C. R. Mecanique 345 (2017) 724–731

Contents lists available at ScienceDirect

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Debonded arc-shaped interface conducting rigid line inclusions in piezoelectric composites

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ARTICLE INFO

Article history:
Received 30 May 2017
Accepted 1 July 2017
Available online 8 August 2017

Keywords:
Debonded arc-shaped conducting rigid line inclusion
Piezoelectric material
Singularity
Riemann–Hilbert problem
Decoupling

ABSTRACT

We consider an arc-shaped conducting rigid line inclusion located at the interface between a circular piezoelectric inhomogeneity and an unbounded piezoelectric matrix subjected to remote uniform anti-plane shear stresses and in-plane electric fields. Moreover, one side of the rigid line inclusion has become fully debonded from the matrix or the inhomogeneity leading to the formation of an insulating crack. After the introduction of two sectionally holomorphic vector functions, the problem is reduced to a vector Riemann–Hilbert problem, which can be decoupled sequentially by repeated application of the orthogonality relations between the eigenvectors for two corresponding generalized eigenvalue problems.

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1. Introduction

The problem of finding stress singularities at an interface crack (including free–free, fixed–fixed, and free–fixed cracks) has attracted considerable attention in the literature (see, for example, Ting [1] for a review). A fixed–fixed crack is more commonly referred to as a rigid line inclusion or anticrack [2], whereas a free–fixed crack corresponds to a debonded rigid line inclusion or debonded anticrack [3]. The corresponding stress singularities can be determined using the method of eigenfunction expansion [4] and complex variable techniques [5,6]. The stress singularities at an interfacial crack tip in piezoelectric solids have also been discussed in detail (see, for example [7–9]).

In this work, we endeavor to study a debonded arc-shaped conducting rigid line inclusion at the interface between a circular piezoelectric inhomogeneity and an infinite piezoelectric matrix when the composite is subjected to remote uniform anti-plane stresses and in-plane electric fields. The conducting rigid line inclusion can be debonded either from the matrix or from the inhomogeneity, resulting in the formation of an insulating crack. Following the introduction of two sectionally holomorphic vector functions, the problem is reduced to a vector Riemann–Hilbert (R–H) problem. By twice applying the orthogonal relations between corresponding eigenvectors, the vector R–H problem is decoupled into four scalar R–H problems, the solutions to which can be obtained by evaluating the corresponding Cauchy integrals. In our discussion, all of the results including those involving singularities in the near tip electroelastic field are obtained analytically. In particular, we also derive a rigorous solution to the degenerate case of equal eigenvalues.

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http://dx.doi.org/10.1016/j.crme.2017.07.001
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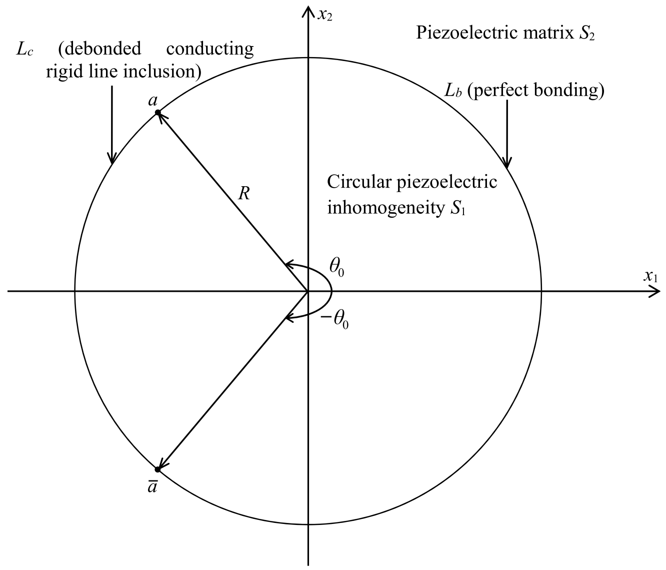

Fig. 1. A debonded arc-shaped conducting rigid line inclusion located at the interface between a circular piezoelectric inhomogeneity and a piezoelectric matrix.

