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Mathematical justification of an elastic elliptic membrane obstacle problem

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Starting from the 3D Signorini problem for a family of elastic elliptic shells, we justify that the obstacle problem of an elastic elliptic membrane is the right approximation posed in a 2D domain, when the thickness tends to zero. Specifically, we provide convergence results in the scaled and de-scaled formulations.

1. Introduction

In the last decade, asymptotic methods have been used to derive and justify contact models for beams and plates and, recently, in [1,2] the authors obtained the first results in the justification of obstacle problems as the two-dimensional limit of unilateral frictionless contact problems for the particular case of shallow shells. Additionally, the rigid foundation/obstacle was assumed to be a plane. More recently, in [3], we developed the formal asymptotic analysis of the problem for general elastic shells in frictionless contact with a rigid foundation, without the previously indicated restrictions. From the work in [3], a classification of different limit problems arose, depending upon the geometry of the middle surface and the region where the Dirichlet condition was placed. This classification is the natural extension of what was found by Ciarlet, Sánchez-Palencia et al. in their works for the case without contact, namely, membranes and flexural shells (see [4] and references therein). This Note aims at justifying rigorously that the obstacle problem of an elastic elliptic membrane is the right two-dimensional approximation of the three-dimensional Signorini problem for a family of elastic elliptic shells, when the thickness tends to zero.

2. The three-dimensional Signorini contact problem for elastic shells: variational formulation in curvilinear coordinates

Let ω be a domain of $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\gamma = \partial\omega$. Let $\boldsymbol{y} = (y_\alpha)$ be a generic point of its closure $\bar{\omega}$ and let ∂_α denote the partial derivative with respect to y_α. Let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha \boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}(\boldsymbol{y})$. We can consider the two vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$ of the same tangent plane defined by the relations

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$\boldsymbol{a}^{\alpha}(\boldsymbol{y})\cdot\boldsymbol{a}_{\beta}(\boldsymbol{y})=\delta^{\alpha}_{\beta}$, which constitute its contravariant basis. We define $\boldsymbol{a}_3(\boldsymbol{y})=\boldsymbol{a}^3(\boldsymbol{y}):=\frac{\boldsymbol{a}_1(\boldsymbol{y})\wedge\boldsymbol{a}_2(\boldsymbol{y})}{|\boldsymbol{a}_1(\boldsymbol{y})\wedge\boldsymbol{a}_2(\boldsymbol{y})|}$ the unit normal vector to S at the point $\boldsymbol{\theta}(\boldsymbol{y})$, where $\wedge$ denotes the vector product in $\mathbb{R}^3$. We can define the first fundamental form, given as the metric tensor, in covariant or contravariant components, respectively, by $a_{\alpha\beta}:=\boldsymbol{a}_{\alpha}\cdot\boldsymbol{a}_{\beta}$, $a^{\alpha\beta}:=\boldsymbol{a}^{\alpha}\cdot\boldsymbol{a}^{\beta}$. Here and in what follows, Greek indices take their values in the set $\{1,2\}$, whereas Latin indices do it in the set $\{1,2,3\}$. The second fundamental form, given as the curvature tensor, in covariant or mixed components, respectively, is given by $b_{\alpha\beta}:=\boldsymbol{a}^3\cdot\partial_{\beta}\boldsymbol{a}_{\alpha}$, $b^{\beta}_{\alpha}:=a^{\beta\sigma}\cdot b_{\sigma\alpha}$, and the Christoffel symbols of the surface S as $\Gamma^{\sigma}_{\alpha\beta}:=\boldsymbol{a}^{\sigma}\cdot\partial_{\beta}\boldsymbol{a}_{\alpha}$. The area element along S is $\sqrt{a}\,\mathrm{d}y$ where $a:=\det(a_{\alpha\beta})$.

