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Mathematical justification of a viscoelastic elliptic membrane problem

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ABSTRACT

We consider a family of linearly viscoelastic elliptic shells, and we use asymptotic analysis to justify that what we have identified as the two-dimensional viscoelastic elliptic membrane problem is an accurate approximation when the thickness of the shell tends to zero. Most noticeable is that the limit problem includes a long-term memory that takes into account the previous history of deformations. We provide convergence results which justify our asymptotic approach.

1. Introduction

In the last decades, many authors have applied the asymptotic methods in three-dimensional elasticity problems in order to derive new reduced one-dimensional or two-dimensional models and justify the existing ones. A complete theory regarding elastic shells can be found in [1], where models for elliptic membranes, generalized membranes, and flexural shells are presented. It contains a full description of the asymptotic procedure that leads to the corresponding sets of two-dimensional equations. Particularly, the existence and uniqueness of the solution to elastic elliptic membrane shell equations can be found in [2] and in [3]. There, the two-dimensional elastic models are completely justified with convergence theorems.

More recently, in [4], the obstacle problem for an elastic elliptic membrane has been identified and justified as the limit problem for a family of unilateral contact problems of elastic elliptic shells. A large number of actual physical and engineering problems have made it necessary to study models that take into account effects such as hardening and memory of the material. An example of these are the viscoelastic models (see, for example, [5,6]). In some of these models, we can find terms that take into account the history of previous deformations or stresses of the body, known as long-term memory. For a family of shells made of a long-term memory viscoelastic material, we can find in [7–9] the use of asymptotic analysis to justify with convergence results the limit two-dimensional membrane, flexural, and Koiter equations.

In this direction, to our knowledge, in [10] we gave the first steps towards the justification of existing models of viscoelastic shells and finding new ones with the starting point being three-dimensional Kelvin–Voigt viscoelastic shell problems. By using the asymptotic expansion method, we found a rich variety of cases for the limit two-dimensional problems, depending on the geometry of the middle surface, the boundary conditions and the order of the applied forces. The most remarkable feature found was that, from the asymptotic analysis of the three-dimensional problems, a long-term

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memory arose in the two-dimensional limit problems, represented by an integral with respect to the time variable. The aim of this Note is to mathematically justify these equations that we identified in [10] as the viscoelastic elliptic membrane problem, by presenting rigorous convergence results.

2. The three-dimensional linearly viscoelastic shell problem

We denote $\mathbb{S}^d$, where $d = 2, 3$ in practice, the space of second-order symmetric tensors on $\mathbb{R}^d$, while " $\cdot$ " will represent the inner product and $|\cdot|$ the usual norm in $\mathbb{S}^d$ and $\mathbb{R}^d$. In what follows, unless the contrary is explicitly written, we will use summation convention on repeated indices. Moreover, Latin indices $i, j, k, l, \ldots$, take their values in the set $\{1, 2, 3\}$, whereas Greek indices $\alpha, \beta, \sigma, \tau, \ldots$ do it in the set $\{1, 2\}$. Also, we use standard notation for the Lebesgue and Sobolev spaces. Moreover, for a time dependent function u, we denote $\dot{u}$ the first derivative of u with respect to the time variable. Recall that "$\to$" denotes strong convergence, while "$\rightharpoonup$" denotes weak convergence.

Let ω be a domain of $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\gamma = \partial\omega$. Let $\boldsymbol{y} = (y_\alpha)$ be a generic point of its closure $\bar{\omega}$ and let ∂_α denote the partial derivative with respect to y_α.

Let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha \boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}(\boldsymbol{y})$. The surface S is uniformly elliptic, in the sense that the two principal radius of curvature are either both positive at all points of S, or both negative at all points of S. We can consider the two vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$ of the same tangent plane defined by the relations $\boldsymbol{a}^\alpha(\boldsymbol{y}) \cdot \boldsymbol{a}_\beta(\boldsymbol{y}) = \delta^\alpha_\beta$, which constitute the contravariant basis. We define the unit vector, $\boldsymbol{a}_3(\boldsymbol{y}) = \boldsymbol{a}^3(\boldsymbol{y}) := \frac{\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})}{|\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})|}$, normal vector to S at the point $\boldsymbol{\theta}(\boldsymbol{y})$, where $\wedge$ denotes the vector product in $\mathbb{R}^3$.

We can define the first fundamental form, given as a metric tensor, in covariant or contravariant components, respectively, by $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$, $a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$, the second fundamental form, given as a curvature tensor, in covariant or mixed components, respectively, by $b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha$, $b^\beta_\alpha := a^{\beta\sigma} \cdot b_{\sigma\alpha}$, and the Christoffel symbols of the surface S by $\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha$. The area element along S is $\sqrt{a}\,\mathrm{d}y$, where $a := \det(a_{\alpha\beta})$.

For each $\varepsilon > 0$, we define the three-dimensional domain $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ and its boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$. We also define the parts of the boundary, $\Gamma^\varepsilon_+ := \omega \times \{\varepsilon\}$, $\Gamma^\varepsilon_- := \omega \times \{-\varepsilon\}$ and $\Gamma^\varepsilon_0 := \gamma \times [-\varepsilon, \varepsilon]$.