2. Problem formulation

The general solution corresponding to the anti-plane shear deformations of a hexagonal piezoelectric material exhibiting 6-mm symmetry with its poling direction along the x_3-axis is given by [10]:

$$\begin{bmatrix} u_3 \\ \phi \end{bmatrix} = \mathrm{Im}\{\mathbf{f}(z)\} \tag{1}$$

$$\begin{bmatrix} 2\varepsilon_{32} + 2\mathrm{i}\varepsilon_{31} \\ -E_2 - \mathrm{i}E_1 \end{bmatrix} = \mathbf{f}'(z), \qquad \begin{bmatrix} \sigma_{32} + \mathrm{i}\sigma_{31} \\ D_2 + \mathrm{i}D_1 \end{bmatrix} = \mathbf{C}\mathbf{f}'(z), \quad \mathbf{C} = \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} C_{44} & e_{15} \\ e_{15} & -\epsilon_{11} \end{bmatrix} \tag{2}$$

where u_3 and ϕ are, respectively, the anti-plane displacement and electric potential; σ_{31} and σ_{32} are the anti-plane shear stresses; D_1 and D_2 are electric displacements; E_1 and E_2 are in-plane electric fields; ε_{31} and ε_{32} are mechanical strains; C_{44}, e_{15} and ϵ_{11} are the elastic stiffness, the piezoelectric constant and the permittivity constant; $\mathbf{f}(z)$ is a 2D analytic vector function of the complex variable $z = x_1 + \mathrm{i}x_2$. In Eq. (2), $\mathbf{C}$ is real symmetric, but not positive definite.

As shown in Fig. 1, we consider the anti-plane shear deformations of an infinite hexagonal piezoelectric matrix reinforced by a circular hexagonal piezoelectric inhomogeneity of radius R with its center at the origin when the composite is subjected to uniform remote anti-plane shear stresses $(\sigma_{31}^{\infty}, \sigma_{32}^{\infty})$ and in-plane electric fields $(E_1^{\infty}, E_2^{\infty})$. The poling directions of the two phases are along the x_3-axis. The inhomogeneity-matrix interface L is composed of two parts: the arc L_b is perfectly bonded whilst the remaining arc L_c is occupied by a debonded arc-shaped conducting rigid line inclusion. The mid-point of the arc L_b lies on the positive x_1-axis and the central angle subtended by L_b is $2\theta_0$. We denote by $a = R\,\mathrm{e}^{\mathrm{i}\theta_0}$ and $\bar{a} = R\,\mathrm{e}^{-\mathrm{i}\theta_0}$ the positions of the two tips of the debonded conducting rigid line inclusion. Throughout the paper, the subscripts 1 and 2 (or the superscripts (1) and (2)) will be used to identify the respective quantities in the inhomogeneity and the matrix.

We introduce two sectionally holomorphic vector functions $\mathbf{h}_1(z)$ and $\mathbf{h}_2(z)$ defined by

$$\mathbf{h}_1(z) = \begin{cases} \mathbf{f}_1(z) + \bar{\mathbf{f}}_2(\frac{R^2}{z}) - \mathbf{k}z - \bar{\mathbf{k}}R^2 z^{-1}, & |z| < R \\ \mathbf{f}_2(z) + \bar{\mathbf{f}}_1(\frac{R^2}{z}) - \mathbf{k}z - \bar{\mathbf{k}}R^2 z^{-1}, & |z| > R \end{cases} \tag{3}$$

$$\mathbf{h}_2(z) = \begin{cases} \mathbf{f}_1(z) - \mathbf{C}_1^{-1}\mathbf{C}_2\bar{\mathbf{f}}_2(\frac{R^2}{z}) - \mathbf{C}_1^{-1}\mathbf{C}_2\mathbf{k}z + \mathbf{C}_1^{-1}\mathbf{C}_2\bar{\mathbf{k}}R^2 z^{-1}, & |z| < R \\ \mathbf{C}_1^{-1}\mathbf{C}_2\mathbf{f}_2(z) - \bar{\mathbf{f}}_1(\frac{R^2}{z}) - \mathbf{C}_1^{-1}\mathbf{C}_2\mathbf{k}z + \mathbf{C}_1^{-1}\mathbf{C}_2\bar{\mathbf{k}}R^2 z^{-1}, & |z| > R \end{cases} \tag{4}$$

where $\mathbf{k}$ is related to the remote electromechanical loading through

$$\mathbf{k} = \begin{bmatrix} \frac{\sigma_{32}^{\infty} + \mathrm{i}\sigma_{31}^{\infty} + e_{15}^{(2)}(E_2^{\infty} + \mathrm{i}E_1^{\infty})}{C_{44}^{(2)}} \\ -(E_2^{\infty} + \mathrm{i}E_1^{\infty}) \end{bmatrix} \tag{5}$$

