already proven in [4].

4. Estimate of an upper bound

Proposition 4.1. *Suppose a set $\Omega' \in \mathcal{A}$ with $\mathbf{P}(\Omega')$ such that $(u,v) \in L^p(\mathcal{O},\mathbb{R}^3) \times V_0$, and a sequence $(u_\varepsilon(\omega,.))_{\varepsilon>0} \in L^p(\mathcal{O},\mathbb{R}^3)$ satisfying, for $\omega \in \Omega$,*

$$\int_{\mathcal{O}} f_0^+(u - \frac{1}{\theta}v)\,\mathrm{d}x + G_0(\frac{1}{\theta}v) = \limsup_{\varepsilon\to 0} H_\varepsilon(\omega,u_\varepsilon(\omega,.))$$

$$(u_\varepsilon(\omega,.), a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon(\omega,.) \rightharpoonup\rightharpoonup (u,v) \quad \text{in } L^p(\mathcal{O},\mathbb{R}^3) \times V_0$$

So, we have almost surely

$$(\Gamma\text{-}\limsup H_\varepsilon)(\omega,u) \le F_0^+(u - \frac{1}{\theta}v) + G_0(\frac{1}{\theta}v)$$

The homogenized energy density f_0^+ depends on the statistical volume ratio θ and on the virtual displacement field v due to fibers.

Proof. The demonstration involves three steps; let us remember that $F_\varepsilon(\omega,.)$ is defined in $L^p(\mathcal{O},\mathbb{R}^3)$ with

$$F_\varepsilon(\omega,u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}) \\ +\infty & \text{else} \end{cases}$$

Step 1. We suppose that $(u,v) \in \mathcal{C}^1_c(\mathcal{O},\mathbb{R}^3) \times \Big(\mathcal{C}^1_c(\mathcal{O},\mathbb{R}^3)\ \cap V_0(\mathcal{O},\mathbb{R}^3)\Big)$; we prove that Ω' with $\mathbf{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$, there exists $(u_\varepsilon(\omega,.))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ satisfying

$$\begin{aligned}
&u_\varepsilon(\omega,.), \rightharpoonup\rightharpoonup (u,v) \quad \text{in } L^p(\mathcal{O},\mathbb{R}^3)\times V_0\\
&\lim_{\varepsilon\to 0} F_\varepsilon(\omega,u_\varepsilon(\omega,.)) = \int_{\mathcal{O}} f_0^+(u-\frac{1}{\theta}v)\,\mathrm{d}x \\
&\lim_{\varepsilon\to 0} G_\varepsilon(\omega,u_\varepsilon(\omega,.)) = G_0(\frac{1}{\theta}\frac{\partial v}{\partial x_3})
\end{aligned} \tag{16}$$

Let $\eta \in \mathbf{Q}^+$ (a positive rational number) and $(Q_{i,\eta})_{i\in I_\eta}$ a finite family of cubes $\mathbb{R}^3$ with diameter η in $\mathcal{O}$ such that

$$\Big|\mathcal{O}\setminus \bigcup_{i\in I_\eta} Q_{i,\eta}\Big| = 0.$$

We define $z_\eta := \sum_{i\in I_\eta} z_{i,\eta} 1_{Q_{i,\eta}}$ and $z_{i,\eta} = (u-\frac{1}{\theta}v)(x_{i,\eta})$ where $x_{i,\eta}$ is arbitrarily chosen in $Q_{i,\eta}$. It is clear that $z_\eta \to u-\frac{1}{\theta}v$ in $L^p(\mathcal{O},\mathbb{R}^3)$ when $\eta\to 0$.

Let $C_{i,\eta,\varepsilon}$ the upper cube $\mathcal{I}$ included in $\frac{1}{\varepsilon}Q_{i,\eta}$ and $w_{i,\eta,\varepsilon} \in \mathrm{adm}_{C_{i,\eta,\varepsilon}}(\omega, z_{i,\eta}(x_{i,\eta}))$ a minimizer of $\tilde{\mathcal{S}}_{C_{i,\eta,\varepsilon}}(\omega, z_{i,\eta}(x_{i,\eta}))$ extend to zero out of $C_{i,\eta,\varepsilon}\setminus T(\omega)$. Note $(C_{i,\eta,\varepsilon})_\varepsilon$ a regular family of $\mathbb{R}^3$. With each cube $Q =]a,b[\in \mathbb{R}^3$, we associate the cube $Q' :=]0,b[$. Take the family $(C'_{i,\eta,\varepsilon})_\varepsilon$. We have:

$$\frac{|C_{i,\eta,\varepsilon}|}{|C'_{i,\eta,\varepsilon}|} = \frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}\times\frac{\left|Q_{i,\eta}\right|}{\left|Q'_{i,\eta}\right|}\times\frac{\left|\frac{1}{\varepsilon}Q'_{i,\eta}\right|}{|C'_{i,\eta,\varepsilon}|}$$

It is easily seen that

$$\lim_{\varepsilon\to 0}\frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|} = \lim_{\varepsilon\to 0}\frac{|C'_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q'_{i,\eta}\right|} = 1$$

so for ε very small $\frac{|C_{i,\eta,\varepsilon}|}{|C'_{i,\eta,\varepsilon}|} \le 2\frac{\left|Q_{i,\eta}\right|}{\left|Q'_{i,\eta}\right|}$. The family $(C'_{i,\eta,\varepsilon})_\varepsilon$ satisfies conditions (i′)–(iv′) of the subadditive theorem [11–13].