Let $\boldsymbol{x}^\varepsilon = (x^\varepsilon_i)$ be a generic point of $\bar{\Omega}^\varepsilon$, and let ∂^ε_i denote the partial derivative with respect to x^ε_i. Note that $x^\varepsilon_\alpha = y_\alpha$ and $\partial^\varepsilon_\alpha = \partial_\alpha$. Let $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ be the mapping defined by

$$\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) := \boldsymbol{\theta}(\boldsymbol{y}) + x^\varepsilon_3 \boldsymbol{a}_3(\boldsymbol{y}) \ \forall \boldsymbol{x}^\varepsilon = (\boldsymbol{y}, x^\varepsilon_3) = (y_1, y_2, x^\varepsilon_3) \in \bar{\Omega}^\varepsilon \tag{1}$$

If the injective mapping $\boldsymbol{\theta} : \bar{\omega} \to \mathbb{R}^3$ is smooth enough, the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is also injective for $\varepsilon > 0$ small enough (see Theorem 3.1-1, [1]). For each ε, $0 < \varepsilon \le \varepsilon_0$ (with ε_0 defined in Theorem 3.1-1, [1]), the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$ is the reference configuration of a viscoelastic shell, with middle surface $S = \boldsymbol{\theta}(\bar{\omega})$ and thickness $2\varepsilon > 0$. Furthermore, for $\varepsilon > 0$, $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon) := \partial^\varepsilon_i \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ are linearly independent, and the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is injective for all ε, $0 < \varepsilon \le \varepsilon_0$, as a consequence of the injectivity of the mapping $\boldsymbol{\theta}$. Hence, the three vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)$ form the covariant basis of the tangent space at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$, and $\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$, defined by the relations $\boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^\varepsilon_j = \delta^i_j$, form the contravariant basis at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. We define the metric tensor, in covariant or contravariant components, respectively, by $g^\varepsilon_{ij} := \boldsymbol{g}^\varepsilon_i \cdot \boldsymbol{g}^\varepsilon_j$, $g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}$, and the Christoffel symbols by $\Gamma^{p,\varepsilon}_{ij} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial^\varepsilon_i \boldsymbol{g}^\varepsilon_j$.

The volume element in the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$ is $\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$, and the surface element in $\boldsymbol{\Theta}(\Gamma^\varepsilon)$ is $\sqrt{g^\varepsilon} d\Gamma^\varepsilon$, where $g^\varepsilon := \det(g^\varepsilon_{ij})$.

Besides, let $T > 0$ be the period of observation and we denote by $u^\varepsilon_i : [0, T] \times \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ the covariant components of the displacement field, i.e. $\boldsymbol{\mathcal{U}}^\varepsilon := u^\varepsilon_i \boldsymbol{g}^{i,\varepsilon} : [0, T] \times \bar{\Omega}^\varepsilon \to \mathbb{R}^3$. For simplicity, we define the vector field $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) : [0, T] \times \Omega^\varepsilon \to \mathbb{R}^3$, which will denote the vector of unknowns.

We assume that the shell is subjected to a boundary condition of place; in particular, we assume that the displacements field vanishes in $\boldsymbol{\Theta}(\Gamma^\varepsilon_0)$, i.e. on the whole lateral face of the shell.

Let us define the space of admissible unknowns,

$$V(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v^\varepsilon_i) \in [H^1(\Omega^\varepsilon)]^3; \ \boldsymbol{v}^\varepsilon = \boldsymbol{0} \ on \ \Gamma^\varepsilon_0\}$$

This is a real Hilbert space with the induced inner product of $[H^1(\Omega^\varepsilon)]^3$. The corresponding norm is denoted by $||\cdot||_{1,\Omega^\varepsilon}$.

We assume that the body is made of a Kelvin–Voigt viscoelastic material, which is homogeneous and isotropic, so that the material is characterized by its Lamé coefficients $\lambda \ge 0$, $\mu > 0$ and its viscosity coefficients, $\theta \ge 0$, $\rho \ge 0$ (see for instance [5,6]). Under the effect of applied forces, the body is deformed, and we can find that $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i)$ verifies the following variational problem of a three-dimensional viscoelastic shell in curvilinear coordinates:

Problem 2.1. Find $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) : [0, T] \times \Omega^\varepsilon \to \mathbb{R}^3$ such that

$$\boldsymbol{u}^\varepsilon(t, \cdot) \in V(\Omega^\varepsilon) \ \forall t \in [0, T]$$

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon + \int_{\Omega^\varepsilon} B^{ijkl,\varepsilon} e^\varepsilon_{k||l}(\dot{\boldsymbol{u}}^\varepsilon) e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$$

$$= \int_{\Omega^\varepsilon} f^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon + \int_{\Gamma^\varepsilon_+\cup\Gamma^\varepsilon_-} h^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon}\,\mathrm{d}\Gamma^\varepsilon \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon),\ a.e. \text{ in } (0,T)$$

$$\boldsymbol{u}^\varepsilon(0,\cdot) = \boldsymbol{u}^\varepsilon_0(\cdot)$$

where the functions

$$A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}) \tag{2}$$

$$B^{ijkl,\varepsilon} := \theta g^{ij,\varepsilon} g^{kl,\varepsilon} + \frac{\rho}{2}(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}) \tag{3}$$

are the contravariant components of the three-dimensional elasticity and viscosity tensors, respectively. We assume that the Lamé coefficients $\lambda \geq 0, \mu > 0$ and the viscosity coefficients $\theta \geq 0, \rho \geq 0$ are all independent of ε. Moreover, the terms $e^\varepsilon_{i||j}(\boldsymbol{u}^\varepsilon) := \frac{1}{2}(u^\varepsilon_{i||j} + u^\varepsilon_{j||i}) = \frac{1}{2}(\partial^\varepsilon_j u^\varepsilon_i + \partial^\varepsilon_i u^\varepsilon_j) - \Gamma^{p,\varepsilon}_{ij} u^\varepsilon_p$ designate the covariant components of the linearized strain tensor associated with the displacement field $\boldsymbol{\mathcal{U}}^\varepsilon$ of the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$. Moreover, $f^{i,\varepsilon}$ denotes the contravariant components of the volumic force densities, $h^{i,\varepsilon}$ denotes contravariant components of surface force densities and $\boldsymbol{u}^\varepsilon_0$ denotes the initial "displacements" (actually, the initial displacement is $\boldsymbol{\mathcal{U}}^\varepsilon_0 := (u^\varepsilon_0)_i \boldsymbol{g}^{i,\varepsilon}$). Note that $\Gamma^{3,\varepsilon}_{\alpha 3} = \Gamma^{p,\varepsilon}_{33} = A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = B^{\alpha\beta\sigma 3,\varepsilon} = B^{\alpha 333,\varepsilon} = 0$ in $\bar{\Omega}^\varepsilon$, by (1).

