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Harmonic elastic inclusions in the presence of point moment

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A B S T R A C T

We employ conformal mapping techniques to design harmonic elastic inclusions when the surrounding matrix is simultaneously subjected to remote uniform stresses and a point moment located at an arbitrary position in the matrix. Our analysis indicates that the uniform and hydrostatic stress field inside the inclusion as well as the constant hoop stress along the entire inclusion–matrix interface (on the matrix side) are independent of the action of the point moment. In contrast, the non-elliptical shape of the harmonic inclusion depends on both the remote uniform stresses and the point moment.

1. Introduction

In the manufacture of composite materials, 'harmonic' inclusions are designed to leave unperturbed everywhere the first invariant of the uncut stress field (or the mean stress) when inserted into a stressed matrix [1–5]. The terminology 'harmonic' is used in this context since the first invariant of the stress tensor is a harmonic function in linear plane elasticity. In previous studies [1–5], the analysis of harmonic inclusions has been undertaken in cases when the matrix is subjected to only uniform or non-uniform stresses at infinity without the possibility of any additional concentrated loading (e.g., a point moment or a circular transformation strain spot). The importance of the analysis of elastic fields subjected to pointwise singularities such as point moment is well-documented. These singular fields are important not only because of their physical significance within micromechanics but also because they often form the basis for fundamental solutions used in, for example, the boundary integral equation method (see, for example, [6]). With this in mind, we pose the following question:

Is it possible to design harmonic inclusions if the matrix is additionally subjected to a point moment?

In this paper, we will address this question. Using complex variable methods, we will demonstrate the design of harmonic elastic inclusions when the matrix is subjected to remote uniform stresses and a point moment applied at an arbitrary position in the matrix. A novel conformal mapping function is first constructed to account for the existence of the point moment. The interface and boundary conditions then allow us to determine analytically the two complex parameters appearing in the mapping function for given loadings. Our results indicate that the internal stress distribution inside the harmonic inclusion is uniform and hydrostatic whilst the hoop stress along the entire inclusion–matrix interface (on the matrix side) remains constant. Both of these stress distributions are found to be independent of the point moment. Consequently, our design also attains the "constant strength" design criterion proposed by Cherepanov [7]. Furthermore, the shape of the harmonic

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inclusion depends on both the remote uniform stresses and the point moment. Several specific examples are presented to illustrate our results.

2. Problem formulation

For plane deformations of an isotropic elastic material, the stresses $(\sigma_{11}, \sigma_{22}, \sigma_{12})$, displacements (u_1, u_2) and stress functions (ϕ_1, ϕ_2) can be expressed in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [8]

$$\begin{aligned} &\sigma_{11} + \sigma_{22} = 2\left[\varphi'(z) + \overline{\varphi'(z)}\right] \\ &\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2\left[\bar{z}\varphi''(z) + \psi'(z)\right] \end{aligned} \tag{1}$$

$$\begin{aligned} &2\mu(u_1 + \mathrm{i}u_2) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \\ &\phi_1 + \mathrm{i}\phi_2 = \mathrm{i}\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\right] \end{aligned} \tag{2}$$

where the constant $\kappa = 3 - 4\nu$ in the case of plane strain deformations (assumed here), $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress and μ, ν $(0 \le \nu \le 1/2)$ are the shear modulus and Poisson's ratio, respectively. In addition, the stresses are related to the stress functions by [9]

$$\begin{aligned} &\sigma_{11} = -\phi_{1,2}, \qquad \sigma_{12} = \phi_{1,1} \\ &\sigma_{21} = -\phi_{2,2}, \qquad \sigma_{22} = \phi_{2,1} \end{aligned} \tag{3}$$

Consider a domain in $\Re^2$, infinite in extent, containing a single internal elastic inclusion with elastic properties different from those of the matrix. We represent the matrix by the domain S_2 and assume that the inclusion occupies a region S_1. The inclusion–matrix interface is denoted by L. The matrix is subjected to remote uniform in-plane stresses $(\sigma_{11}^{\infty}, \sigma_{22}^{\infty}, \sigma_{12}^{\infty})$ and a point moment M at some arbitrary point $z = z_0$ in the matrix. The criterion to be satisfied in the design of a harmonic elastic inclusion is that the mean stress $\sigma_{11} + \sigma_{22}$ in the matrix is not disturbed when the inclusion is inserted into the matrix. In what follows, the subscripts 1 and 2 are used to identify the respective quantities in S_1 and S_2.