By Theorem 2.1,

$$\lim_{\varepsilon\to 0}\frac{\tilde{\mathcal{S}}_{C_{i,\eta,\varepsilon}}(\omega, z_{i,\eta}(x_{i,\eta}))}{\left|C_{i,\eta,\varepsilon}\right|} = \lim_{\varepsilon\to 0}\frac{1}{\left|C_{i,\eta,\varepsilon}\right|}\int_{C_{i,\eta,\varepsilon}\setminus T(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,y))\,\mathrm{d}y = f_0^+(z_{i,\eta}(x_{i,\eta})) \tag{17}$$

let $\omega\in\Omega_{i,\eta}$ with $\mathbf{P}(\Omega_{i,\eta}) = 1$, note $\Omega' := \bigcap_{\eta\in\mathbf{Q}^+}\bigcap_{i\in I_\eta}\Omega_{i,\eta}$ and fix $\omega\in\Omega'$. With (17), we can write

$$\begin{aligned}
\int_{\mathcal{O}} f_0^+(z_\eta)\,\mathrm{d}x &= \sum_{i\in I_\eta}\left|Q_{i,\eta}\right| f_0^+(z_{i,\eta}(x_{i,\eta}))\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\left|Q_{i,\eta}\right|\frac{1}{\left|C_{i,\eta,\varepsilon}\right|}\int_{C_{i,\eta,\varepsilon}\setminus T(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,y))\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\frac{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}{|C_{i,\eta,\varepsilon}|}\int_{Q_{i,\eta}\setminus\varepsilon T(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\int_{Q_{i,\eta}\setminus T_\varepsilon(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))\,\mathrm{d}y
\end{aligned} \tag{18}$$

Indeed, $\lim_{\varepsilon\to 0}\frac{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}{|C_{i,\eta,\varepsilon}|} = 1$ and $w_{i,\eta,\varepsilon} = 0$ out of $C_{i,\varepsilon,\eta}\setminus T(\omega)$.

Now we can define the function $u_{\eta,\varepsilon}$ by

$$u_{\eta,\varepsilon} := (\frac{1}{\theta}v + \varepsilon\xi_{\varepsilon,\eta}(\omega,\frac{\hat{x}}{\varepsilon})) + \sum_{i\in I_\eta} w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})1_{Q_{i,\eta}}(x)$$

where

$$\theta\int\limits_{\mathcal{O}} g^{\perp}(\frac{1}{\theta}\frac{\partial u}{\partial x_3})\,\mathrm{d}x = \theta\int\limits_{\mathcal{O}} \inf_{\xi\in\mathbf{M}^{3\times 2}} g\left(\xi+\frac{1}{\theta}\hat{\nabla}u,\ \frac{1}{\theta}\frac{\partial u}{\partial x_3}\right)\mathrm{d}x$$
$$\geq \theta\int\limits_{\mathcal{O}} g\left(\xi_\eta+\hat{\nabla}u,\frac{1}{\theta}\frac{\partial u}{\partial x_3}\right)\mathrm{d}x-\eta \tag{19}$$

The measurability of $x\mapsto\xi_\eta(x)$ is due to the coercivity of g and to the Measurability Theorem (see [5]). Also, from the density of $\mathcal{C}_c^1(\mathcal{O},\mathbf{M}^{3\times 2})$ in $L^p(\mathcal{O},\mathbf{M}^{3\times 2})$ and from the Lipschitz property of the convex function g, we can suppose that $\xi_\eta\in\mathcal{C}_c^1(\mathcal{O},\mathbf{M}^{3\times 2})$. So we note $\xi_{\varepsilon,\eta}:=\varepsilon\rho(\omega,.)\xi_\eta$ where $\rho(\omega,.)\in\mathcal{C}_c^1(\mathbb{R}^2,\mathbb{R}^2)$ and $\rho(\omega,\hat{y})=\hat{y}$ for all $\hat{y}\in D(\omega)$.

It is easily seen that $a(\omega,\frac{\cdot}{\varepsilon})u_{\eta,\varepsilon}=a(\omega,\frac{\cdot}{\varepsilon})(\frac{1}{\theta}v+\varepsilon\xi_{\varepsilon,\eta}(\omega,\frac{\hat{x}}{\varepsilon}))$. On the other hand,

$$\lim_{\varepsilon\to 0} a(\omega,\frac{\cdot}{\varepsilon})u_{\eta,\varepsilon}=v\ \text{ weakly in } L^p(\mathcal{O},\mathbb{R}^3)$$
$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0} u_{\eta,\varepsilon}=\frac{1}{\theta}v+u-\frac{1}{\theta}v=u\ \text{ weakly in } L^p(\mathcal{O},\mathbb{R}^3)$$