The existence and uniqueness of the solution to Problem 2.1 for $\varepsilon > 0$ small enough can be consulted in [10]. There we find that, under suitable regularity hypotheses for the applied forces and initial condition, there exists a unique solution such that $\boldsymbol{u}^\varepsilon \in W^{1,2}(0,T; V(\Omega^\varepsilon))$.

3. The scaled three-dimensional shell problem

For convenience, we consider a reference domain independent of the small parameter ε. Hence, let us define the three-dimensional domain $\Omega := \omega \times (-1,1)$ and its boundary $\Gamma = \partial\Omega$. We also define the parts of the boundary, $\Gamma_+ := \omega \times \{1\}, \Gamma_- := \omega \times \{-1\}$ and $\Gamma_0 := \gamma \times [-1,1]$. Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be a generic point in $\bar{\Omega}$ and consider the notation ∂_i for the partial derivative with respect to x_i. We define the projection map, $\pi^\varepsilon : \boldsymbol{x} = (x_1,x_2,x_3) \in \bar{\Omega} \longrightarrow \pi^\varepsilon(\boldsymbol{x}) = \boldsymbol{x}^\varepsilon = (x^\varepsilon_i) = (x^\varepsilon_1, x^\varepsilon_2, x^\varepsilon_3) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon$; hence, $\partial^\varepsilon_\alpha = \partial_\alpha$ and $\partial^\varepsilon_3 = \frac{1}{\varepsilon}\partial_3$. We consider the scaled unknown and vector fields defined as $u^\varepsilon_i(t, \boldsymbol{x}^\varepsilon) =: u_i(\varepsilon)(t, \boldsymbol{x})$ and $v^\varepsilon_i(\boldsymbol{x}^\varepsilon) =: v_i(\boldsymbol{x})\ \forall \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon,\ \forall\, t \in [0,T]$.

Also, we define the scaled functions $\Gamma^p_{ij}(\varepsilon)(\boldsymbol{x}) := \Gamma^{p,\varepsilon}_{ij}(\boldsymbol{x}^\varepsilon), g(\varepsilon)(\boldsymbol{x}) := g^\varepsilon(\boldsymbol{x}^\varepsilon),\ A^{ijkl}(\varepsilon)(\boldsymbol{x}) := A^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon),\ B^{ijkl}(\varepsilon)(\boldsymbol{x}) := B^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon)$, for all $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon$. For all $\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3$, we define the scaled linearized strains components $e_{i||j}(\varepsilon)(\boldsymbol{v}) \in L^2(\Omega)$ by

$$e_{\alpha||\beta}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p \tag{4}$$

$$e_{\alpha||3}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}\left(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3\right) - \Gamma^p_{\alpha 3}(\varepsilon) v_p \tag{5}$$

$$e_{3||3}(\varepsilon; \boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3 \tag{6}$$

Note that with these definitions, it is verified that $e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)(\pi^\varepsilon(\boldsymbol{x})) = e_{i||j}(\varepsilon; \boldsymbol{v})(\boldsymbol{x})\ \forall \boldsymbol{x} \in \Omega$.

Remark 1. The functions $\Gamma^p_{ij}(\varepsilon)$, $g(\varepsilon)$, $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ converge in $\mathcal{C}^0(\bar{\Omega})$ when ε tends to zero.

When we consider $\varepsilon = 0$, the functions will be defined with respect to $\boldsymbol{y} \in \bar{\omega}$. Also, we shall distinguish the three-dimensional Christoffel symbols from the two-dimensional ones by using $\Gamma^\sigma_{\alpha\beta}(\varepsilon)$ and $\Gamma^\sigma_{\alpha\beta}$, respectively.

The next result is an adaptation of (b) in Theorem 3.3-2, [1] to the viscoelastic case. We will study the asymptotic behaviour of the scaled contravariant components $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ of the three-dimensional elasticity and viscosity tensors defined above, as $\varepsilon \to 0$. We show their uniform positive definiteness not only with respect to $\boldsymbol{x} \in \bar{\Omega}$, but also with respect to ε, $0 < \varepsilon \leq \varepsilon_0$. Besides, their limits are functions of $\boldsymbol{y} \in \bar{\omega}$ only, i.e. they are independent of the transversal variable x_3.