It is seen from the above definitions that $\mathbf{h}_1(z)$ and $\mathbf{h}_2(z)$ are continuous across the arc L_b and are analytic in $|z| < R$ and $|z| > R$, respectively, including the point at infinity; they are discontinuous only across the arc L_c. It then follows from

Eqs. (3) and (4) that the original analytic vector functions and their analytical continuations $\mathbf{f}_1(z), \mathbf{f}_2(z), \bar{\mathbf{f}}_1(\frac{R^2}{z}), \bar{\mathbf{f}}_2(\frac{R^2}{z})$ can be expressed in terms of the newly introduced $\mathbf{h}_1(z)$ and $\mathbf{h}_2(z)$ as follows:

$$\begin{aligned}
&\mathbf{f}_1(z) = (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_2\mathbf{h}_1(z) + (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_1\mathbf{h}_2(z) + 2(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_2\mathbf{k}z \\
&\bar{\mathbf{f}}_2\left(\frac{R^2}{z}\right) = (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_1\mathbf{h}_1(z) - (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_1\mathbf{h}_2(z) + (\mathbf{C}_1+\mathbf{C}_2)^{-1}(\mathbf{C}_1-\mathbf{C}_2)\mathbf{k}z + \bar{\mathbf{k}}R^2z^{-1}, \quad |z|<R
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\bar{\mathbf{f}}_1\left(\frac{R^2}{z}\right) = (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_2\mathbf{h}_1(z) - (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_1\mathbf{h}_2(z) + 2(\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_2\bar{\mathbf{k}}R^2z^{-1} \\
&\mathbf{f}_2(z) = (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_1\mathbf{h}_1(z) + (\mathbf{C}_1+\mathbf{C}_2)^{-1}\mathbf{C}_1\mathbf{h}_2(z) + \mathbf{k}z + (\mathbf{C}_1+\mathbf{C}_2)^{-1}(\mathbf{C}_1-\mathbf{C}_2)\bar{\mathbf{k}}R^2z^{-1}, \quad |z|>R
\end{aligned} \tag{7}$$

In the following two sections, we will discuss in detail two configurations: (i) the inner side of the conducting rigid line inclusion is perfectly bonded to the piezoelectric inhomogeneity, whilst its outer side is fully debonded from the piezoelectric matrix; (ii) the inner side of the conducting rigid line inclusion is fully debonded from the piezoelectric inhomogeneity, whilst its outer side is perfectly bonded to the piezoelectric matrix.

3. A rigid line inclusion debonded from the matrix

In the first configuration, the boundary conditions on the conducting rigid line inclusion (which is debonded from the matrix) can be expressed in terms of the functions $\mathbf{f}_1(z), \mathbf{f}_2(z), \bar{\mathbf{f}}_1(\frac{R^2}{z}), \bar{\mathbf{f}}_2(\frac{R^2}{z})$ as follows

$$\mathbf{f}_1^+(z) - \bar{\mathbf{f}}_1^-\left(\frac{R^2}{z}\right) = \mathbf{0}, \qquad \mathbf{f}_2^-(z) + \bar{\mathbf{f}}_2^+\left(\frac{R^2}{z}\right) = \mathbf{0}, \quad z\in L_c \tag{8}$$

Substitution of Eqs. (6) and (7) into the above yields the following vector Riemann–Hilbert problem:

$$\begin{bmatrix} \mathbf{C}_2 & \mathbf{C}_1 \\ \mathbf{C}_1 & -\mathbf{C}_1 \end{bmatrix}\begin{bmatrix} \mathbf{h}_1^+(z) \\ \mathbf{h}_2^+(z) \end{bmatrix} + \begin{bmatrix} -\mathbf{C}_2 & \mathbf{C}_1 \\ \mathbf{C}_1 & \mathbf{C}_1 \end{bmatrix}\begin{bmatrix} \mathbf{h}_1^-(z) \\ \mathbf{h}_2^-(z) \end{bmatrix} = -2\begin{bmatrix} \mathbf{C}_2 \\ \mathbf{C}_1 \end{bmatrix}\mathbf{k}z + 2\begin{bmatrix} \mathbf{C}_2 \\ -\mathbf{C}_1 \end{bmatrix}\bar{\mathbf{k}}R^2z^{-1}, \quad z\in L_c \tag{9}$$