The first limit is a consequence of Proposition 5.3 in [14] (i.e. $a(\omega,\frac{\cdot}{\varepsilon})\rightharpoonup\theta$). To establish the second one, we only need to consider a $w_{i,\eta,\varepsilon}\in\mathrm{adm}_{C_{i,\eta,\varepsilon}}(\omega,z_{i,\eta})$,

$$\fint\limits_{Q_{i,\eta}} w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})\,\mathrm{d}x=\frac{1}{|Q_{i,\eta}|}\int\limits_{\varepsilon C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})\,\mathrm{d}x$$
$$=\frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}\fint\limits_{C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega,x)\,\mathrm{d}x$$
$$=\frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}z_{i,\eta}$$

Letting successively $\varepsilon\to 0$ and $\eta\to 0$, we obtain $\lim\limits_{\eta\to 0}\lim\limits_{\varepsilon\to 0}u_{\eta,\varepsilon}=u$.

By the definition of $u_{\eta,\varepsilon}$,

$$\int\limits_{\mathcal{O}} f_0^+(z_{\delta,\eta})\,\mathrm{d}x=\lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\int\limits_{Q_{i,\eta}\setminus T_\varepsilon(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon}))\,\mathrm{d}x$$
$$=\lim_{\varepsilon\to 0}\int\limits_{\mathcal{O}\setminus T_\varepsilon(\omega)} f^{\infty,p}(\nabla u_{\eta,\varepsilon}(\omega,\frac{x}{\varepsilon}))\,\mathrm{d}x=\lim_{\varepsilon\to 0}F_\varepsilon(\omega,u_{\eta,\varepsilon}(\omega,.)$$

When $\eta\to 0$ and noticing that the function $w\mapsto\int_{\mathcal{O}} f_0^+(w)\,\mathrm{d}x$ is continuous in $L^p(\mathcal{O},\mathbb{R}^3)$, we obtain:

$$\int\limits_{\mathcal{O}} f_0^+(u-\frac{1}{\theta}v)=\lim_{\eta\to 0}\lim_{\varepsilon\to O}F_\varepsilon(\omega,u_{\eta,\varepsilon}(\omega,.)) \tag{20}$$

Thanks to (19)

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}G_\varepsilon(\omega,a(\omega,\frac{\cdot}{\varepsilon})u_{\delta,\eta,\varepsilon})=\theta\int\limits_{\mathcal{O}}(g^{\perp})^{**}(\frac{1}{\theta}\frac{\partial v}{\partial x_3}) \tag{21}$$

Finally, a vector argument diagonalization yields:

$$\begin{aligned}&(u_\varepsilon(\omega,.),a(\omega,\tfrac{\cdot}{\varepsilon})u_\varepsilon)\rightharpoonup\rightharpoonup(u,v)\ \text{ in } L^p(\mathcal{O},\mathbb{R}^3)\times V_0\\&\lim\nolimits_{\varepsilon\to 0}F_\varepsilon(\omega,u_\varepsilon(\omega,.))=\textstyle\int_{\mathcal{O}} f_0^+(u-\frac{1}{\theta}v)\,\mathrm{d}x\\&\lim\nolimits_{\varepsilon\to 0}G_\varepsilon(\omega,u_\varepsilon(\omega,.))=G_0(\tfrac{1}{\theta}\tfrac{\partial v}{\partial x_3})\end{aligned} \tag{22}$$

Step 2. The result of Step 1 is established assuming only $(u,v)\in L^p(\mathcal{O},\mathbb{R}^3)\times\left(\mathcal{C}_c^1(\mathcal{O},\mathbb{R}^3)\ \cap V_0(\mathcal{O},\mathbb{R}^3)\right)$.

We construct a sequence $(u_n(\omega,\cdot),v)$ in $\mathcal{C}_c^1(\mathcal{O},\mathbb{R}^3)\times\left(\mathcal{C}_c^1(\mathcal{O},\mathbb{R}^3)\ \cap V_0(\mathcal{O},\mathbb{R}^3)\right)$ where u_n weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$. After step 1, we can construct a sequence $(u_{\varepsilon,n})_{\varepsilon>0}$ that weakly converges to u_n, satisfying (16). We then obtain our result by diagonalization.

Step 3. Now let us apply (22) for $(u, v_\eta) \in L^p(\mathcal{O}, \mathbb{R}^3) \times V_0$, where v_η is a minimizer sequence of $H_0(u)$, then there exits a sequence $u_{\varepsilon,\eta} \in W_0^{1,p}(\mathcal{O}, \mathbb{R}^3)$ weakly converging to u in $L^p(\mathcal{O}, \mathbb{R}^3)$ such that

$$\lim_{\varepsilon\to 0} H_\varepsilon(\omega, u_{\varepsilon,\eta}(\omega, .)) = \int_{\mathcal{O}} f_0^+(u - \frac{1}{\theta} v)\,\mathrm{d}x + G_0(\frac{1}{\theta}\frac{\partial v}{\partial x_3}) + \eta$$

We end the proof by $\eta \to 0$ and again using a diagonalization argument. $\square$