Theorem 3.1. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$; let $a^{\alpha\beta}$ denote the contravariant components of the metric tensor of $S = \boldsymbol{\theta}(\bar{\omega})$. In addition to that, let the other assumptions on the mapping $\boldsymbol{\theta}$ and the definition of ε_0 be as in Theorem 3.1-1, [1]. The contravariant components $A^{ijkl}(\varepsilon)$,*

$B^{ijkl}(\varepsilon)$ of the scaled three-dimensional elasticity and viscosity tensors, respectively, satisfy $A^{ijkl}(\varepsilon) = A^{ijkl}(0) + O(\varepsilon)$ and $A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0$, $B^{ijkl}(\varepsilon) = B^{ijkl}(0) + O(\varepsilon)$ and $B^{\alpha\beta\sigma 3}(\varepsilon) = B^{\alpha 333}(\varepsilon) = 0$, *for all* ε, $0 < \varepsilon \leq \varepsilon_0$ *and*

$$\begin{aligned}
&A^{\alpha\beta\sigma\tau}(0) = \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}), && A^{\alpha\beta 33}(0) = \lambda a^{\alpha\beta}\\
&A^{\alpha 3\sigma 3}(0) = \mu a^{\alpha\sigma}, && A^{3333}(0) = \lambda + 2\mu\\
&B^{\alpha\beta\sigma\tau}(0) = \theta a^{\alpha\beta} a^{\sigma\tau} + \frac{\rho}{2}(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}), && B^{\alpha\beta 33}(0) = \theta a^{\alpha\beta}\\
&B^{\alpha 3\sigma 3}(0) = \frac{\rho}{2} a^{\alpha\sigma}, && B^{3333}(0) = \theta + \rho
\end{aligned}$$

$A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) = B^{\alpha\beta\sigma 3}(0) = B^{\alpha 333}(0) = 0$. *Moreover, there exist two constants* $C_e > 0$ *and* $C_v > 0$, *independent of the variables and* ε, *such that*

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijkl}(\varepsilon)(\boldsymbol{x}) t_{kl} t_{ij}, \quad \sum_{i,j} |t_{ij}|^2 \leq C_v B^{ijkl}(\varepsilon)(\boldsymbol{x}) t_{kl} t_{ij} \tag{7}$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, *for all* $\boldsymbol{x} \in \bar{\Omega}$ *and all* $\boldsymbol{t} = (t_{ij}) \in \mathbb{S}^2$.

Let the scaled applied forces be defined by $\boldsymbol{f}^\varepsilon =: \boldsymbol{f}(\varepsilon) = (f^i(\varepsilon))(t, \boldsymbol{x}) = \boldsymbol{f}^0(t, \boldsymbol{x})$ $\forall \boldsymbol{x} \in \Omega$ and $\boldsymbol{h}^\varepsilon =: \boldsymbol{h}(\varepsilon) = (h^i(\varepsilon))(t, \boldsymbol{x}) = \varepsilon \boldsymbol{h}^1(t, \boldsymbol{x})$ $\forall \boldsymbol{x} \in \Gamma_+ \cup \Gamma_-$ and $\forall t \in [0, T]$ where $\boldsymbol{f}^0$ and $\boldsymbol{h}^1$ are functions independent of ε. Also, we introduce $\boldsymbol{u}_0(\varepsilon) : \Omega \longrightarrow \mathbb{R}^3$ by $\boldsymbol{u}_0(\varepsilon)(\boldsymbol{x}) := \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}^\varepsilon)$ $\forall \boldsymbol{x} \in \Omega$, where $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon$ and define the space $V(\Omega) := \{\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3; \boldsymbol{v} = \boldsymbol{0}$ *on* $\Gamma_0\}$, which is a Hilbert space with the inner product of $[H^1(\Omega)]^3$. The corresponding norm is denoted by $||\cdot||_{1,\Omega}$. Then, the scaled variational problem can be written as follows.

Problem 3.2. Find $\boldsymbol{u}(\varepsilon) : [0, T] \times \Omega \longrightarrow \mathbb{R}^3$ such that

$$\begin{aligned}
&\boldsymbol{u}(\varepsilon)(t, \cdot) \in V(\Omega)\ \forall\, t \in [0, T]\\
&\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon, \boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon, \boldsymbol{v}) \sqrt{g(\varepsilon)}\, \mathrm{d}x + \int_\Omega B^{ijkl}(\varepsilon) e_{k||l}(\varepsilon, \dot{\boldsymbol{u}}(\varepsilon)) e_{i||j}(\varepsilon, \boldsymbol{v}) \sqrt{g(\varepsilon)}\, \mathrm{d}x\\
&\quad = \int_\Omega f^{i,0} v_i \sqrt{g(\varepsilon)}\, \mathrm{d}x + \int_{\Gamma_+ \cup \Gamma_-} \varepsilon h^{i,1} v_i \sqrt{g(\varepsilon)}\, \mathrm{d}\Gamma \quad \forall \boldsymbol{v} \in V(\Omega),\ a.e. \text{ in } (0, T)\\
&\boldsymbol{u}(\varepsilon)(0, \cdot) = \boldsymbol{u}_0(\varepsilon)(\cdot)
\end{aligned} \tag{8}$$

We can prove the existence and uniqueness of the solution to Problem 3.2 (see [10]). Moreover, under suitable regularity conditions, $\boldsymbol{u}(\varepsilon) \in W^{1,2}(0, T; V(\Omega))$.

In Theorem 3.3-1, [1], we find that the limits of the scaled Christoffel symbols are independent of x_3. Moreover, $g(\varepsilon) = a + O(\varepsilon)$.

4. Asymptotic analysis. Convergence results as $\varepsilon \to 0$

In the next theorems, we recall, for the benefit of the reader, the following three- and two-dimensional inequalities of Korn's type for a family of elliptic membrane shells (see for example Theorem 4.3-1 and Theorem 2.7-3, [1], respectively).