The corresponding boundary value problem thus takes the form

$$\begin{aligned} &\varphi_2(z) + z\overline{\varphi_2'(z)} + \overline{\psi_2(z)} = \varphi_1(z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} \\ &\kappa_2\varphi_2(z) - z\overline{\varphi_2'(z)} - \overline{\psi_2(z)} = \Gamma\kappa_1\varphi_1(z) - \Gamma z\overline{\varphi_1'(z)} - \Gamma\overline{\psi_1(z)}, \quad z \in L \end{aligned} \tag{4}$$

$$\varphi_2(z) \equiv \frac{\sigma_{11}^{\infty} + \sigma_{22}^{\infty}}{4} z, \quad z \in S_2 \tag{5}$$

$$\psi_2(z) \cong \frac{\sigma_{22}^{\infty} - \sigma_{11}^{\infty} + 2\sigma_{12}^{\infty}}{2} z + O(1), \quad |z| \to \infty \tag{6}$$

$$\psi_2(z) \cong \frac{\mathrm{i}M}{2\pi}\frac{1}{z - z_0} + O(1), \quad \text{as } z \to z_0 \tag{7}$$

where $\Gamma = \mu_2/\mu_1$. The singular behavior in Eq. (7) for a point moment follows from the analysis in [6]. We note here that the point moment will not induce any singular behavior in $\varphi_2(z)$ [6].

3. Harmonic elastic inclusions

The conformal mapping function is assumed to take the following form

$$z = \omega(\xi) = R\left(\xi + \frac{p}{\xi} + \frac{q}{\xi - \bar{\xi}_0^{-1}}\right), \quad \xi = \omega^{-1}(z),\ |\xi| \ge 1 \tag{8}$$

where R is a real scaling constant, p and q are complex constants to be determined, and $\xi_0 = \omega^{-1}(z_0)$. The region occupied by the matrix in the z-plane is mapped onto $|\xi| \ge 1$ in the ξ-plane, and the interface L is mapped onto the unit circle $|\xi| = 1$. The appearance of the first-order pole in Eq. (8) is to account for the existence of the point moment at $z = z_0$.

In order to ensure that the mean stress in the surrounding matrix is not disturbed by the inclusion, the two analytic functions defined in S_1 should take the following form

$$\varphi_1(z) = \frac{A}{R} z, \qquad \psi_1(z) = 0, \quad z \in S_1 \tag{9}$$

where A is a real constant.

It follows from Eqs. (4), (8) and (9) that

$$\begin{aligned}&\varphi_2(z)=\frac{\Gamma(\kappa_1-1)+2}{\kappa_2+1}\frac{A}{R}z, \quad z\in S_2\\&\psi_2(\xi)=\psi_2\big(\omega(\xi)\big)=\frac{2[\kappa_2-1-\Gamma(\kappa_1-1)]}{\kappa_2+1}A\left(\frac{1}{\xi}+\bar{p}\xi-\frac{\bar{q}\xi_0^2}{\xi-\xi_0}\right), \quad |\xi|>1\end{aligned} \tag{10}$$

A comparison of Eq. (10) with Eqs. (5) and (6) leads to

$$A=\frac{R(\kappa_2+1)(\sigma_{11}^\infty+\sigma_{22}^\infty)}{4[\Gamma(\kappa_1-1)+2]}, \qquad p=\frac{[\Gamma(\kappa_1-1)+2](\sigma_{22}^\infty-\sigma_{11}^\infty-2\mathrm{i}\sigma_{12}^\infty)}{[\kappa_2-1-\Gamma(\kappa_1-1)](\sigma_{11}^\infty+\sigma_{22}^\infty)} \tag{11}$$

which indicates that the two constants A and p are independent of the presence of the point moment.