In order to solve this problem, we first consider the following generalized eigenvalue problem:

$$\mathbf{C}_2\mathbf{w} = \rho\mathbf{C}_1\mathbf{w} \tag{10}$$

where ρ is the eigenvalue and $\mathbf{w}$ the associated eigenvector. The two eigenvalues and the associated eigenvectors of Eq. (10) are given explicitly as

$$\rho_{1,2} = \frac{C_{44}^{(1)}\epsilon_{11}^{(2)} + C_{44}^{(2)}\epsilon_{11}^{(1)} + 2e_{15}^{(1)}e_{15}^{(2)} \pm \sqrt{(C_{44}^{(1)}\epsilon_{11}^{(2)} - C_{44}^{(2)}\epsilon_{11}^{(1)})^2 + 4(C_{44}^{(1)}e_{15}^{(2)} - C_{44}^{(2)}e_{15}^{(1)})(\epsilon_{11}^{(2)}e_{15}^{(1)} - \epsilon_{11}^{(1)}e_{15}^{(2)})}}{2[C_{44}^{(1)}\epsilon_{11}^{(1)} + (e_{15}^{(1)})^2]} \tag{11}$$

$$\mathbf{w}_1 = \begin{bmatrix} \rho_1 e_{15}^{(1)} - e_{15}^{(2)} \\ C_{44}^{(2)} - \rho_1 C_{44}^{(1)} \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} \rho_2 e_{15}^{(1)} - e_{15}^{(2)} \\ C_{44}^{(2)} - \rho_2 C_{44}^{(1)} \end{bmatrix} \tag{12}$$

The two eigenvalues in Eq. (11) can be positive real or complex conjugates with positive real parts. When the two eigenvalues are distinct, we have the following orthogonality relations between the two eigenvectors

$$\begin{bmatrix} \mathbf{w}_1^{\mathrm{T}} \\ \mathbf{w}_2^{\mathrm{T}} \end{bmatrix}\mathbf{C}_1\begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{w}_1^{\mathrm{T}} \\ \mathbf{w}_2^{\mathrm{T}} \end{bmatrix}\mathbf{C}_2\begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \delta_1\rho_1 & 0 \\ 0 & \delta_2\rho_2 \end{bmatrix} \tag{13}$$

where δ_1 and δ_2 are non-zero coefficients.

By introducing the following transform

$$\mathbf{h}_1(z) = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix}\mathbf{p}(z), \qquad \mathbf{h}_2(z) = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix}\mathbf{q}(z) \tag{14}$$

and making use of the orthogonality relations in Eq. (13), Eq. (9) can be expressed in terms of $\mathbf{p}(z) = \begin{bmatrix} p_1(z) & p_2(z) \end{bmatrix}^T$ and $\mathbf{q}(z) = \begin{bmatrix} q_1(z) & q_2(z) \end{bmatrix}^T$ as follows

$$\begin{aligned}
&\begin{bmatrix} \rho_1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1^+(z) \\ q_1^+(z) \end{bmatrix} + \begin{bmatrix} -\rho_1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p_1^-(z) \\ q_1^-(z) \end{bmatrix} = -2z\begin{bmatrix} Y_1(\rho_1,\delta_1,k_1,k_2) \\ Y_2(\rho_1,\delta_1,k_1,k_2) \end{bmatrix} + 2R^2z^{-1}\begin{bmatrix} Y_1(\rho_1,\delta_1,\bar{k}_1,\bar{k}_2) \\ -Y_2(\rho_1,\delta_1,\bar{k}_1,\bar{k}_2) \end{bmatrix} \\
&\begin{bmatrix} \rho_2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_2^+(z) \\ q_2^+(z) \end{bmatrix} + \begin{bmatrix} -\rho_2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p_2^-(z) \\ q_2^-(z) \end{bmatrix} = -2z\begin{bmatrix} Y_1(\rho_2,\delta_2,k_1,k_2) \\ Y_2(\rho_2,\delta_2,k_1,k_2) \end{bmatrix} + 2R^2z^{-1}\begin{bmatrix} Y_1(\rho_2,\delta_2,\bar{k}_1,\bar{k}_2) \\ -Y_2(\rho_2,\delta_2,\bar{k}_1,\bar{k}_2) \end{bmatrix} \\
&z\in L_c
\end{aligned} \tag{15}$$