Theorem 4.1. *Assume that* $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$ *and consider* ε_0 *defined as in Theorem 3.1-1 [1]. We consider a family of elliptic membrane shells with thickness* 2ε *with each one having the same middle surface* $S = \boldsymbol{\theta}(\bar{\omega})$. *Then there exist a constant* ε_1 *verifying* $0 < \varepsilon_1 < \varepsilon_0$ *and a constant* $C > 0$ *such that, for all* ε, $0 < \varepsilon \leq \varepsilon_1$, *the following three-dimensional inequality of Korn's type holds,*

$$\left(\sum_\alpha ||v_\alpha||^2_{1,\Omega} + |v_3|^2_{0,\Omega} \right)^{1/2} \leq C \left(\sum_{i,j} |e_{i||j}(\varepsilon; \boldsymbol{v})|^2_{0,\Omega} \right)^{1/2} \quad \forall \boldsymbol{v} = (v_i) \in V(\Omega) \tag{9}$$

Theorem 4.2. *Let* ω *be a domain in* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\bar{\omega}; \mathbb{R}^3)$ *be an injective mapping such that the two vectors* $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ *are linearly independent at all points of* $\bar{\omega}$ *and such that the surface* $S = \boldsymbol{\theta}(\bar{\omega})$ *is elliptic. Then, the following inequality is verified*

$$\left(\sum_\alpha ||\eta_\alpha||^2_{1,\omega} + |\eta_3|^2_{0,\omega} \right)^{1/2} \leq C_M \left(\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} \right)^{1/2} \quad \forall \boldsymbol{\eta} \in V_M(\omega) \tag{10}$$

where $V_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$ *and* $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}\left(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha\right) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3$ *denote the covariant components of the linearized change of metric tensor associated with a displacement* $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ *of the middle surface.*

We recall the two-dimensional equations obtained for a viscoelastic membrane shell as a consequence of the formal asymptotic study made in [10]. For the case of elliptic membranes, the right space where the problem is well posed is $V_M(\omega)$. Moreover, the space defined by

$$V_0(\omega) := \{\boldsymbol{\eta} \in [H^1(\omega)]^3;\ \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma,\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ on } \omega\}$$

is such that only contains the element $\boldsymbol{\eta} = \mathbf{0}$ (see (10)).

From the asymptotic analysis made in [10], we show that, if the applied body force density is $O(1)$ with respect to ε and the surface traction density is $O(\varepsilon)$ as in Problem 3.2, we obtain in the limit the two-dimensional variational problem for a viscoelastic membrane. Let us remind the definition of the two-dimensional fourth-order tensors that appeared naturally in [10]:

$$a^{\alpha\beta\sigma\tau} := \frac{2\lambda\rho^2 + 4\mu\theta^2}{(\theta+\rho)^2} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}) \tag{11}$$

$$b^{\alpha\beta\sigma\tau} := \frac{2\theta\rho}{\theta+\rho} a^{\alpha\beta} a^{\sigma\tau} + \rho(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}) \tag{12}$$

$$c^{\alpha\beta\sigma\tau} := \frac{2\,(\theta\Lambda)^2}{\theta+\rho} a^{\alpha\beta} a^{\sigma\tau} \tag{13}$$

where $\Lambda := \left(\frac{\lambda}{\theta} - \frac{\lambda+2\mu}{\theta+\rho}\right)$. Therefore, we can enunciate the two-dimensional variational problem for a linear viscoelastic elliptic membrane as follows.

Problem 4.3. Find $\boldsymbol{\xi} : [0,T] \times \omega \longrightarrow \mathbb{R}^3$ such that: $\boldsymbol{\xi}(t,\cdot) \in V_M(\omega)\ \forall\, t \in [0,T]$,

$$\begin{aligned}
&\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y + \int_\omega b^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\dot{\boldsymbol{\xi}})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \\
&\quad - \int_0^t e^{-k(t-s)} \int_\omega c^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi}(s))\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_\omega p^{i,0}\eta_i\sqrt{a}\,\mathrm{d}y \quad \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega),\ a.e.\ t \in (0,T) \\
&\boldsymbol{\xi}(0,\cdot) = \boldsymbol{\xi}_0(\cdot)
\end{aligned}$$

where we introduced the constant $k > 0$ defined by $k := \frac{\lambda+2\mu}{\theta+\rho}$, and

$$p^{i,0}(t) := \int_{-1}^{1} f^{i,0}(t)\,\mathrm{d}x_3 + h_+^{i,1}(t) + h_-^{i,1}(t), \text{ with } h_\pm^{i,1}(t) = h^{i,1}(t,\cdot,\pm 1)$$

Remark 2. As the reader can check in [10], assuming an asymptotic expansion of the unknowns and substituting them into the equations of Problem 3.2, we found that the leading terms of the components $e_{\alpha||3}(\varepsilon)$ vanish. Besides, denoting by $e^0_{i||j}$ the corresponding leading-order terms of the components $e_{i||j}$, we obtained the following ordinary differential equation:

$$\lambda a^{\alpha\beta} e^0_{\alpha||\beta} + (\lambda+2\mu)e^0_{3||3} + \theta a^{\alpha\beta}\dot{e}^0_{\alpha||\beta} + (\theta+\rho)\dot{e}^0_{3||3} = 0 \tag{14}$$

Hence, assuming that $\theta > 0$, we found that the terms $e^0_{3||3}$ can be expressed in function of the components $e^0_{\alpha||\beta}$ through a long-term memory. Moreover, the latter ones can be identified with the covariant components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ of the linearized change of metric tensor, where $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ represents a displacement of the middle surface S of the shell. As a result, we derived the two-dimensional equations given in Problem 4.3.

The existence and uniqueness of the solution to Problem 4.3 makes use of the Korn's inequality (10) (see [10] for details).