A comparison of Eq. (10) with Eq. (7) leads to

$$q=\frac{\mathrm{i}M(\kappa_2+1)}{4\pi A[\kappa_2-1-\Gamma(\kappa_1-1)]}\frac{1}{\overline{\xi_0^2\omega'(\xi_0)}}=\frac{\mathrm{i}M[\Gamma(\kappa_1-1)+2]}{\pi R(\sigma_{11}^\infty+\sigma_{22}^\infty)[\kappa_2-1-\Gamma(\kappa_1-1)]}\frac{1}{\overline{\xi_0^2\omega'(\xi_0)}} \tag{12}$$

Insertion of Eq. (12) into Eq. (8) yields

$$\omega(\xi)=R\left(\xi+\frac{p}{\xi}\right)+\frac{\mathrm{i}M[\Gamma(\kappa_1-1)+2]}{\pi(\sigma_{11}^\infty+\sigma_{22}^\infty)[\kappa_2-1-\Gamma(\kappa_1-1)]\overline{\xi_0^2\omega'(\xi_0)}}\frac{1}{\xi-\bar{\xi}_0^{-1}} \tag{13}$$

The satisfaction of the consistency condition for $\omega'(\xi_0)$ will result in the following non-linear equation for $\omega'(\xi_0)$

$$\omega'(\xi_0)+\frac{\mathrm{i}M[\Gamma(\kappa_1-1)+2]}{\pi(\sigma_{11}^\infty+\sigma_{22}^\infty)[\kappa_2-1-\Gamma(\kappa_1-1)](|\xi_0|^2-1)^2}\frac{1}{\overline{\omega'(\xi_0)}}=R\left(1-\frac{p}{\xi_0^2}\right) \tag{14}$$

from which we obtain

$$\begin{aligned}&\left|\omega'(\xi_0)\right|^2=R^2\frac{|1-p\xi_0^{-2}|^2}{2}\pm R^2\sqrt{\frac{|1-p\xi_0^{-2}|^4}{4}-\frac{M^2[\Gamma(\kappa_1-1)+2]^2}{\pi^2R^4(\sigma_{11}^\infty+\sigma_{22}^\infty)^2[\kappa_2-1-\Gamma(\kappa_1-1)]^2(|\xi_0|^2-1)^4}}>0\\&\arg\{\omega'(\xi_0)\}=\arg\{1-p\xi_0^{-2}\}-\arctan\left\{\frac{M[\Gamma(\kappa_1-1)+2]}{\pi(\sigma_{11}^\infty+\sigma_{22}^\infty)[\kappa_2-1-\Gamma(\kappa_1-1)](|\xi_0|^2-1)^2|\omega'(\xi_0)|^2}\right\}\end{aligned} \tag{15}$$

with

$$\frac{|M|}{R^2|\sigma_{11}^\infty+\sigma_{22}^\infty|}\le\frac{\pi|1-p\xi_0^{-2}|^2(|\xi_0|^2-1)^2|\kappa_2-1-\Gamma(\kappa_1-1)|}{2[\Gamma(\kappa_1-1)+2]} \tag{16}$$

In order to ensure that the mapping function in Eq. (8) is one-to-one, the following condition should be satisfied

$$\omega'(\xi)\neq 0 \quad \text{for } |\xi|>1 \tag{17}$$

or equivalently all the four roots of the following quartic equation should be located either inside or on the unit circle:

$$\xi^4-2\bar{\xi}_0^{-1}\xi^3+\left(\bar{\xi}_0^{-2}-p-q\right)\xi^2+2p\bar{\xi}_0^{-1}\xi-p\bar{\xi}_0^{-2}=0 \tag{18}$$

We deduce from Eqs. (9) and (11) that the internal stress field is given by

$$\sigma_{11}=\sigma_{22}=\frac{(\kappa_2+1)(\sigma_{11}^\infty+\sigma_{22}^\infty)}{2[\Gamma(\kappa_1-1)+2]}, \qquad \sigma_{12}=0, \quad z\in S_1 \tag{19}$$