5. Estimate of a lower bound

5.1. Estimation of the lower bound in the matrix

Proposition 5.1. *For all $(u_\varepsilon, \mathbf{1}_{\mathcal{O}\cap T_\varepsilon} u_\varepsilon)$ weakly converging (u, v) in $L^p(\mathcal{O}, \mathbb{R}^3) \times V_0$ with $\sup_{\varepsilon>0} H_\varepsilon(\omega, u_\varepsilon) < +\infty$, we have for almost every ω in Ω*

$$F_0^-(u - \frac{1}{\theta} v) \le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega, u_\varepsilon)$$

Proof. We can assume

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\omega, u_\varepsilon) < +\infty \tag{23}$$

and with (3), $F_\varepsilon(\omega, u_\varepsilon) = \varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\nabla u_\varepsilon)\,\mathrm{d}x$. The homogeneity of $f^{\infty,p}$ gives us:

$$\liminf_{\varepsilon\to 0} \varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\nabla u_\varepsilon)\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x$$

We want to show that, for all $n \in \mathbb{N}^*$:

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x \ge \int_{\mathcal{O}} \oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big(u - \frac{1}{\theta} v\Big)\,\mathrm{d}x$$

Note Step$(\mathcal{O})$ the set of step functions $w = \sum_{i\in I} z_i^* \mathbf{1}_{\mathcal{O}_i}$ where $(\mathcal{O}_i)_{i\in I}$ is a family included in cubes $\mathcal{O}$ such as $|\mathcal{O} \setminus \bigcup_{i\in I} \mathcal{O}_i| = 0$ and $z_i^* \in \mathbf{Q}^3$. Furthermore Step$(\mathcal{O})$ is a dense subspace of $L^q(\mathcal{O}, \mathbb{R}^3)$.

Consider $w = \sum_{i\in I} z^* \mathbf{1}_{\mathcal{O}_i}$ in Step$(\mathcal{O})$; we establish $n \in \mathbb{N}^*$. Let $\sigma_{i,n}$ a minimizer of $\Big(\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^*(z_i^*)$ which is extended by 0 in $D \cap n\hat{Y}$. Then we extend by covariance the strain field $\sigma_{i,n}$ in $\mathbb{R}^2 \setminus D$, meaning:

$$\bar{\sigma}_{i,n}(\omega, \hat{x}) := \sigma_{i,n}(\tau_z\omega, \hat{x} - z) \text{ when } \hat{x} \in n\hat{Y} + z,\ z \in n\mathbf{Z}^2$$

It is easy to see that $\bar{\sigma}_{i,n}$ satisfies *the covariance property*: for all $\hat{x} \in \mathbb{R}^2$ and all $z \in n\mathbf{Z}^2$,

$$\bar{\sigma}_{i,n}(\omega, \hat{x} + z) = \bar{\sigma}_{i,n}(\tau_z\omega, \hat{x}) \tag{24}$$

and, $\bar{\sigma}_{i,n}(\omega, .) = 0$ in $D(\omega)$. Thanks to the boundary conditions satisfied by $\sigma_{i,n}$, we have:

$$-\mathrm{div}\,\bar{\sigma}_{i,n} = z_i^* \text{ on } \mathbb{R}^2 \setminus D$$

From the generalization of Birkhoff's ergodic theorem and from (24), we deduce for almost every $\omega \in \Omega$

$$\int_{\mathcal{O}_i\setminus T_\varepsilon} (f^{\infty,p})^*(\bar{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), 0)\,\mathrm{d}x = \int_{\mathcal{O}_i} (f^{\infty,p})^*(\bar{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), 0)\,\mathrm{d}x \to |\mathcal{O}_i| \mathbf{E} \fint_{n\hat{Y}} (f^{\infty,p})^*(\sigma_{i,n}, 0)\,\mathrm{d}\hat{x}$$

when $\varepsilon \to 0$, i.e., by definition of $\sigma_{i,n}$,

$$\int_{\mathcal{O}_i\setminus T_\varepsilon} f^{\infty,p}(\bar{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), 0)\,\mathrm{d}x \to |\mathcal{O}_i| \mathbf{E}\Big(\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^*(z_i^*) \tag{25}$$

Let $\Omega'' = \bigcup_{z^*\in Q^3} \Omega_z$ where $P(\Omega_{z^*}) = 1$ and the set of sour probability with convergence (25) for $z^* \in \mathbf{Q}^3$ fixed. Convergence (25) is valid for all ω in Ω'. With the Fenchel inequality, for almost every $x \in \mathcal{O}\setminus T_\varepsilon$

$$f^{\infty,p}(\varepsilon\nabla u_\varepsilon(x)) \geq f^{\infty,p}(\varepsilon\nabla u_\varepsilon(x))\varphi_{i,\delta}(x)$$
$$\geq \varepsilon\hat{\nabla}u_\varepsilon(x) : \overline{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon})\varphi_{i,\delta}(x) - (f^{\infty,p})^*(\overline{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), 0)\varphi_{i,\delta}(x)$$