Theorem 4.4. *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$. Let $f^{i,0} \in L^2(0,T;L^2(\Omega))$, $h^{i,1} \in L^2(0,T;L^2(\Gamma_1))$, where $\Gamma_1 := \Gamma_+ \cup \Gamma_-$. Let $\boldsymbol{\xi}_0 \in V_M(\omega)$. Then Problem 4.3, has a unique solution $\boldsymbol{\xi} \in W^{1,2}(0,T;V_M(\omega))$. In addition to that, if $\dot{f}^{i,0} \in L^2(0,T;L^2(\Omega))$, $\dot{h}^{i,1} \in L^2(0,T;L^2(\Gamma_1))$, then $\boldsymbol{\xi} \in W^{2,2}(0,T;V_M(\omega))$.*

From now on, we shall use the short-hand notation $e_{i||j}(\varepsilon) := e_{i||j}(\varepsilon; \boldsymbol{u}(\varepsilon))$. For each $\varepsilon > 0$, we assume that the initial condition for the scaled linear strains is

$$e_{i||j}(\varepsilon)(0, \cdot) = 0 \tag{15}$$

i.e. the domain is in its natural state with no strains on it at the beginning of the period of observation.

Now, we present here the main result of this paper, namely that the scaled three-dimensional unknown $\boldsymbol{u}(\varepsilon)$ converges, as ε tends to zero, towards a limit $\boldsymbol{u}$ independent of the transversal variable x_3. Moreover, this limit can be identified with the solution $\boldsymbol{\xi} = \bar{\boldsymbol{u}} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3$ of Problem 4.3, posed over the set ω.

Theorem 4.5. *Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$. Consider a family of viscoelastic elliptic membrane shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \boldsymbol{\theta}(\bar{\omega})$, and let the assumptions on the data be as in Theorem 4.4. For all ε, $0 < \varepsilon \le \varepsilon_0$ let $\boldsymbol{u}(\varepsilon)$ be the solution to the associated three-dimensional scaled Problem 3.2. Then, there exist functions $u_\alpha \in W^{1,2}(0, T, H^1(\Omega))$ satisfying $u_\alpha = 0$ on $\gamma \times [-1, 1]$ and a function $u_3 \in W^{1,2}(0, T, L^2(\Omega))$, such that*

(i) *$u_\alpha(\varepsilon) \to u_\alpha$ in $W^{1,2}(0, T, H^1(\Omega))$ and $u_3(\varepsilon) \to u_3$ in $W^{1,2}(0, T, L^2(\Omega))$ when $\varepsilon \to 0$,*
(ii) *$\boldsymbol{u} := (u_i)$ is independent of the transversal variable x_3.*

Furthermore, the average $\bar{\boldsymbol{u}}$ verifies Problem 4.3.

The proof of this theorem can be found in full detail in [11]. In this Note, we describe the scheme of the proof when the proposed problem is subjected only to volume forces (the inclusion of traction forces needs the definition of a trace in $X(0, T; \Omega) := \{\boldsymbol{v} \in W^{1,2}(0, T, L^2(\Omega)); \partial_3 v \in W^{1,2}(0, T, L^2(\Omega))\}$).

(i) *A priori boundedness and extraction of weak convergent sequences.*
The norms $|e_{i||j}(\varepsilon)|_{W^{1,2}(0,T,L^2(\Omega))}$, $||u_\alpha(\varepsilon)||_{W^{1,2}(0,T,H^1(\Omega))}$, and $|u_3(\varepsilon)|_{W^{1,2}(0,T,L^2(\Omega))}$ are bounded independently of $\varepsilon, 0 < \varepsilon \le \varepsilon_1$, where $\varepsilon_1 > 0$ is given in Theorem 4.1. Consequently, there exist a subsequence, also denoted $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$, and functions $e_{i||j} \in W^{1,2}(0, T, L^2(\Omega)), u_\alpha \in W^{1,2}(0, T, H^1(\Omega))$, satisfying $u_\alpha = 0$ on Γ_0, and $u_3 \in W^{1,2}(0, T, L^2(\Omega))$, such that $e_{i||j}(\varepsilon) \rightharpoonup e_{i||j}$ in $W^{1,2}(0, T, L^2(\Omega)), u_\alpha(\varepsilon) \rightharpoonup u_\alpha$ in $W^{1,2}(0, T, H^1(\Omega))$, and hence $u_\alpha(\varepsilon) \to u_\alpha$ in $W^{1,2}(0, T, L^2(\Omega))$, $u_3(\varepsilon) \rightharpoonup u_3$ in $W^{1,2}(0, T, L^2(\Omega))$.
For the proof of this step, we take $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)(t, \cdot)$ and $\boldsymbol{v} = \dot{\boldsymbol{u}}(\varepsilon)(t, \cdot)$ in (8), alternately. Then, using the ellipticity of $A^{ijkl}(\varepsilon)$ and $B^{ijkl}(\varepsilon)$, the initial condition (15), the Korn's type inequality (9), and the Cauchy–Schwartz inequality, the conclusion is achieved.

(ii) *The limits of the scaled unknown found in step (i) are independent of x_3.*
To do this, we use the definition of the scaled strains $e_{\alpha||3}(\varepsilon)$ and $e_{3||3}(\varepsilon)$. After some calculations, we get the independence of x_3 of those functions in the sense of distributions. Applying a generalization of the Theorem 4.2-1 (*a*), [1] the conclusion follows.