Clearly, the internal stress field is uniform and hydrostatic and is independent of the action of the point moment. In addition, the hoop stress is constant along the entire inclusion–matrix interface on the matrix side and is given by

$$\sigma_{tt}=\frac{[2\Gamma(\kappa_1-1)+3-\kappa_2](\sigma_{11}^\infty+\sigma_{22}^\infty)}{2[\Gamma(\kappa_1-1)+2]}, \quad z\in S_2 \tag{20}$$

which is also independent of the point moment. It is noteworthy that the constant hoop stress has satisfied the "constant strength" design criterion first advanced by Cherepanov [7] and later studied in [10,11].

In contrast, the shape of the harmonic inclusion depends on both the remote uniform stresses and the point moment. More precisely, $z/R=\omega(\xi)/R$, $|\xi|=1$ is completely determined by three material parameters Γ, κ_1, κ_2, two loading parameters $\frac{\sigma_{22}^\infty-\sigma_{11}^\infty-2\mathrm{i}\sigma_{12}^\infty}{\sigma_{11}^\infty+\sigma_{22}^\infty}$, $\frac{M}{R^2(\sigma_{11}^\infty+\sigma_{22}^\infty)}$ and one geometric parameter ξ_0. The complex constant p in the mapping function is given by Eq. (11), and the other complex constant q in the mapping function is given by Eq. (12) with $\omega'(\xi_0)$ being determined by Eq. (15).

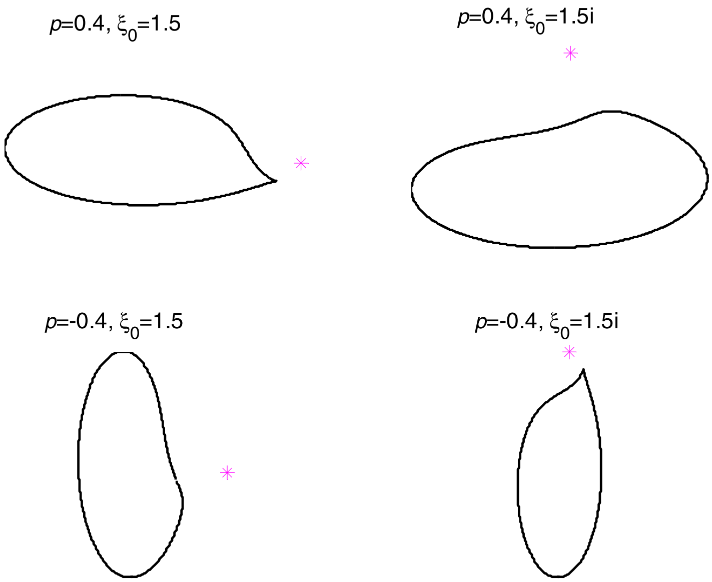

Fig. 1. The non-elliptical shapes of the harmonic inclusions for different values of p and ξ_0 with $\Gamma = 3$, $\kappa_1 = \kappa_2 = 2$, $\rho = 0.2$. The star in each subplot is the location of the point moment.

4. Examples

In this section, several examples will be presented to illustrate the design of harmonic elastic inclusions in the presence of a point moment. In the first example, we set

$$\Gamma = 3, \qquad \kappa_1 = \kappa_2 = 2, \qquad \rho = \frac{M}{R^2(\sigma_{11}^\infty + \sigma_{22}^\infty)} = 0.2 \tag{21}$$

In this example, the inclusion is softer than the matrix ($\Gamma > 1$). The complex constant q can be determined uniquely for different combinations of p and ξ_0 as follows

$$\begin{aligned} &q = 0.0133 - 0.0860\mathrm{i} \quad \text{when } p = 0.4,\ \xi_0 = 1.5 \\ &q = -0.0044 + 0.0601\mathrm{i} \quad \text{when } p = 0.4,\ \xi_0 = 1.5\mathrm{i} \\ &q = 0.0044 - 0.0601\mathrm{i} \quad \text{when } p = -0.4,\ \xi_0 = 1.5 \\ &q = -0.0133 + 0.0860\mathrm{i} \quad \text{when } p = -0.4,\ \xi_0 = 1.5\mathrm{i} \end{aligned} \tag{22}$$

The corresponding non-elliptical shapes of the harmonic inclusions are shown in Fig. 1.