We define the three-dimensional domain $\Omega^{\varepsilon}:=\omega\times(-\varepsilon,\varepsilon)$ and its boundary $\Gamma^{\varepsilon}=\partial\Omega^{\varepsilon}$. We also define the following parts of the boundary, $\Gamma^{\varepsilon}_{+}:=\omega\times\{\varepsilon\}$, $\Gamma^{\varepsilon}_{C}:=\omega\times\{-\varepsilon\}$, $\Gamma^{\varepsilon}_{0}:=\gamma\times[-\varepsilon,\varepsilon]$. Let $\boldsymbol{x}^{\varepsilon}=(x^{\varepsilon}_i)$ be a generic point of $\bar{\Omega}^{\varepsilon}$ and let ∂^{ε}_i denote the partial derivative with respect to x^{ε}_i. Note that $x^{\varepsilon}_{\alpha}=y_{\alpha}$ and $\partial^{\varepsilon}_{\alpha}=\partial_{\alpha}$. Let $\boldsymbol{\Theta}:\bar{\Omega}^{\varepsilon}\to\mathbb{R}^3$ be the mapping defined by

$$\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon}):=\boldsymbol{\theta}(\boldsymbol{y})+x^{\varepsilon}_3\boldsymbol{a}_3(\boldsymbol{y})\ \forall\boldsymbol{x}^{\varepsilon}=(\boldsymbol{y},x^{\varepsilon}_3)=(y_1,y_2,x^{\varepsilon}_3)\in\bar{\Omega}^{\varepsilon} \tag{1}$$

In [4, Th. 3.1-1], it is shown that if the injective mapping $\boldsymbol{\theta}:\bar{\omega}\to\mathbb{R}^3$ is smooth enough, the mapping $\boldsymbol{\Theta}:\bar{\Omega}^{\varepsilon}\to\mathbb{R}^3$ is also injective for $0<\varepsilon<\varepsilon_0$ small enough and the vectors $\boldsymbol{g}^{\varepsilon}_i(\boldsymbol{x}^{\varepsilon}):=\partial^{\varepsilon}_i\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})$ are linearly independent. Therefore, the three vectors $\boldsymbol{g}^{\varepsilon}_i(\boldsymbol{x}^{\varepsilon})$ form the covariant basis at the point $\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})$, and $\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^{\varepsilon})$, defined by the relations $\boldsymbol{g}^{i,\varepsilon}\cdot\boldsymbol{g}^{\varepsilon}_j=\delta^i_j$, form the contravariant basis at the point $\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})$. The covariant and contravariant components of the metric tensor are defined, respectively, as $g^{\varepsilon}_{ij}:=\boldsymbol{g}^{\varepsilon}_i\cdot\boldsymbol{g}^{\varepsilon}_j$, $g^{ij,\varepsilon}:=\boldsymbol{g}^{i,\varepsilon}\cdot\boldsymbol{g}^{j,\varepsilon}$, and Christoffel symbols as $\Gamma^{p,\varepsilon}_{ij}:=\boldsymbol{g}^{p,\varepsilon}\cdot\partial^{\varepsilon}_i\boldsymbol{g}^{\varepsilon}_j$. The volume element in the set $\boldsymbol{\Theta}(\bar{\Omega}^{\varepsilon})$ is $\sqrt{g^{\varepsilon}}\,\mathrm{d}x^{\varepsilon}$ and the surface element in $\boldsymbol{\Theta}(\Gamma^{\varepsilon})$ is $\sqrt{g^{\varepsilon}}\,\mathrm{d}\Gamma^{\varepsilon}$, where $g^{\varepsilon}:=\det(g^{\varepsilon}_{ij})$. Let $\boldsymbol{n}^{\varepsilon}(\boldsymbol{x}^{\varepsilon})$ denote the unit outward normal vector on $\boldsymbol{x}^{\varepsilon}\in\Gamma^{\varepsilon}$ and $\hat{\boldsymbol{n}}^{\varepsilon}(\hat{\boldsymbol{x}}^{\varepsilon})$ the unit outward normal vector on $\hat{\boldsymbol{x}}^{\varepsilon}=\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})\in\boldsymbol{\Theta}(\Gamma^{\varepsilon})$. It is verified that (see, [5, p. 41]) $\hat{\boldsymbol{n}}^{\varepsilon}(\hat{\boldsymbol{x}}^{\varepsilon})=\frac{\mathrm{Cof}(\nabla\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon}))\boldsymbol{n}^{\varepsilon}(\boldsymbol{x}^{\varepsilon})}{|\mathrm{Cof}(\nabla\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon}))\boldsymbol{n}^{\varepsilon}(\boldsymbol{x}^{\varepsilon})|}$. We are particularly interested in the normal components of vectors on $\boldsymbol{\Theta}(\Gamma^{\varepsilon}_C)$. Recall that on Γ^{ε}_C, it is verified that $\boldsymbol{n}^{\varepsilon}=(0,0,-1)$. Also, note that from (1), we deduce that $\boldsymbol{g}^{\varepsilon}_3=\boldsymbol{g}^{3,\varepsilon}=\boldsymbol{a}_3$, and therefore $g^{33,\varepsilon}=|\boldsymbol{g}^{3,\varepsilon}|=1$. These arguments imply that, in particular, $\hat{\boldsymbol{n}}^{\varepsilon}(\hat{\boldsymbol{x}}^{\varepsilon})=-\boldsymbol{g}_3(\boldsymbol{x}^{\varepsilon})=-\boldsymbol{a}_3(\boldsymbol{y})$, where $\hat{\boldsymbol{x}}^{\varepsilon}=\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})$ and $\boldsymbol{x}^{\varepsilon}=(\boldsymbol{y},-\varepsilon)\in\Gamma^{\varepsilon}_C$. Now, for a field $\hat{\boldsymbol{v}}^{\varepsilon}$ defined in $\boldsymbol{\Theta}(\bar{\Omega}^{\varepsilon})$, where the Cartesian basis is denoted by $\{\hat{\boldsymbol{e}}^i\}_{i=1}^3$, we define its covariant curvilinear coordinates (v^{ε}_i) in $\bar{\Omega}^{\varepsilon}$ as $\hat{\boldsymbol{v}}^{\varepsilon}(\hat{\boldsymbol{x}}^{\varepsilon})=\hat{v}^{\varepsilon}_i(\hat{\boldsymbol{x}}^{\varepsilon})\hat{\boldsymbol{e}}^i=:v^{\varepsilon}_i(\boldsymbol{x}^{\varepsilon})\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^{\varepsilon})$ with $\hat{\boldsymbol{x}}^{\varepsilon}=\boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})$. Therefore, on Γ^{ε}_C, we have:

$$\hat{v}_n:=\hat{\boldsymbol{v}}^{\varepsilon}\cdot\hat{\boldsymbol{n}}^{\varepsilon}=(\hat{v}^{\varepsilon}_i\hat{n}^{i,\varepsilon})=(\hat{v}^{\varepsilon}_i\hat{\boldsymbol{e}}^i)\cdot(-\boldsymbol{g}_3)=(v^{\varepsilon}_i\boldsymbol{g}^{i,\varepsilon})\cdot(-\boldsymbol{g}_3)=-v^{\varepsilon}_3$$

Also, since $v^{\varepsilon}_i n^{i,\varepsilon}=-v^{\varepsilon}_3$ on Γ^{ε}_C, it is verified in particular that $\hat{v}_n=(\hat{v}^{\varepsilon}_i\hat{n}^{i,\varepsilon})=v^{\varepsilon}_i n^{i,\varepsilon}=-v^{\varepsilon}_3$.

We assume that $\boldsymbol{\Theta}(\bar{\Omega}^{\varepsilon})$ is a natural state of a shell made of an elastic material, which is homogeneous and isotropic, so that the material is characterized by its Lamé coefficients $\lambda\geq0$, $\mu>0$. We assume that these constants are independent of ε. We also assume that the shell is subjected to a boundary condition of place; in particular, the displacements field vanishes on $\boldsymbol{\Theta}(\Gamma^{\varepsilon}_0)$, this is, the whole lateral face of the shell. Further, under the effect of applied volume forces of density $\hat{\boldsymbol{f}}^{\varepsilon}=(\hat{f}^{i,\varepsilon})$ acting in $\boldsymbol{\Theta}(\Omega^{\varepsilon})$ and tractions of density $\hat{\boldsymbol{h}}^{\varepsilon}=(\hat{h}^{i,\varepsilon})$ acting upon $\boldsymbol{\Theta}(\Gamma^{\varepsilon}_{+})$, the elastic shell is deformed and may enter in contact with a rigid foundation, which, initially, is at a known distance s^{ε} measured along the direction of $\hat{\boldsymbol{n}}^{\varepsilon}$ on $\boldsymbol{\Theta}(\Gamma^{\varepsilon}_C)$. For simplicity, we take $s^{\varepsilon}=0$ in the following.