(iii) *The limits $e_{i||j}$ found in (i) are independent of the variable x_3. Moreover, they are related with the limits $\boldsymbol{u} := (u_i)$ by*

$$e_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) := \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^\sigma_{\alpha\beta} u_\sigma - b_{\alpha\beta} u_3, \quad e_{\alpha||3} = 0$$

$$e_{3||3}(t) = -\frac{\theta}{\theta + \rho}\left(a^{\alpha\beta} e_{\alpha||\beta}(t) + \Lambda \int_0^t e^{-k(t-s)} a^{\alpha\beta} e_{\alpha||\beta}(s)\, ds\right), \text{ in } \Omega, \ \forall t \in [0, T]$$

with $\Lambda = \left(\frac{\lambda}{\theta} - \frac{\lambda+2\mu}{\theta+\rho}\right)$ and $k = \frac{\lambda+2\mu}{\theta+\rho}$.
Considering $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in (4) and $\boldsymbol{\eta} = \boldsymbol{u}$ in the definition of $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ (see Theorem 10), taking into account step (*i*) and the convergences of the Christoffel symbols, we have that

$$e_{\alpha||\beta}(\varepsilon) = \frac{1}{2}(\partial_\beta u_\alpha(\varepsilon) + \partial_\alpha u_\beta(\varepsilon)) - \Gamma^p_{\alpha\beta}(\varepsilon) u_p(\varepsilon) \rightharpoonup e_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) \text{ in } W^{1,2}(0, T, L^2(\Omega))$$

Then, we take alternately particular test functions $\boldsymbol{v} \in V(\Omega)$ in (8), expand the resulting terms and use a result of calculus of variations (see Theorem 3.4-1, [1]). On the one hand, we obtain that $2\mu a e_{\alpha||3} + \rho a \dot{e}_{\alpha||3} = 0$, which, using (15), implies that $e_{\alpha||3} = 0$. On the other hand, we obtain the differential equation:

$$\lambda a^{\alpha\beta} e_{\alpha||\beta} + (\lambda + 2\mu) e_{3||3} + \theta a^{\alpha\beta} \dot{e}_{\alpha||\beta} + (\theta + \rho) \dot{e}_{3||3} = 0 \tag{16}$$

In order to obtain the expression of $e_{3||3}$ in the most general case, we need to assume that $\theta > 0$. Therefore, the viscoelastic case can not generalize the elastic case from now on. Then, from (16) together with (15), the expression announced yields.

(iv) *The function $\bar{\boldsymbol{u}} = (\bar{\boldsymbol{u}}_i)$ satisfies the two-dimensional variational Problem 4.3 with $p^{i,0} := \int_{-1}^{1} f^i \, dx_3$. In particular, since the solution to this problem is unique, the convergences on (i) are verified for all the family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$. We have that $\bar{\boldsymbol{u}}(t,\cdot) = (\bar{u}_i(t,\cdot)) \in V_M(\omega)$, $\forall\, t \in [0,T]$.*
To do this, let $\boldsymbol{v} = (v_i) \in V(\Omega)$ be independent of x_3 in (8) and take the limit when $\varepsilon \to 0$. Then, using the asymptotic behaviour of the functions involved and the findings from previous steps, we obtain that $\bar{\boldsymbol{u}}$ satisfies Problem 4.3 for all $\boldsymbol{v} = (v_i) \in V(\Omega)$ independent of x_3. Then, applying a generalization of the Theorem 4.2-1, [1] the conclusion follows.

(v) *The weak convergences $e_{i||j}(\varepsilon)(t,\cdot) \rightharpoonup e_{i||j}(t,\cdot)$ in $W^{1,2}(0,T,L^2(\Omega))$ are, in fact, strong.*
To do this, we define:

$$\begin{aligned}\Psi(\varepsilon) :=& \int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon) - e_{k||l})(e_{i||j}(\varepsilon) - e_{i||j})\sqrt{g(\varepsilon)}\,dx \\ &+ \int_\Omega B^{ijkl}(\varepsilon)(\dot{e}_{k||l}(\varepsilon) - \dot{e}_{k||l})(e_{i||j}(\varepsilon) - e_{i||j})\sqrt{g(\varepsilon)}\,dx \\ =& \int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}\,dx - \int_\Omega A^{ijkl}(\varepsilon)(2e_{k||l}(\varepsilon) - e_{k||l})e_{i||j}\sqrt{g(\varepsilon)}\,dx \\ &+ \int_\Omega B^{ijkl}(\varepsilon)\left(\dot{e}_{k||l}e_{i||j} - \frac{\partial}{\partial t}(e_{k||l}(\varepsilon)e_{i||j})\right)\sqrt{g(\varepsilon)}\,dx\end{aligned}$$

Using the ellipticity of the fourth order tensors (7), the initial condition (15) and the Cauchy–Schwartz inequality, we obtain that

$$C_e^{-1} g_0^{1/2} \int_0^T \left(\sum_{i,j} |e_{i||j}(\varepsilon)(t) - e_{i||j}(t)|_{0,\Omega}^2 \right) dt \leq \int_0^T \Psi(\varepsilon)\,dt$$

Then, we show that $\Psi := \lim_{\varepsilon\to 0}\Psi(\varepsilon) = 0$, having in mind the step (i) and the asymptotic behaviour of the functions involved. Hence, thanks to the Lebesgue dominated convergence theorem, the strong convergences in $L^2(0,T,L^2(\Omega))$ are satisfied. Analogously, we define:

$$\begin{aligned}\tilde{\Psi}(\varepsilon) :=& \int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon) - e_{k||l})(\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j})\sqrt{g(\varepsilon)}\,dx \\ &+ \int_\Omega B^{ijkl}(\varepsilon)(\dot{e}_{k||l}(\varepsilon) - \dot{e}_{k||l})(\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j})\sqrt{g(\varepsilon)}\,dx \\ =& \int_\Omega f^i \dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}\,dx + \int_\Omega A^{ijkl}(\varepsilon)\left(e_{k||l}\dot{e}_{i||j} - \frac{\partial}{\partial t}(e_{k||l}(\varepsilon)e_{i||j})\right)\sqrt{g(\varepsilon)}\,dx \\ &- \int_\Omega B^{ijkl}(\varepsilon)(2\dot{e}_{k||l}(\varepsilon) - \dot{e}_{k||l})\dot{e}_{i||j}\sqrt{g(\varepsilon)}\,dx\end{aligned}$$

Using similar arguments, we show that

$$C_v^{-1} g_0^{1/2} \int_0^T \left(\sum_{i,j} |\dot{e}_{i||j}(\varepsilon)(t) - \dot{e}_{i||j}(t)|_{0,\Omega}^2 \right) dt \leq \int_0^T \tilde{\Psi}(\varepsilon)\,dt$$

and that $\tilde{\Psi} := \lim_{\varepsilon\to 0}\tilde{\Psi}(\varepsilon) = 0$. Hence, again by the Lebesgue dominated convergence theorem, we conclude that the strong convergences hold in $W^{1,2}(0,T,L^2(\Omega))$.