In the second example, we set

$$\Gamma = \frac{1}{3}, \qquad \kappa_1 = \kappa_2 = 2, \qquad \rho = \frac{M}{R^2(\sigma_{11}^\infty + \sigma_{22}^\infty)} = 0.1 \tag{23}$$

In this example, the inclusion is stiffer than the matrix ($\Gamma < 1$). The complex constant q can also be determined uniquely for different combinations of p and ξ_0 as follows:

$$\begin{aligned} &q = 0.0064 + 0.0602\mathrm{i} \quad \text{when } p = 0.4,\ \xi_0 = 1.5 \\ &q = -0.0022 - 0.0420\mathrm{i} \quad \text{when } p = 0.4,\ \xi_0 = 1.5\mathrm{i} \\ &q = 0.0022 + 0.0420\mathrm{i} \quad \text{when } p = -0.4,\ \xi_0 = 1.5 \\ &q = -0.0064 - 0.0602\mathrm{i} \quad \text{when } p = -0.4,\ \xi_0 = 1.5\mathrm{i} \end{aligned} \tag{24}$$

The corresponding non-elliptical shapes of the harmonic inclusions are shown in Fig. 2.

If we set $\Gamma = 3$, $\kappa_1 = \kappa_2 = 2$, $\xi_0 = 1.5$ and p is real valued (i.e. $\sigma_{12}^\infty = 0$), the two parameters p and $\rho = \frac{M}{R^2(\sigma_{11}^\infty + \sigma_{22}^\infty)}$ should lie within the region enclosed by the curve shown in Fig. 3 in order to ensure that the mapping function (8) is one-to-one. It is seen from Fig. 3 that p lies in the range $[-1, 1]$, whilst the range of permissible $|\rho|$ decreases from 0.417 to 0 as $|p|$ increases from 0 to 1.

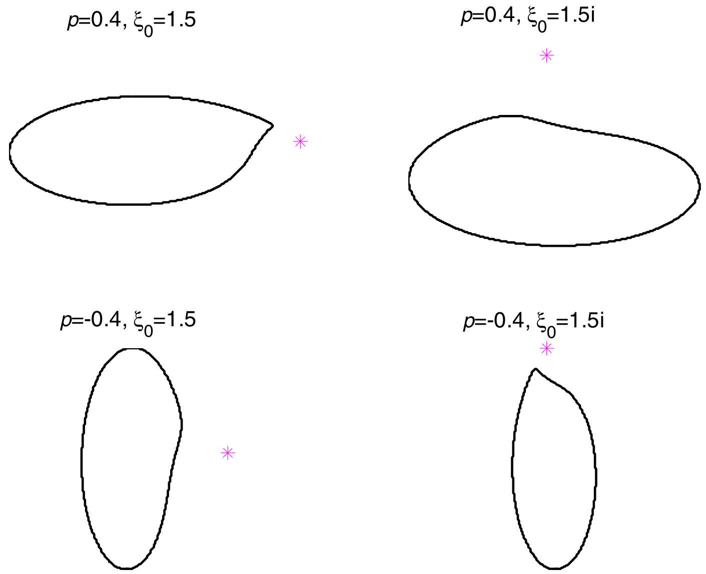

Fig. 2. The non-elliptical shapes of the harmonic inclusions for different values of p and ξ_0 with $\Gamma = 1/3$, $\kappa_1 = \kappa_2 = 2$, $\rho = 0.1$. The star in each subplot is the location of the point moment.