where

$$\begin{aligned} Y_1(\rho,\delta,k_1,k_2) &= \frac{\rho}{\delta}\left[k_1\left(C_{44}^{(2)}e_{15}^{(1)} - C_{44}^{(1)}e_{15}^{(2)}\right) + k_2\left(C_{44}^{(1)}\epsilon_{11}^{(2)} + e_{15}^{(1)}e_{15}^{(2)}\right)\right] - \frac{k_2}{\delta}\left[C_{44}^{(2)}\epsilon_{11}^{(2)} + \left(e_{15}^{(2)}\right)^2\right] \\ Y_2(\rho,\delta,k_1,k_2) &= \frac{\rho k_2}{\delta}\left[C_{44}^{(1)}\epsilon_{11}^{(1)} + \left(e_{15}^{(1)}\right)^2\right] + \frac{k_1}{\delta}\left(C_{44}^{(2)}e_{15}^{(1)} - C_{44}^{(1)}e_{15}^{(2)}\right) - \frac{k_2}{\delta}\left(C_{44}^{(2)}\epsilon_{11}^{(1)} + e_{15}^{(1)}e_{15}^{(2)}\right) \end{aligned} \tag{16}$$

It is seen from Eq. (15) that the vector R–H problem for $\left[\, p_1(z) \;\; q_1(z) \,\right]^T$ is completely decoupled from that for $\left[\, p_2(z) \;\; q_2(z) \,\right]^{\mathrm{T}}$. We next consider the following generalized eigenvalue problem:

$$\begin{bmatrix} -\rho & 1 \\ 1 & 1 \end{bmatrix}\mathbf{v} = \lambda \begin{bmatrix} \rho & 1 \\ 1 & -1 \end{bmatrix}\mathbf{v} \tag{17}$$

where λ is the eigenvalue and $\mathbf{v}$ the associated eigenvector. The two eigenvalues and the associated eigenvectors of Eq. (17) are determined as

$$\lambda_1 = \frac{1+\mathrm{i}\sqrt{\rho}}{1-\mathrm{i}\sqrt{\rho}}, \qquad \lambda_2 = \frac{1-\mathrm{i}\sqrt{\rho}}{1+\mathrm{i}\sqrt{\rho}} \tag{18}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \mathrm{i}\sqrt{\rho} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\mathrm{i}\sqrt{\rho} \end{bmatrix} \tag{19}$$

In addition, we have the following orthogonality relations between the two eigenvectors

$$\begin{aligned} \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \rho & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} &= \begin{bmatrix} 2(\rho+\mathrm{i}\sqrt{\rho}) & 0 \\ 0 & 2(\rho-\mathrm{i}\sqrt{\rho}) \end{bmatrix} \\ \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} -\rho & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} &= \begin{bmatrix} 2(-\rho+\mathrm{i}\sqrt{\rho}) & 0 \\ 0 & 2(-\rho-\mathrm{i}\sqrt{\rho}) \end{bmatrix} \end{aligned} \tag{20}$$

Upon introduction of the following transform,

$$\begin{bmatrix} p_1(z) \\ q_1(z) \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^{(1)} & \mathbf{v}_2^{(1)} \end{bmatrix}\begin{bmatrix} \xi_1(z) \\ \eta_1(z) \end{bmatrix}, \qquad \begin{bmatrix} p_2(z) \\ q_2(z) \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^{(2)} & \mathbf{v}_2^{(2)} \end{bmatrix}\begin{bmatrix} \xi_2(z) \\ \eta_2(z) \end{bmatrix} \tag{21}$$

(with superscripts (1) and (2) indicating the quantities associated with ρ_1 and ρ_2, respectively) and application of Eq. (20), Eq. (15) can be decoupled into the following four independent scalar R–H problems:

$$\begin{aligned} \xi_j^+(z) + \lambda_1^{(j)}\xi_j^-(z) &= -z\frac{Y_1(\rho_j,\delta_j,k_1,k_2) + \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,k_1,k_2)}{\rho_j + \mathrm{i}\sqrt{\rho_j}} \\ &\quad + z^{-1}\frac{R^2[Y_1(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2) - \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2)]}{\rho_j + \mathrm{i}\sqrt{\rho_j}} \\ \eta_j^+(z) + \lambda_2^{(j)}\eta_j^-(z) &= -z\frac{Y_1(\rho_j,\delta_j,k_1,k_2) - \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,k_1,k_2)}{\rho_j - \mathrm{i}\sqrt{\rho_j}} \\ &\quad + z^{-1}\frac{R^2[Y_1(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2) + \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2)]}{\rho_j - \mathrm{i}\sqrt{\rho_j}}, \quad z \in L_c,\ j = 1,2 \end{aligned} \tag{22}$$