We deduce that the unilateral contact condition $\hat{v}_n\leq0$ in the well-known definition of the set of admissible displacements in Cartesian coordinates is equivalent to $v^{\varepsilon}_3\geq0$ in curvilinear coordinates. Therefore, let us define the set of admissible unknowns as follows:

$$K(\Omega^{\varepsilon})=\{\boldsymbol{v}^{\varepsilon}=(v^{\varepsilon}_i)\in V(\Omega^{\varepsilon});\ v^{\varepsilon}_3\geq0\text{ on }\Gamma^{\varepsilon}_C\}$$

where $V(\Omega^{\varepsilon})=\{\boldsymbol{v}^{\varepsilon}=(v^{\varepsilon}_i)\in[H^1(\Omega^{\varepsilon})]^3;\ \boldsymbol{v}^{\varepsilon}=\boldsymbol{0}\text{ on }\Gamma^{\varepsilon}_0\}$ is a real Hilbert space with the induced inner product of $[H^1(\Omega^{\varepsilon})]^3$. The corresponding norm is denoted by $||\cdot||_{1,\Omega^{\varepsilon}}$. Note that $K(\Omega^{\varepsilon})$ is a non-empty, closed and convex subset of $V(\Omega^{\varepsilon})$. We now give in contravariant components the volume forces $f^{i,\varepsilon}(\boldsymbol{x}^{\varepsilon})\boldsymbol{g}^{\varepsilon}_i(\boldsymbol{x}^{\varepsilon})$, and tractions $h^{i,\varepsilon}(\boldsymbol{x}^{\varepsilon})\boldsymbol{g}^{\varepsilon}_i(\boldsymbol{x}^{\varepsilon})\sqrt{g^{\varepsilon}(\boldsymbol{x}^{\varepsilon})}\mathrm{d}\Gamma^{\varepsilon}$. With these definitions, it is straightforward to derive the variational formulation of the Signorini problem in curvilinear coordinates:

Problem 2.1. Find $\boldsymbol{u}^{\varepsilon}=(u^{\varepsilon}_i):\Omega^{\varepsilon}\to\mathbb{R}^3$ such that

$$\boldsymbol{u}^{\varepsilon}\in K(\Omega^{\varepsilon}),\quad\int_{\Omega^{\varepsilon}}A^{ijkl,\varepsilon}e^{\varepsilon}_{k||l}(\boldsymbol{u}^{\varepsilon})(e^{\varepsilon}_{i||j}(\boldsymbol{v}^{\varepsilon})-e^{\varepsilon}_{i||j}(\boldsymbol{u}^{\varepsilon}))\sqrt{g^{\varepsilon}}\,\mathrm{d}x^{\varepsilon}$$
$$\geq\int_{\Omega^{\varepsilon}}f^{i,\varepsilon}(v^{\varepsilon}_i-u^{\varepsilon}_i)\sqrt{g^{\varepsilon}}\,\mathrm{d}x^{\varepsilon}+\int_{\Gamma^{\varepsilon}_{+}}h^{i,\varepsilon}(v^{\varepsilon}_i-u^{\varepsilon}_i)\sqrt{g^{\varepsilon}}\,\mathrm{d}\Gamma^{\varepsilon}\quad\forall\boldsymbol{v}^{\varepsilon}\in K(\Omega^{\varepsilon})$$

where the functions $A^{ijkl,\varepsilon} = A^{jikl,\varepsilon} = A^{klij,\varepsilon} \in \mathcal{C}^1(\bar{\Omega}^\varepsilon)$, defined by $A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon})$ represent the contravariant components of the three-dimensional elasticity tensor, and the functions $e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon) = e^\varepsilon_{j\|i}(\boldsymbol{v}^\varepsilon) \in L^2(\Omega^\varepsilon)$ are defined for all $\boldsymbol{v}^\varepsilon \in [H^1(\Omega^\varepsilon)]^3$ by $e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon) := \frac{1}{2}(\partial^\varepsilon_j v^\varepsilon_i + \partial^\varepsilon_i v^\varepsilon_j) - \Gamma^{p,\varepsilon}_{ij} v^\varepsilon_p$, and ∂^ε_i denotes the partial derivative with respect to x^ε_i. In [4, Th. 1.8-1] it is shown the uniform ellipticity of $A^{ijkl,\varepsilon}$ for $\varepsilon > 0$ small enough. Moreover, in [4, Th. 1.7-4] a Korn inequality is provided. We can cast Problem 2.1 in the framework of the elliptic variational inequalities theory and conclude the existence and uniqueness of a solution $\boldsymbol{u}^\varepsilon \in K(\Omega^\varepsilon)$.