Summing for $i \in I$ and integrating over $\mathcal{O}\backslash T_\varepsilon$, we get

$$\int_{\mathcal{O}\backslash T_\varepsilon} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x \geq \sum_{i\in I}\Big(\int_{\mathcal{O}\backslash T_\varepsilon} \varepsilon\nabla u_\varepsilon : \overline{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon})\varphi_{i,\delta}\,\mathrm{d}x - \int_{\mathcal{O}_i} (f^{\infty,p})^*(\overline{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}))\,\mathrm{d}x\Big)$$

Integrating by parts the first terms of the right-hand side, and noting that

$$-\varepsilon \operatorname{div} \bar{\sigma}_{i,n}(\frac{\cdot}{\varepsilon}) = z_i^* \text{ in } \mathcal{O} \setminus T_\varepsilon$$

we obtain:

$$\int_{\mathcal{O}\backslash T_\varepsilon} f^{\infty,p}\,(\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x \geq \sum_{i\in I}\Big(\int_{\mathcal{O}\backslash T_\varepsilon} u_\varepsilon \cdot z_i^*\varphi_{i,\delta}\,\mathrm{d}x - \int_{\mathcal{O}\backslash T_\varepsilon} \varepsilon u_\varepsilon \cdot \bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\mathrm{grad}\,\varphi_{i,\delta}\,\mathrm{d}x$$
$$+ \int_{\partial T_\varepsilon\cap\mathcal{O}} \varepsilon u_\varepsilon.\sigma_{i,n}(\frac{\hat{x}}{\varepsilon})\nu\varphi_{i,\delta}\,\mathrm{d}\mathcal{H}^2\Big) - \sum_{i\in I}\int_{\mathcal{O}_i}(f^{\infty,p})^*(\bar{\sigma}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), 0)\,\mathrm{d}x \tag{26}$$

where ν is the normal unit vector on T. We can now show that the second term

$$\int_{\mathcal{O}\backslash T_\varepsilon} \varepsilon u_\varepsilon \cdot \bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\mathrm{grad}\,\varphi_{i,\delta}\,\mathrm{d}x$$

on the right-hand side of (26) almost surely approaches 0 when $\varepsilon \to 0$. Indeed, the function $\mathbf{1}_{\mathcal{O}\backslash T_\varepsilon}u_\varepsilon \rightharpoonup u - v$ in $L^p(\mathcal{O}, \mathbb{R}^3)$. On the other hand,

$$\left|\frac{1}{\varepsilon}\widehat{\mathcal{O}} \setminus \sum_{z\in I_\varepsilon}(n\hat{Y} + z)\right| = 0$$

$\#(I_\varepsilon) = \frac{|\widehat{\mathcal{O}}|}{n^2\varepsilon^2}$ and

$$\int_{\mathcal{O}\backslash T_\varepsilon} \left|\bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\right|^q \mathrm{d}x = h\varepsilon^2 \int_{\frac{1}{\varepsilon}\widehat{\mathcal{O}}\backslash D} \left|\bar{\sigma}_{i,n}(\hat{x})\right|^q \mathrm{d}\hat{x}$$
$$\leq \sum_{z\in I_\varepsilon} h\varepsilon^2 \int_{n\hat{Y}+z\backslash D} \left|\bar{\sigma}_{i,n}(\hat{x})\right|^q \mathrm{d}\hat{x}$$
$$= \sum_{z\in I_\varepsilon} h\varepsilon^2 \int_{n\hat{Y}\backslash D(\tau_z\omega)} \left|\bar{\sigma}_{i,n}(\tau_z\omega, \hat{x})\right|^q \mathrm{d}\hat{x}$$
$$= \frac{|\widehat{\mathcal{O}}|}{n^2}\,\frac{1}{\#(I_\varepsilon)}\sum_{z\in I_\varepsilon}\int_{n\hat{Y}\backslash D(\tau_z\omega)} \left|\bar{\sigma}_{i,n}(\tau_z\omega, \hat{x})\right|^q \mathrm{d}\hat{x}$$
$$\leq C\frac{1}{\#(I_\varepsilon)}\sum_{z\in I_\varepsilon}\frac{1}{n^2}\int_{nY\backslash T(\tau_z\omega)} (f^{\infty,p})^*(\bar{\sigma}_{i,n}(\tau_z\omega, y), 0)\,y$$
$$= C\frac{1}{\#(I_\varepsilon)}\sum_{z\in I_\varepsilon}\Big(\frac{\mathcal{S}^-_{n\hat{Y}}(\tau_z\omega, .)}{n^2}\Big)^*(z_i^*) \tag{27}$$

from coercivity $(f^{\infty,p})^*$ and from the fact that the constant C is positive and does not depend on $\widehat{\mathcal{O}}$, we obtain (27). Moreover, thanks to Proposition 5.3 in [14] we have for almost every $\omega \in \Omega$

$$\lim_{\varepsilon\to 0}\frac{1}{\#(I_\varepsilon)}\sum_{z\in I_\varepsilon}\Big(\frac{\mathcal{S}^-_{n\hat{Y}}(\tau_z\omega, .)}{n^2}\Big)^*(z_i^*) = \mathbf{E}\Big(\Big(\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^*(z_i^*)\Big)$$

Thus, $\sup_{\varepsilon>0}\int_{\mathcal{O}\backslash T_\varepsilon}\left|\bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\right|^q \mathrm{d}x < +\infty$, and the assertion is proved.