By evaluating the corresponding Cauchy integrals, the solutions to the above are found to be

$$\begin{aligned} \xi_j'(z) &= -\frac{Y_1(\rho_j,\delta_j,k_1,k_2) + \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,k_1,k_2)}{(1+\lambda_1^{(j)})(\rho_j + \mathrm{i}\sqrt{\rho_j})}\left\{1 - \chi_1^{(j)}(z)\left[z - \mathrm{Re}\{a\} + 2\varepsilon_1^{(j)}\,\mathrm{Im}\{a\}\right]\right\} \\ &\quad - \frac{R^2[Y_1(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2) - \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2)]}{(1+\lambda_1^{(j)})(\rho_j + \mathrm{i}\sqrt{\rho_j})}\left[\frac{1}{z^2} - \frac{\chi_1^{(j)}(z)}{\chi_1^{(j)}(0)z^2} + \frac{\chi_1^{(j)}(z)\chi_1'^{(j)}(0)}{[\chi_1^{(j)}(0)]^2 z}\right] \\ \eta_j'(z) &= -\frac{Y_1(\rho_j,\delta_j,k_1,k_2) - \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,k_1,k_2)}{(1+\lambda_2^{(j)})(\rho_j - \mathrm{i}\sqrt{\rho_j})}\left\{1 - \chi_2^{(j)}(z)\left[z - \mathrm{Re}\{a\} + 2\varepsilon_2^{(j)}\,\mathrm{Im}\{a\}\right]\right\} \\ &\quad - \frac{R^2[Y_1(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2) + \mathrm{i}\sqrt{\rho_j}Y_2(\rho_j,\delta_j,\bar{k}_1,\bar{k}_2)]}{(1+\lambda_2^{(j)})(\rho_j - \mathrm{i}\sqrt{\rho_j})}\left[\frac{1}{z^2} - \frac{\chi_2^{(j)}(z)}{\chi_2^{(j)}(0)z^2} + \frac{\chi_2^{(j)}(z)\chi_2'^{(j)}(0)}{[\chi_2^{(j)}(0)]^2 z}\right], \quad j = 1,2 \end{aligned} \tag{23}$$

where

$$\chi_k^{(j)}(z)=(z-a)^{-\frac{1}{2}-\mathrm{i}\varepsilon_k^{(j)}}(z-\bar{a})^{-\frac{1}{2}+\mathrm{i}\varepsilon_k^{(j)}},\qquad \varepsilon_k^{(j)}=-\frac{\ln\lambda_k^{(j)}}{2\pi},\quad j,\ k=1,2 \tag{24}$$

The branch cuts for $\chi_k^{(j)}(z)$ are taken along $L_c(z)$ such that $\chi_k^{(j)}(z)\cong z^{-1}$ as $|z|\to\infty$. Once $\xi_j'(z)$ and $\eta_j'(z)$ are known, $p_j'(z)$ and $q_j'(z)$ can be determined from Eq. (21). Consequently, $\mathbf{h}_1'(z)$ and $\mathbf{h}_2'(z)$ are arrived at from Eq. (14), and the original analytic vector functions $\mathbf{f}_1'(z)$ defined in the inhomogeneity and $\mathbf{f}_2'(z)$ defined in the matrix are ultimately determined from Eqs. $(6)_1$ and $(7)_2$.

When $\rho_1=\rho_2$ in Eq. (11) and $\mathbf{C}_1$ is not proportional to $\mathbf{C}_2$ (this situation corresponds to the degenerate case), it is more convenient to consider the following transform

$$\begin{bmatrix}\mathbf{h}_1(z)\\ \mathbf{h}_2(z)\end{bmatrix}=\begin{bmatrix}\mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4\end{bmatrix}\mathbf{g}(z) \tag{25}$$

where $\mathbf{g}(z)=\begin{bmatrix}g_1(z) & g_2(z) & g_3(z) & g_4(z)\end{bmatrix}^T$ is a 4D analytic vector function, and the generalized eigenvectors $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{y}_4$ are determined by