3. The scaled three-dimensional shell Signorini contact problem

For convenience, we consider a reference domain independent of the small parameter ε. Hence, let us define the three-dimensional domain $\Omega := \omega \times (-1, 1)$ and its boundary $\Gamma = \partial\Omega$. We also define the following parts of the boundary, $\Gamma_+ := \omega \times \{1\}$, $\Gamma_C := \omega \times \{-1\}$, $\Gamma_0 := \gamma \times [-1, 1]$. Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be a generic point in $\bar{\Omega}$ and consider the notation ∂_i for the partial derivative with respect to x_i. We define the projection map $\pi^\varepsilon : \bar{\Omega} \longrightarrow \bar{\Omega}^\varepsilon$, such that $\pi^\varepsilon(\boldsymbol{x}) = \boldsymbol{x}^\varepsilon = (x^\varepsilon_i) = (x^\varepsilon_1, x^\varepsilon_2, x^\varepsilon_3) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon$, hence, $\partial^\varepsilon_\alpha = \partial_\alpha$ and $\partial^\varepsilon_3 = \frac{1}{\varepsilon}\partial_3$. We consider the scaled unknown and the scaled vector fields $u^\varepsilon_i(\boldsymbol{x}^\varepsilon) =: u_i(\varepsilon)(\boldsymbol{x})$ and $v^\varepsilon_i(\boldsymbol{x}^\varepsilon) =: v_i(\boldsymbol{x})$ for all $\boldsymbol{x} \in \bar{\Omega}$, $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon$. Also, we define the scaled versions of other functions: $\Gamma^p_{ij}(\varepsilon)(\boldsymbol{x}) := \Gamma^{p,\varepsilon}_{ij}(\boldsymbol{x}^\varepsilon)$, $g(\varepsilon)(\boldsymbol{x}) := g^\varepsilon(\boldsymbol{x}^\varepsilon)$, $A^{ijkl}(\varepsilon)(\boldsymbol{x}) := A^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon)$ and the scaled linearized strains $(e_{i\|j}(\varepsilon)(\boldsymbol{v})) \in L^2(\Omega)$, which we also denote as $(e_{i\|j}(\varepsilon; \boldsymbol{v}))$, defined by

$$e_{\alpha\|\beta}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p$$

$$e_{\alpha\|3}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3) - \Gamma^p_{\alpha 3}(\varepsilon) v_p, \quad e_{3\|3}(\varepsilon; \boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3$$

Note that with these definitions, it is verified that $e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)(\pi^\varepsilon(\boldsymbol{x})) = e_{i\|j}(\varepsilon; \boldsymbol{v})(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$. In [4, Th. 3.3-2], it is shown the uniform positive definiteness of $A^{ijkl}(\varepsilon)$ with respect to $\boldsymbol{x} \in \bar{\Omega}$ and ε, $0 < \varepsilon \le \varepsilon_0$. Moreover, the limits $A^{ijkl}(0)$ are found and shown independent of the transversal variable x_3. Further, in [4, Th. 3.3-1], the limits of the scaled Christoffel symbols are given and shown independent of x_3 as well. Besides, $g(\varepsilon) = a + O(\varepsilon)$. Following the insight given by the formal asymptotic analysis developed in [3], we define $\boldsymbol{f}^\varepsilon = (f^{i,\varepsilon})(\boldsymbol{x}^\varepsilon) =: \boldsymbol{f}(\varepsilon) = (f^i)(\boldsymbol{x})$, independent of ε and $\boldsymbol{h}^\varepsilon = (h^{i,\varepsilon})(\boldsymbol{x}^\varepsilon) =: \boldsymbol{h}(\varepsilon) = \varepsilon(h^i(\varepsilon))(\boldsymbol{x})$. The scaled variational problem can then be written as follows.

Problem 3.1. Find $\boldsymbol{u}(\varepsilon) : \Omega \longrightarrow \mathbb{R}^3$ such that

$$\boldsymbol{u}(\varepsilon) \in K(\Omega) := \{\boldsymbol{v} = (v_i) \in V(\Omega);\ v_3 \ge 0\ on\ \Gamma_C\}$$

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon; \boldsymbol{u}(\varepsilon))(e_{i\|j}(\varepsilon; \boldsymbol{v}) - e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon)))\sqrt{g(\varepsilon)}\,\mathrm{d}x$$

$$\ge \int_\Omega f^i(v_i - u_i(\varepsilon))\sqrt{g(\varepsilon)}\,\mathrm{d}x + \int_{\Gamma_+} h^i(v_i - u_i(\varepsilon))\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma \quad \forall \boldsymbol{v} \in K(\Omega)$$

where $V(\Omega) = \{\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3;\ \boldsymbol{v} = \boldsymbol{0}$ on $\Gamma_0\}$ is a Hilbert space.