From (26) and (25), we obtain:

$$\liminf_{\varepsilon\to 0}\int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x \geq \sum_{i\in I}\int_{\mathcal{O}}(u-v).z_i^*\varphi_{i,\delta}\,\mathrm{d}x - \sum_{i\in I}|\mathcal{O}_i|\mathbf{E}\Big(\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^*(z_i^*)$$
$$+\sum_{i\in I}\liminf_{\varepsilon\to 0}\int_{\partial T_\varepsilon\cap\mathcal{O}}\varepsilon u_\varepsilon.\bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\nu\varphi_{i,\delta}\,\mathrm{d}\mathcal{H}^2 \tag{28}$$

Considering the fact that the energy in the fibers is uniformly bounded, we now estimate the limit

$$\liminf_{\varepsilon\to 0}\int_{\partial T_\varepsilon\cap\mathcal{O}}\varepsilon u_\varepsilon\cdot\bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\nu\varphi_{i,\delta}\,\mathrm{d}\mathcal{H}^2$$

Lemma 5.1.

$$\liminf_{\varepsilon\to 0}\int_{\partial T_\varepsilon\cap\mathcal{O}}\varepsilon u_\varepsilon.\bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\nu\varphi_{i,\delta}\,\mathrm{d}\mathcal{H}^2 = \int_{\mathcal{O}} v\big(1-\frac{|n\hat{Y}|}{|n\hat{Y}\cap D|}\big)z_i^*\varphi_{i,\delta} \tag{29}$$

Proof. Consider the (random) inhomogeneous problem of Neumann defined in $n\hat{Y}\cap D$ by:

$$\begin{cases} -\operatorname{div}\big(|\nabla U|^{p-2}\nabla U\big) = \big(1-\frac{|n\hat{Y}|}{|n\hat{Y}\cap D|}\big)z_i^* \text{ in } n\hat{Y}\cap D \\ |\nabla U|^{p-2}\nabla U.\nu = -\sigma_{i,n}.\nu \text{ in } \partial D\cap n\hat{Y} \end{cases} \tag{30}$$

Note that problem (30) is posed by the criterion of compatibility and is verified through

$$\int_{\partial D\cap n\hat{Y}} -\sigma_{i,n}\cdot\nu\,\mathrm{d}\mathcal{H}^1 + \int_{D\cap n\hat{Y}}\big(1-\frac{|n\hat{Y}|}{|n\hat{Y}\cap D|}\big)z_i^*\,\mathrm{d}\hat{x} = 0$$

i.e.

$$\int_{\partial D\cap n\hat{Y}} -\sigma_{i,n}\cdot\nu\,\mathrm{d}\mathcal{H}^1 + \big(|n\hat{Y}\cap D|-|n\hat{Y}|\big)z_i^* = 0$$

Indeed, $\sigma_{i,n}\in\widetilde{\operatorname{adm}^*_{n\hat{Y}}}(z_i^*)$, which means that

$$-\operatorname{div}\sigma_{i,n} = z_i^* \;\text{ in }\; n\hat{Y}\setminus D \;\text{ and }\; \sigma_{i,n}\mu = 0 \;\text{ on }\; \partial n\hat{Y}$$

where μ is the outgoing normal vector $\partial n\hat{Y}$; so, integrating over $n\hat{Y}\setminus D$ and using Green's formula,

$$-\int_{\partial D\cap n\hat{Y}}\sigma_{i,n}\mu\,\mathrm{d}\mathcal{H}^1 = |n\hat{Y}\setminus D|z_i^* = \big(|n\hat{Y}|-|n\hat{Y}\cap D|\big)z_i^*$$

For further details, see [9], Chapter 15. So was proved the existence of at least one solution to problem (30).

Let $\xi_{i,\eta}=\nabla U$, which is extended on D by covariance:

$$\bar{\xi}_{i,n}(\omega,\hat{x}) := \xi_{i,n}(\tau_z\omega,\hat{x}-z) \quad \text{for } \hat{x}\in n\hat{Y}+z,\ z\in n\mathbf{Z}^2$$

By problem (30) and Green's formula,

$$\int_{\partial T_\varepsilon\cap\mathcal{O}}\varepsilon u_\varepsilon\cdot\bar{\sigma}_{i,n}(\frac{\hat{x}}{\varepsilon})\nu\varphi_{i,\delta}\,\mathrm{d}\mathcal{H}^2 = \int_{T_\varepsilon\cap\mathcal{O}}\varepsilon\hat{\nabla}u_\varepsilon:\bar{\xi}_{i,n}(\frac{\hat{x}}{\varepsilon})\varphi_{i,\delta}\,\mathrm{d}x - \int_{\mathcal{O}\cap T_\varepsilon}\varepsilon u_\varepsilon\cdot\bar{\xi}_{i,n}(\frac{\hat{x}}{\varepsilon})\operatorname{grad}\varphi_{i,\delta}\,\mathrm{d}x$$
$$+\int_{\mathcal{O}\cap T_\varepsilon}u_\varepsilon\big(1-\frac{|n\hat{Y}|}{|n\hat{Y}\cap D|}\big)z_i^*\varphi_{i,\delta}\,\mathrm{d}x \tag{31}$$