(vi) *The family $(\bar{\boldsymbol{u}}(\varepsilon))_{\varepsilon>0}$ converges strongly to $\bar{\boldsymbol{u}}$ (when $\varepsilon \to 0$) in $W^{1,2}(0,T,V_M(\omega))$, i.e.* $\bar{u}_\alpha(\varepsilon) \to \bar{u}_\alpha$ in $W^{1,2}(0,T,H^1(\omega))$, $\bar{u}_3(\varepsilon) \to \bar{u}_3$ in $W^{1,2}(0,T,L^2(\omega))$.
This proof is a corollary of the step (vi) in Th. 4.4-1 [1]. In order to do that, we follow the same arguments made there to prove that $\bar{u}_\alpha(\varepsilon) \to \bar{u}_\alpha$ in $L^2(0,T,H^1(\omega))$, $\bar{u}_3(\varepsilon) \to \bar{u}_3$ in $L^2(0,T,L^2(\omega))$ and the corresponding convergences of the time derivatives in the same spaces. Then the conclusion follows.

(vii) *The convergence $u_3(\varepsilon) \rightharpoonup u_3$ in $W^{1,2}(0,T,L^2(\Omega))$ is, in fact, strong.*
Indeed, by (6) and step (i), we have $\partial_3 u_3(\varepsilon) = \varepsilon e_{3||3}(\varepsilon) \to 0$ in $W^{1,2}(0,T,L^2(\Omega))$. On the other hand, we have $\bar{u}_3(\varepsilon) \to \bar{u}_3$ in $W^{1,2}(0,T,L^2(\omega))$. Hence, by a generalization of the Theorem 4.2-1 (c), the conclusion follows.

It remains to be proved an analogous result to the previous theorem, but in terms of de-scaled unknowns. The convergences $u_\alpha(\varepsilon) \to u_\alpha$ in $W^{1,2}(0,T,H^1(\Omega))$ and $u_3(\varepsilon) \to u_3$ in $W^{1,2}(0,T,L^2(\Omega))$ from Theorem 4.5, the scaling proposed in Section 3, the de-scalings $\xi_i^\varepsilon := \xi_i$ for each $\varepsilon > 0$, and a generalization of Theorem 4.2-1, [1] together lead to the convergences $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \, dx_3^\varepsilon \to \xi_\alpha$ in $W^{1,2}(0,T,H^1(\omega))$, $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \, dx_3^\varepsilon \to \xi_3$ in $W^{1,2}(0,T,L^2(\omega))$. Furthermore, we can prove the following theorem regarding the convergences of the averages of the tangential and normal components of the three-dimensional displacement vector field.

Theorem 4.6. *Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega};\mathbb{R}^3)$. Consider a family of viscoelastic elliptic membrane shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \boldsymbol{\theta}(\bar{\omega})$, and let the assumptions on the data be as in Theorem 4.4.*

Let $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) \in W^{1,2}(0,T,V(\Omega^\varepsilon))$ and $\boldsymbol{\xi}^\varepsilon = (\xi_i^\varepsilon) \in W^{1,2}(0,T,V_M(\omega))$ respectively denote for each $\varepsilon > 0$ the solutions to the three-dimensional and two-dimensional Problems 2.1 and 4.3 de-scaled version. Moreover, let $\boldsymbol{\xi} = (\xi_i) \in W^{1,2}(0,T,V_M(\omega))$ denote the solution to Problem 4.3. Then we have that

$$\xi_\alpha^\varepsilon = \xi_\alpha \text{ and thus } \xi_\alpha^\varepsilon \boldsymbol{a}^\alpha = \xi_\alpha \boldsymbol{a}^\alpha \text{ in } W^{1,2}(0,T,H^1(\omega)), \forall \varepsilon > 0$$

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \boldsymbol{g}^{\alpha,\varepsilon} \, dx_3^\varepsilon \to \xi_\alpha \boldsymbol{a}^\alpha \text{ in } W^{1,2}(0,T,H^1(\omega)) \text{ as } \varepsilon \to 0$$

and

$$\xi_3^\varepsilon = \xi_3 \text{ and thus } \xi_3^\varepsilon \boldsymbol{a}^3 = \xi_3 \boldsymbol{a}^3 \text{ in } W^{1,2}(0,T,L^2(\omega)), \forall \varepsilon > 0$$

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon} \, dx_3^\varepsilon \to \xi_3 \boldsymbol{a}^3 \text{ in } W^{1,2}(0,T,L^2(\omega)) \text{ as } \varepsilon \to 0$$

As a conclusion, we have found and mathematically justified a two-dimensional model for viscoelastic elliptic membranes. To this end, we used the insight provided by the asymptotic expansion method (presented in our previous work [10]) and we have justified this approach by obtaining convergence theorems. The main novelty is that from the asymptotic analysis of the three-dimensional problems, which include a short-term memory represented by a time derivative, a long-term memory arises in the two-dimensional limit problems, represented by an integral with respect to the time variable. As future work, we shall present convergence theorems in forthcoming papers for the remaining cases of the limit problems found in [10], namely the generalized membrane and flexural shells.

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