$$\begin{aligned}&\begin{bmatrix}-\mathbf{C}_2 & \mathbf{C}_1\\ \mathbf{C}_1 & \mathbf{C}_1\end{bmatrix}\mathbf{y}_j=\lambda_j\begin{bmatrix}\mathbf{C}_2 & \mathbf{C}_1\\ \mathbf{C}_1 & -\mathbf{C}_1\end{bmatrix}\mathbf{y}_j\\ &\begin{bmatrix}-\mathbf{C}_2 & \mathbf{C}_1\\ \mathbf{C}_1 & \mathbf{C}_1\end{bmatrix}\mathbf{y}_{j+1}=\lambda_j\begin{bmatrix}\mathbf{C}_2 & \mathbf{C}_1\\ \mathbf{C}_1 & -\mathbf{C}_1\end{bmatrix}\mathbf{y}_{j+1}+\begin{bmatrix}\mathbf{C}_2 & \mathbf{C}_1\\ \mathbf{C}_1 & -\mathbf{C}_1\end{bmatrix}\mathbf{y}_j,\quad j=1,3\end{aligned} \tag{26}$$

where

$$\lambda_1=\frac{1+\mathrm{i}\sqrt{\rho_1}}{1-\mathrm{i}\sqrt{\rho_1}},\qquad \lambda_3=\frac{1-\mathrm{i}\sqrt{\rho_1}}{1+\mathrm{i}\sqrt{\rho_1}} \tag{27}$$

The following quasi-orthogonal relationships among the four generalized eigenvectors are shown to be true:

$$\begin{aligned}&\begin{bmatrix}\mathbf{y}_2^T\\ \mathbf{y}_1^T\\ \mathbf{y}_4^T\\ \mathbf{y}_3^T\end{bmatrix}\begin{bmatrix}\mathbf{C}_2 & \mathbf{C}_1\\ \mathbf{C}_1 & -\mathbf{C}_1\end{bmatrix}\begin{bmatrix}\mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4\end{bmatrix}=\begin{bmatrix}d_1 & d_2 & 0 & 0\\ 0 & d_1 & 0 & 0\\ 0 & 0 & d_3 & d_4\\ 0 & 0 & 0 & d_3\end{bmatrix}\\ &\begin{bmatrix}\mathbf{y}_2^T\\ \mathbf{y}_1^T\\ \mathbf{y}_4^T\\ \mathbf{y}_3^T\end{bmatrix}\begin{bmatrix}-\mathbf{C}_2 & \mathbf{C}_1\\ \mathbf{C}_1 & \mathbf{C}_1\end{bmatrix}\begin{bmatrix}\mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4\end{bmatrix}=\begin{bmatrix}d_1 & d_2 & 0 & 0\\ 0 & d_1 & 0 & 0\\ 0 & 0 & d_3 & d_4\\ 0 & 0 & 0 & d_3\end{bmatrix}\begin{bmatrix}\lambda_1 & 1 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_3 & 1\\ 0 & 0 & 0 & \lambda_3\end{bmatrix}\end{aligned} \tag{28}$$

where d_1,d_2,d_3,d_4 are non-zero coefficients. It is deduced from Eqs. (26)–(28) that

$$\mathbf{y}_3=\bar{\mathbf{y}}_1,\qquad \mathbf{y}_4=\bar{\mathbf{y}}_2,\qquad \lambda_3=\bar{\lambda}_1,\qquad d_3=\bar{d}_1,\qquad d_4=\bar{d}_2 \tag{29}$$

By considering Eqs. (25) and (28), Eq. (9) can be rewritten in the following form

$$\begin{aligned}&\begin{bmatrix}g_1^+(z)\\ g_2^+(z)\end{bmatrix}+\begin{bmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{bmatrix}\begin{bmatrix}g_1^-(z)\\ g_2^-(z)\end{bmatrix}=-2\begin{bmatrix}\frac{1}{d_1}\mathbf{y}_2^T-\frac{d_2}{d_1^2}\mathbf{y}_1^{\mathrm{T}}\\ \frac{1}{d_1}\mathbf{y}_1^{\mathrm{T}}\end{bmatrix}\left\{\begin{bmatrix}\mathbf{C}_2\\ \mathbf{C}_1\end{bmatrix}\mathbf{k}z-\begin{bmatrix}\mathbf{C}_2\\ -\mathbf{C}_1\end{bmatrix}\bar{\mathbf{k}}R^2z^{-1}\right\}\\ &\begin{bmatrix}g_3^+(z)\\ g_4^+(z)\end{bmatrix}+\begin{bmatrix}\lambda_3 & 1\\ 0 & \lambda_3\end{bmatrix}\begin{bmatrix}g_3^-(z)\\ g_4^-(z)\end{bmatrix}=-2\begin{bmatrix}\frac{1}{d_3}\mathbf{y}_4^T-\frac{d_4}{d_3^2}\mathbf{y}_3^{\mathrm{T}}\\ \frac{1}{d_3}\mathbf{y}_3^{\mathrm{T}}\end{bmatrix}\left\{\begin{bmatrix}\mathbf{C}_2\\ \mathbf{C}_1\end{bmatrix}\mathbf{k}z-\begin{bmatrix}\mathbf{C}_2\\ -\mathbf{C}_1\end{bmatrix}\bar{\mathbf{k}}R^2z^{-1}\right\},\quad z\in L_c\end{aligned} \tag{30}$$