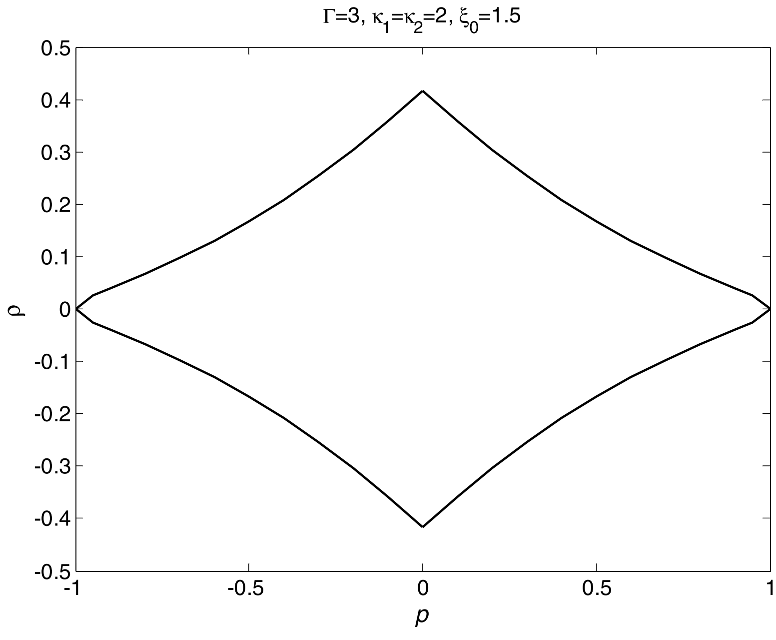

Fig. 3. Condition for the existence of harmonic elastic inclusions with $\Gamma = 3$, $\kappa_1 = \kappa_2 = 2$, $\xi_0 = 1.5$.

In Fig. 4, we show that $\max\{|\rho|\}$ is a decreasing function of real-valued ξ_0 with $\Gamma = 3$, $\kappa_1 = \kappa_2 = 2$, $p = 0$ in order to ensure that the mapping function (8) is one-to-one. In this case, the remote loading is hydrostatic. Roughly, the magnitude of the point moment $|M|$ decreases as the point moment moves closer to the inclusion. We illustrate in Fig. 5 $\max\{|\rho|\}$ as a function of Γ taking $\kappa_1 = \kappa_2 = 2$, $p = 0$, $\xi_0 = 1.5$ in order to ensure that the mapping function (8) is one-to-one. It is observed from Fig. 5 that: (*i*) $\max\{|\rho|\}$ is a decreasing function of Γ when $\Gamma < 1$ and increases when $\Gamma > 1$; (*ii*) $\max\{|\rho|\} = 1.042$ as $\Gamma \to \infty$ (the inclusion becomes a hole), $\max\{|\rho|\} = 0.521$ when $\Gamma = 0$ (the inclusion becomes rigid); (*iii*) when the inclusion and the matrix have identical elastic properties (same elastic constants), it is not possible to design harmonic elastic inclusions in the presence of a point moment with $\rho \neq 0$. Finally, we plot in Fig. 6 a harmonic hole with $\Gamma = \infty$, $\kappa_1 = \kappa_2 = 2$, $p = 0$, $\xi_0 = 1.5$, $\rho = 1.042$, and, in Fig. 7, a harmonic rigid inclusion with $\Gamma = 0$, $\kappa_1 = \kappa_2 = 2$, $p = 0$, $\xi_0 = 1.5$, $\rho = 0.521$. It is clear from Figs. 6 and 7 that the inclusion–matrix interface has a sharp corner which can be attributed to the action of the nearby point moment.