By repeating the arguments that led to estimate (27), we obtain, for almost every ω on Ω:

$$\int_{T_\varepsilon \cap \mathcal{O}} \left|\bar{\xi}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon})\right|^q \mathrm{d}x \to \mathbf{E}\Big(\int_{n\hat{Y} \cap D} \left|\xi_{i,n}(.,x)\right|^q \mathrm{d}x\Big)$$

so that $\sup_{\varepsilon>0} \int_{T_\varepsilon \cap \mathcal{O}} \left|\bar{\xi}_{i,n}(\omega, \frac{\hat{x}}{\varepsilon})\right|^q \mathrm{d}x < +\infty$.

In addition, from $\sup_{\varepsilon>0} H_\varepsilon(\omega, u_\varepsilon) < +\infty$, we deduce:

$$\sup_{\varepsilon>0} \int_{T_\varepsilon \cap \mathcal{O}} |\nabla u_\varepsilon|^p \, \mathrm{d}x < +\infty$$

and Poincaré's inequality leads to

$$\sup_{\varepsilon>0} \int_{T_\varepsilon \cap \mathcal{O}} |u_\varepsilon|^p \, \mathrm{d}x < +\infty$$

The estimated values (29) are therefore obtained when $\varepsilon \to 0$ in equality (31). □

Come back to (28), by result (29), we obtain

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\varepsilon \nabla u_\varepsilon) \, \mathrm{d}x \geq \sum_{i\in I} \int_{\mathcal{O}} (u-v) \cdot z_i^* \varphi_{i,\delta} \, \mathrm{d}x + \sum_{i\in I} \int_{\mathcal{O}} v\Big(1 - \frac{|n\hat{Y}|}{|n\hat{Y} \cap D|}\Big) z_i^*$$
$$- \sum_{i\in I} |\mathcal{O}_i| \mathbf{E}\Big(\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^* (z_i^*)$$

So, when $\delta \to 0$,

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\varepsilon \nabla u_\varepsilon) \, \mathrm{d}x \geq \int_{\mathcal{O}} \Big(u - \frac{|n\hat{Y}|}{|n\hat{Y} \cap D|} v\Big) \cdot w \, \mathrm{d}x - \int_{\mathcal{O}} \mathbf{E}\Big(\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^* (w) \, \mathrm{d}x \tag{32}$$

The Valadier result (cf. [10]) gives the equality:

$$\mathbf{E}\Big(\frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^* = \Big(\oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big)^*$$

from which, by passing on the functions Supremum $w \in \mathrm{Step}(\mathcal{O})$ in (32), we deduce

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\varepsilon \nabla u_\varepsilon) \, \mathrm{d}x \geq \int_{\mathcal{O}} \oint \frac{\mathcal{S}^-_{nY}}{n^2}\Big(u - \frac{|n\hat{Y}|}{|n\hat{Y} \cap D|} v\Big) \, \mathrm{d}x$$

Moreover, it is easy to show that $\oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}$ is locally Lipschitz:

$$\left|\oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}(\xi) - \oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}(\xi')\right| \leq L'|\xi - \xi'|(|\xi|^{p-1} + |\xi'|^{p-1})$$

with $L' > 0$, so

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon} f^{\infty,p}(\varepsilon \nabla u_\varepsilon) \, \mathrm{d}x \geq \int_{\mathcal{O}} \oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big(u - \frac{|n\hat{Y}|}{|n\hat{Y} \cap D|} v\Big) \, \mathrm{d}x$$
$$\geq \int_{\mathcal{O}} \oint \frac{\mathcal{S}^-_{n\hat{Y}}}{n^2}\Big(u - \frac{1}{\theta} v\Big) \, \mathrm{d}x - L' \left| \frac{|n\hat{Y}|}{|n\hat{Y} \cap D|} - \frac{1}{\theta} \right| \Big(\Big(\frac{|n\hat{Y}|}{|n\hat{Y} \cap D|}\Big)^{p-1} + \Big(\frac{1}{\theta}\Big)^{p-1}\Big)$$

We obtain the final result when $n \in \mathbb{N}^*$ goes to ∞, with Corollary 2.2 and ergodic theorem (cf. [6]). Indeed, for almost ω in Ω

$$\lim_{n\to+\infty}\frac{|n\hat{Y}\cap D|}{|n\hat{Y}|}=\lim_{n\to+\infty}\frac{1}{n^2}\sum_{z\in I_n}|(\hat{Y}+z)\cap D|$$
$$=\lim_{n\to+\infty}\frac{1}{n^2}\sum_{z\in I_n}|\hat{Y}\cap D(\tau_z(\omega))|=\mathbf{E}(|\hat{Y}\cap D|):=\theta \quad \square$$

6. The periodic case

It is located in the particular case where the sections of the fibers are distributed periodically, in the case of a random chessboard with $\#(\omega_0)=1$. We will show that this is an equality in particular cases.