4. Asymptotic analysis. Convergence results for the elliptic case

We recall the two-dimensional variational formulation of the obstacle problem for an elastic membrane shell, as was derived from the formal asymptotic study made in [3]. For the case of elliptic membranes, the right space where the problem is well posed is $V_M(\omega) := H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$. Therefore, we have the following.

Problem 4.1. Find $\boldsymbol{\xi} : \omega \longrightarrow \mathbb{R}^3$ such that

$$\boldsymbol{\xi} \in K_M(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in V_M(\omega);\ \eta_3 \ge 0 \text{ in } \omega\}$$

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta} - \boldsymbol{\xi})\sqrt{a}\,\mathrm{d}y \ge \int_\omega p^i(\eta_i - \xi_i)\sqrt{a}\,\mathrm{d}y\ \forall \boldsymbol{\eta} = (\eta_i) \in K(\omega)$$

where, $p^i := \int_{-1}^1 f^i\,\mathrm{d}x_3 + h^i_+$, with $h^i_+ = h^i(\cdot, +1)$. Also, $a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$ is the two-dimensional fourth-order elasticity tensor, and given $\boldsymbol{\eta} = (\eta_i) \in [H^1(\omega)]^3$, then $\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$ denotes the covariant components of the linearized change of metric tensor associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the surface S.

For this type of membranes, it is verified the following two-dimensional Korn inequality (see, for example, [4, Th. 2.7-3]): there exists a constant $c_M = c_M(\omega, \boldsymbol{\theta})$ such that

$$\left(\sum_{\alpha} ||\eta_\alpha||_{1,\omega}^2 + ||\eta_3||_{0,\omega}^2\right)^{1/2} \leq c_M \left(\sum_{\alpha,\beta} ||\gamma_{\alpha\beta}(\boldsymbol{\eta})||_{0,\omega}^2\right)^{1/2} \quad \forall \boldsymbol{\eta} \in V_M(\omega) \tag{2}$$

As a consequence, Problem 4.1 is well posed and it has existence and uniqueness of solution (see [3]). Now, we present here the main result of this paper, namely that the solution $\boldsymbol{u}(\varepsilon)$ of the scaled three-dimensional Problem 3.1 converges, as ε tends to zero, towards a limit $\boldsymbol{u}$ independent of the transversal variable. Moreover, this limit can be identified with the solution $\boldsymbol{\xi}$ of Problem 4.1, posed over the set ω.

Theorem 4.2. *Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$. Consider a family of elastic elliptic shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \boldsymbol{\theta}(\bar{\omega})$. For all ε, $0 < \varepsilon \leq \varepsilon_0$, let $\boldsymbol{u}(\varepsilon)$ be the solution to the associated three-dimensional scaled Problem 3.1. Then, there exist functions $u_\alpha \in H^1(\Omega)$ satisfying $u_\alpha = 0$ on $\gamma \times [-1, 1]$ and a function $u_3 \in L^2(\Omega)$, such that*

(1) $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(\Omega)$ and $u_3(\varepsilon) \to u_3$ in $L^2(\Omega)$ when $\varepsilon \to 0$,
(2) $\boldsymbol{u} := (u_i)$ is independent of the transversal variable x_3.

Furthermore, the average $\bar{\boldsymbol{u}} := \dfrac{1}{2}\displaystyle\int_{-1}^{1} \boldsymbol{u}\, dx_3$ verifies Problem 4.1.

The proof of this result will be published in a forthcoming paper [6]. In this Note, we describe the main steps of it for the case when no traction is applied (the inclusion of traction needs the definition of a trace in $X(\Omega) := \{\boldsymbol{v} \in L^2(\Omega); \partial_3 v \in L^2(\Omega)\}$).