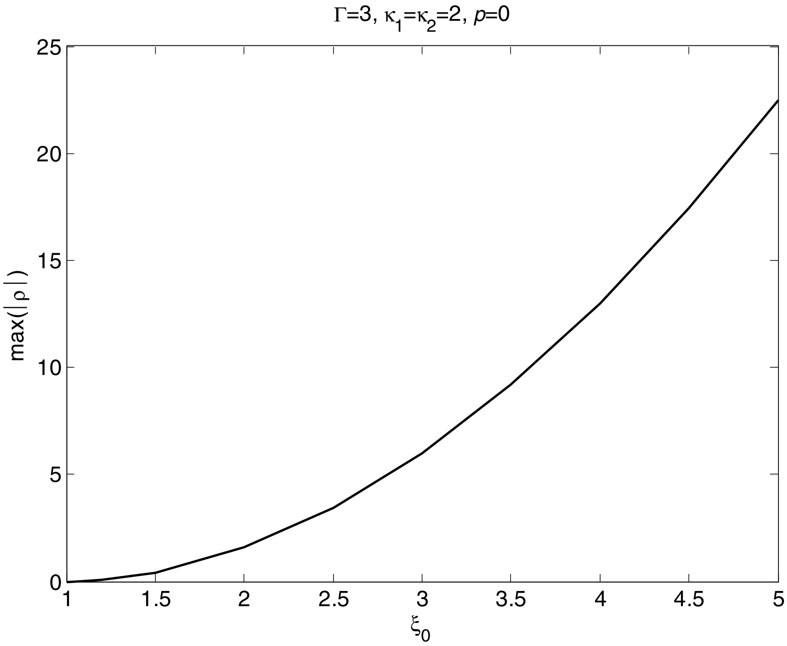

Fig. 4. $\max\{|\rho|\}$ as a function of real-valued ξ_0 with $\Gamma = 3$, $\kappa_1 = \kappa_2 = 2$, $p = 0$ for the existence of harmonic elastic inclusions.

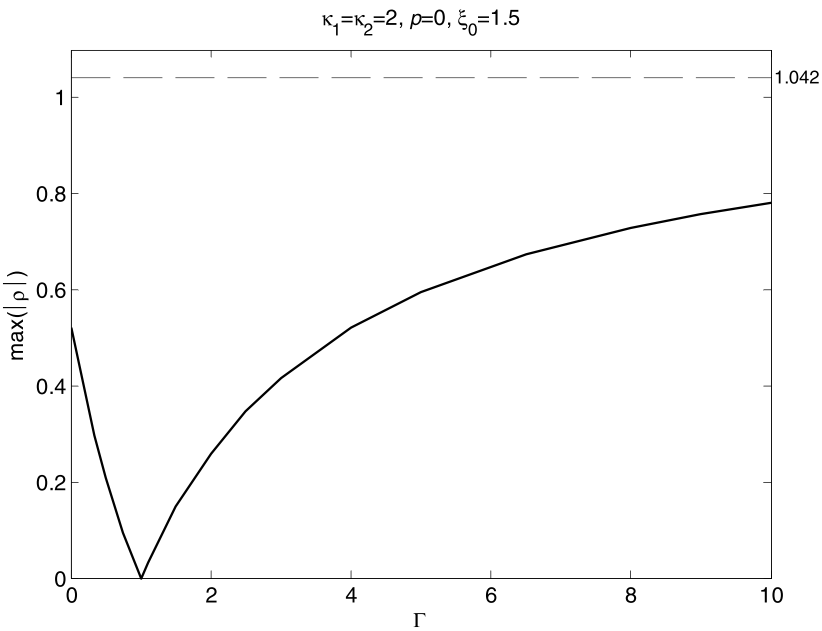

Fig. 5. $\max\{|\rho|\}$ as a function of Γ with $\kappa_1 = \kappa_2 = 2$, $p = 0$, $\xi_0 = 1.5$ for the existence of harmonic elastic inclusions.

5. Conclusions

We consider an inverse problem in linear elasticity associated with the design of harmonic elastic inclusions in the presence of a point moment located in the matrix. For the first time, the design of harmonic inclusions is achieved when the matrix is simultaneously subjected to a point moment and remote uniform stresses. A first-order pole is inserted into the mapping function (8) to account for the point moment at $z = z_0$. Our numerical results demonstrate the feasibility of the proposed design method.