We define the density $f_\#^-$ and $f_\#^+$ associated with f_0^- and f_0^+

$$f_\#^-(\xi)=\inf\left\{\int_{\hat{Y}} f^{\infty,p}(\nabla w,0)\,\mathrm{d}x : w\in \mathrm{adm}_\#^-\right\}$$

$$f_\#^+(\xi)=\inf\left\{\int_{\hat{Y}\times(0,1)} f^{\infty,p}(\nabla w)\,\mathrm{d}x : w\in \mathrm{adm}_\#^+\right\}$$

with $\xi\in\mathbb{R}^3$ and

$$\mathrm{adm}_\#^-:=\left\{w\in W_\#(\hat{Y},\mathbb{R}^3): \int_{\hat{Y}} w\,\mathrm{d}\hat{x}=\xi,\ \ w=0 \text{ in } D\right\}$$

$$\mathrm{adm}_\#^+:=\left\{w\in W_\#(Y,\mathbb{R}^3): \int_{Y} w\,\mathrm{d}x=\xi,\ \ w=0 \text{ in } D\right\}$$

and we define the following functional energies for all $u\in L^p(\mathcal{O},\mathbb{R}^3)$

$$F_\#^-(u)=\int_{\mathcal{O}} f_\#^-(u)\,\mathrm{d}x,\quad F_\#^+(u)=\int_{\mathcal{O}} f_\#^+(u)\,\mathrm{d}x$$

Energy G_0 is defined as in the framework with the stochastic $\theta=|\hat{Y}\cap D|$.

We want to show:

i) $f_\#^-=f_0^-=f_\#^+$,
ii) $F_\#^- \underset{e}{+} G_0\le\Gamma\text{-}\liminf H_\varepsilon\le\Gamma\text{-}\limsup H_\varepsilon\le F_\#^+\underset{e}{+}G_0$.

Then the global energy $(H_\varepsilon)_{\varepsilon>0}$ Γ-converges to $F_\#^-\underset{e}{+}G_0=F_\#^+\underset{e}{+}G_0$.

Proof i) It is clear that $f_\#^-=f_0^-$. For all $\xi\in\mathbb{R}^3$ fixed, by Jensen's inequality, $f_\#^+(\xi)\ge f_\#^-(\xi)$. On the other hand, for every function $w\in\mathrm{adm}_\#^-$, the function $\tilde{w}$ is defined by $\tilde{w}(x):=w(\hat{x})$ of $w\in\mathrm{adm}_\#^+$ with $f_\#^+(\xi)\le f_\#^-(\xi)$.

Proof ii) The inequality $F_\#^-\underset{e}{+}G_0\le\Gamma\text{-}\liminf H_\varepsilon$ is due to $F_0^-\underset{e}{+}G_0\le\Gamma\text{-}\liminf H_\varepsilon(\omega,.)$ (previous result), and to $f_\#^-=f_0^-$. Moreover, with the notation of the proof of Proposition 4.1, Step 1, let $w_{i,\eta}\in\mathrm{adm}_\#^+$ satisfying

$$\int_Y f^{\infty,p}(\nabla w_{i,\eta})\,\mathrm{d}x=f_\#^+(z_{i,\eta}(x_{i,\eta}))$$

extended by Y-periodicity. As

$$f^{\infty,p}(\nabla w_{i,\eta}(\frac{y}{\varepsilon}))\rightharpoonup\int_Y f^{\infty,p}(\nabla w_{i,\eta})\,\mathrm{d}x$$

$\sigma(L^1, L^\infty)$, we obtain the following inequality associated with (18)

$$\int_{\mathcal{O}} f_{\#}^{+}(z_\eta)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\sum_{i\in I_\eta}\int_{Q_{i,\eta}\setminus T_\varepsilon} f^{\infty,p}(\nabla w_{i,\eta}(\frac{y}{\varepsilon}))\,\mathrm{d}y$$

Define the function $u_{\eta,\varepsilon}$ by

$$u_{\eta,\varepsilon}(x) :== (\frac{1}{\theta}v + \varepsilon\xi_{\varepsilon,\eta}(\frac{\hat{x}}{\varepsilon})) + \sum_{i\in I_\eta} w_{i,\eta}(\frac{x}{\varepsilon})1_{Q_{i,\eta}}(x)$$

where $\xi_{\varepsilon,\eta}$ is defined as in (19), with some clear modifications. This ends the proof as in the proof of Proposition 4.1 to obtain $F^{-}_{\#\,\underset{e}{+}}G_0 \le \Gamma\text{-}\liminf H_\varepsilon$.

7. Conclusion

In this study, we obtained a deterministic and nonlocal model of a randomly reinforced material based on a homogenization technique. From a mechanical perspective, in the elastic case, this energy density allows us to obtain upper and lower bounds of the homogenized parameters. This work is a first step; we will hereafter try to validate these results through an extensive numerical study, which will be presented in an upcoming paper in the dynamic case with an asymptotic area fraction of fibers $\theta(t)$ computed with dynamic covariance [15]. Last but not least, both terminals are identical in the case of a periodic distribution. Then, in the periodic case, given that both limits are identical, such results encourage us to continue our research.

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