(i) *We show that the norms $|e_{i||j}(\varepsilon)|_{0,\Omega}$, $||u_\alpha(\varepsilon)||_{1,\Omega}$, and $|u_3(\varepsilon)|_{0,\Omega}$ are bounded independently of ε, $0 < \varepsilon \leq \varepsilon_1 < \varepsilon_0$. Consequently, there exists a subsequence, also denoted $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$, and functions $e_{i||j} \in L^2(\Omega)$, $u_\alpha \in H^1(\Omega)$, satisfying $u_\alpha = 0$ on Γ_0, and $u_3 \in L^2(\Omega)$, such that $e_{i||j}(\varepsilon) \rightharpoonup e_{i||j}$ in $L^2(\Omega)$, $u_\alpha(\varepsilon) \rightharpoonup u_\alpha$ in $H^1(\Omega)$, hence $u_\alpha(\varepsilon) \to u_\alpha$ in $L^2(\Omega)$, and $u_3(\varepsilon) \rightharpoonup u_3$ in $L^2(\Omega)$:*
to do this, we take $\boldsymbol{v} = 2\boldsymbol{u}(\varepsilon)$ and $\boldsymbol{v} = \boldsymbol{0}$ in Problem 3.1. Then we combine the use of a convenient Korn inequality (see [4, Th. 4.3-1]), only valid for the elliptic shells case, the ellipticity of $A^{ijkl}(\varepsilon)$ and Cauchy–Schwartz inequalities.

(ii) *The limits of the scaled unknown, u_i, found in step* (i) *are independent of x_3:*
to do this, we combine the use of the definitions of $e_{i||3}(\varepsilon)$ with the convergences of the various scaled functions and the results in the previous step.

(iii) *The limits $e_{i||j}$ found in* (i) *are independent of the variable x_3. Moreover, they are related with the limits $\boldsymbol{u} := (u_i)$ by*

$$e_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) := \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma_{\alpha\beta}^{\sigma} u_\sigma - b_{\alpha\beta} u_3, \quad e_{\alpha||3} = 0$$
$$e_{3||3} = -\frac{\lambda}{\lambda + 2\mu} a^{\alpha\beta} e_{\alpha||\beta} \tag{3}$$

We take particular cases of test functions $\boldsymbol{v} \in K(\Omega)$ in Problem 3.1, expand the resulting terms and use the results of the calculus of variations. It is important to notice that to find (3), from inequalities, the following abstract lemma had to be derived (see also [3]).

Lemma 4.3. *Let ω be a domain in $\mathbb{R}^2$ with boundary γ, let $\Omega = \omega \times (-1, 1)$, and let $g \in L^p(\Omega)$, $p > 1$, be a function such that*

$$\int_\Omega g \partial_3 v \, dx \geq 0, \text{ for all } v \in \mathcal{C}^\infty(\bar{\Omega}) \text{ with } v = 0 \text{ on } \gamma \times [-1, 1]$$

and $v \geq 0$ in Ω. Then $g = 0$ a.e. in Ω.

(iv) *The function $\bar{\boldsymbol{u}} = (\bar{\boldsymbol{u}}_i)$ satisfies the two-dimensional variational Problem 4.1 with $p^i := \int_{-1}^{1} f^i\, dx_3$. In particular, since the solution to this problem is unique, the convergences on* (i) *are verified for all the family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$. We also show that $\bar{\boldsymbol{u}} = (\bar{u}_i) \in K_M(\omega)$.*
We take $\boldsymbol{v}$ independent of x_3 and pass to the limit in Problem 3.1 by having in mind the results of the previous steps. Besides, the property that

$$\liminf_{\varepsilon \to 0} \int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon) e_{i||j}(\varepsilon) \sqrt{g(\varepsilon)}\, dx \geq \int_\Omega A^{ijkl}(0) e_{k||l} e_{i||j} \sqrt{a}\, dx$$

is needed.

For the second part, we have to show that since $u_3(\varepsilon) \geq 0$ a.e. on Γ_C, $u_3(\varepsilon) \rightharpoonup u_3$ in $L^2(\Omega)$ and $\partial_3 u_3(\varepsilon) \to 0$ in $L^2(\Omega)$, then $u_3 \geq 0$ a.e. in Ω.

(v) *The weak convergences $e_{i||j}(\varepsilon) \rightharpoonup e_{i||j}$ in $L^2(\Omega)$ are, in fact, strong.*
To do this, we define the quantity

$$\Lambda(\varepsilon) := \int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon) - e_{k||l})(e_{i||j}(\varepsilon) - e_{i||j})\sqrt{g(\varepsilon)}\,\mathrm{d}x$$
$$= \int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}\,\mathrm{d}x - \int_\Omega A^{ijkl}(\varepsilon)(2e_{k||l}(\varepsilon) - e_{k||l})e_{i||j}\sqrt{g(\varepsilon)}\,\mathrm{d}x$$

and show that $\Lambda(\varepsilon) \geq C\sum_{i,j}|e_{i||j}(\varepsilon) - e_{i||j}|^2_{0,\Omega}$, $C > 0$, and that $\lim_{\varepsilon\to 0}\Lambda(\varepsilon) = 0$.