We note here that our method can be extended to accommodate the design of a harmonic elastic inhomogeneity interacting with a circular Eshelby's thermal inclusion [6,12] under uniform stresses at infinity. In this case, the mean stress

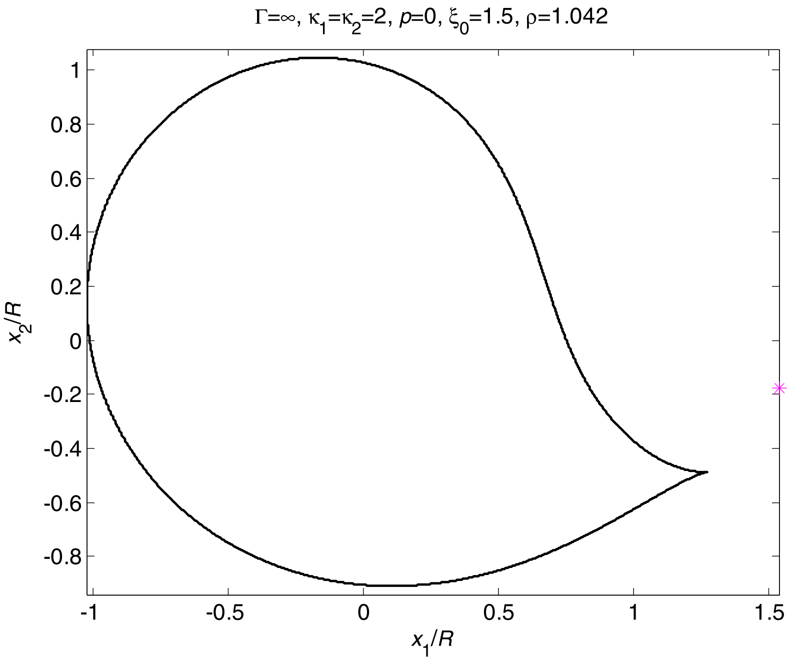

Fig. 6. A harmonic hole with $\Gamma = \infty$, $\kappa_1 = \kappa_2 = 2$, $p = 0$, $\xi_0 = 1.5$, $\rho = 1.042$. The star is the location of the point moment.

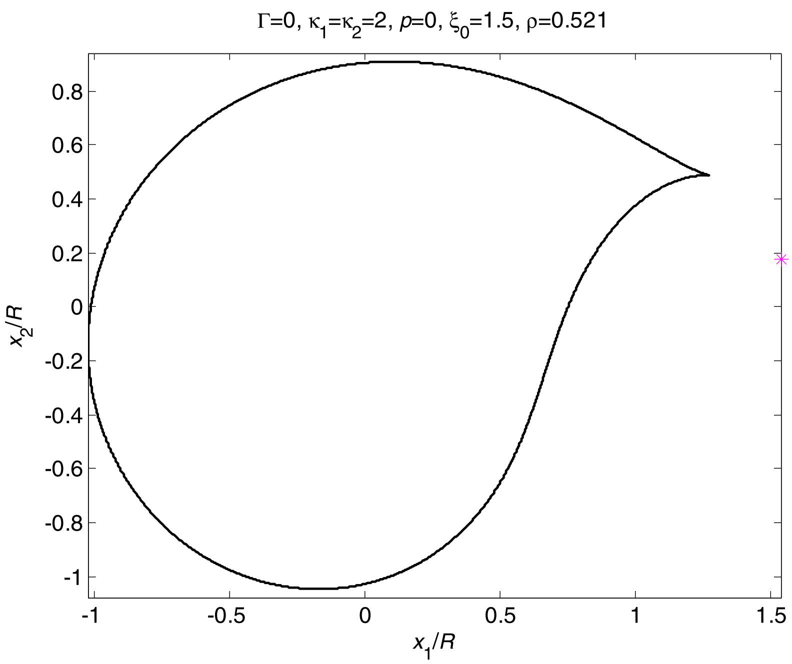

Fig. 7. A harmonic rigid inclusion with $\Gamma = 0$, $\kappa_1 = \kappa_2 = 2$, $p = 0$, $\xi_0 = 1.5$, $\rho = 0.521$. The star is the location of the point moment.

outside both the elastic inhomogeneity and the thermal inclusion is not disturbed. In addition, the stress field inside the elastic inhomogeneity is uniform and hydrostatic whilst the mean stress inside the thermal inclusion remains constant.

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