(vi) *The family of averages $(\bar{\boldsymbol{u}}(\varepsilon))_{\varepsilon>0}$ converges strongly to $\bar{\boldsymbol{u}}$ (when $\varepsilon \to 0$) in $V_M(\omega)$, that is,*

$$\bar{u}_\alpha(\varepsilon) \to \bar{u}_\alpha \text{ in } H^1(\omega),\ \bar{u}_3(\varepsilon) \to \bar{u}_3 \text{ in } L^2(\omega)$$

We combine steps (iii) and (v) with [4, Th. 4.2-1] (part(d)) to show that $\gamma_{\alpha\beta}(\bar{\boldsymbol{u}}(\varepsilon)) \to \gamma_{\alpha\beta}(\bar{\boldsymbol{u}})$, and then use (2).

(vii) *The convergence $u_3(\varepsilon) \rightharpoonup u_3$ in $L^2(\Omega)$ is, in fact, strong.*
Since $\partial_3 u_3(\varepsilon) = \varepsilon e_{3||3}(\varepsilon) \to 0$ and since $\bar{u}_3(\varepsilon) \to \bar{u}_3$ in $L^2(\omega)$ (step (vi)), the conclusion follows from [4, Th. 4.2-1] (part(c)).

(viii) *The convergences $u_\alpha(\varepsilon) \to u_\alpha$ are strong in $H^1(\Omega)$:*
This part is based on the use of the classical Korn inequality in Cartesian coordinates in $V(\Omega)$ for $\boldsymbol{u}'(\varepsilon) = (u_1(\varepsilon), u_2(\varepsilon), 0)$, $\boldsymbol{u}' = (u_1, u_2, 0)$. We want to show that $|e_{ij}(\boldsymbol{u}'(\varepsilon) - e_{ij}(\boldsymbol{u}')|_{0,\Omega} \to 0$. To do that we combine relations of $e_{ij}(\boldsymbol{u}(\varepsilon))$ with $e_{i||j}(\boldsymbol{u}(\varepsilon))$ and a Lemma of J.-L. Lions.

5. Conclusions

It remains to prove a result analogous to the previous theorem, but in terms of de-scaled unknowns. The scalings in Section 3 suggest the de-scalings $\xi_i^\varepsilon(\boldsymbol{y}) = \xi_i(\boldsymbol{y})$ for all $\boldsymbol{y} \in \bar{\omega}$. The convergences $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(\Omega)$ and $u_3(\varepsilon) \to u_3$ in $L^2(\Omega)$ from Theorem 4.2 and [4, Th. 4.2-1] together lead to the following convergences.

Theorem 5.1. *Let $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) \in K(\Omega^\varepsilon)$ and $\boldsymbol{\xi}^\varepsilon = (\xi_i^\varepsilon) \in K_M(\omega)$ denote for each $\varepsilon > 0$ the solutions to the three-dimensional Problem 2.1 and the de-scaled version of two-dimensional Problem 4.1, respectively. Then we have that $\xi_\alpha^\varepsilon = \xi_\alpha$, and thus $\xi_\alpha^\varepsilon \boldsymbol{a}^\alpha = \xi_\alpha \boldsymbol{a}^\alpha$ in $H^1(\omega)$ for all $\varepsilon > 0$, and*

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \boldsymbol{g}^{\alpha,\varepsilon}\,\mathrm{d}x_3^\varepsilon \to \xi_\alpha \boldsymbol{a}^\alpha \text{ in } H^1(\omega) \text{ as } \varepsilon \to 0$$

Also, $\xi_3^\varepsilon = \xi_3$, and thus $\xi_3^\varepsilon \boldsymbol{a}^3 = \xi_3 \boldsymbol{a}^3$ in $L^2(\omega)$ for all $\varepsilon > 0$ and

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon}\,\mathrm{d}x_3^\varepsilon \to \xi_3 \boldsymbol{a}^3 \text{ in } L^2(\omega) \text{ as } \varepsilon \to 0$$

As a conclusion, we have found and mathematically justified an obstacle model for elastic elliptic membranes. To this end, we used the insight provided by the asymptotic expansion method (presented in our previous work [3]) and we have justified this approach by obtaining convergence theorems. Let us notice that in the process we have shown that the limit of contact problems (with the conditions *on* the boundary) is an obstacle problem (with the conditions *in* the domain).

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