The above can be solved by means of the following quasi-decoupling method: $g_2'(z)$ and $g_4'(z)$ can first be determined by solving two independent scalar R–H problems, $g_1'(z)$ and $g_3'(z)$ can then be determined by inserting the expressions for $g_2'(z)$ and $g_4'(z)$, and solving the resulting two independent scalar R–H problems. The final results are then

$$\begin{aligned}g_j'(z)=&\frac{2}{d_j(1+\lambda_j)}\left[\left(\frac{d_{j+1}}{d_j}+\frac{1}{1+\lambda_j}\right)\mathbf{y}_j^T-\mathbf{y}_{j+1}^T\right]\left\{\begin{aligned}&\begin{bmatrix}\mathbf{C}_2\\ \mathbf{C}_1\end{bmatrix}\mathbf{k}\{1-\chi_j(z)[z-\mathrm{Re}\{a\}+2\varepsilon_j\,\mathrm{Im}\{a\}]\}\\ &+\begin{bmatrix}\mathbf{C}_2\\ -\mathbf{C}_1\end{bmatrix}\bar{\mathbf{k}}R^2\left[\frac{1}{z^2}-\frac{\chi_j(z)}{\chi_j(0)z^2}+\frac{\chi_j(z)\chi_j'(0)}{[\chi_j(0)]^2z}\right]\end{aligned}\right\}\\ &+\frac{\chi_j(z)}{\mathrm{i}\pi d_j\lambda_j(1+\lambda_j)}\int\limits_{L_c}\frac{1}{t-z}\mathbf{y}_j^T\left\{\begin{bmatrix}\mathbf{C}_2\\ \mathbf{C}_1\end{bmatrix}\mathbf{k}[t-\mathrm{Re}\{a\}+2\varepsilon_j\,\mathrm{Im}\{a\}]+\begin{bmatrix}\mathbf{C}_2\\ -\mathbf{C}_1\end{bmatrix}\bar{\mathbf{k}}R^2\left[\frac{1}{\chi_j(0)t^2}-\frac{\chi_j'(0)}{[\chi_j(0)]^2t}\right]\right\}\mathrm{d}t\end{aligned}$$

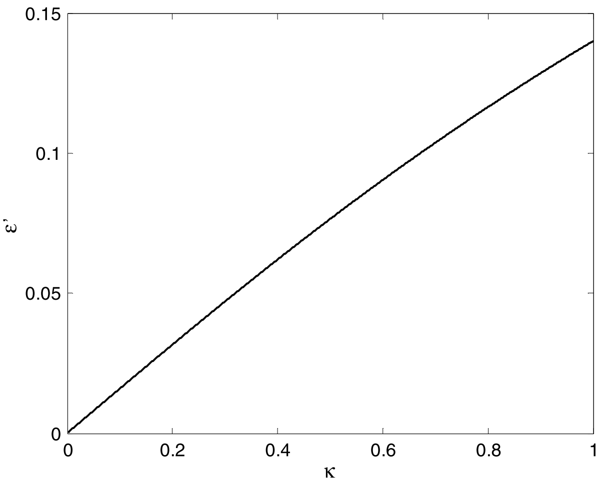

Fig. 2. The oscillatory index ε' as a function of the coupling factor κ.

$$g'_{j+1}(z) = -\frac{2}{d_j(1+\lambda_j)}\mathbf{y}_j^T \left\{ \begin{array}{l} \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{C}_1 \end{bmatrix} \mathbf{k}\big\{1-\chi_j(z)\big[z-\mathrm{Re}\{a\}+2\varepsilon_j \,\mathrm{Im}\{a\}\big]\big\} \\ +\begin{bmatrix} \mathbf{C}_2 \\ -\mathbf{C}_1 \end{bmatrix} \bar{\mathbf{k}} R^2 \left[\frac{1}{z^2} - \frac{\chi_j(z)}{\chi_j(0)z^2} + \frac{\chi_j(z)\chi'_j(0)}{[\chi_j(0)]^2 z}\right] \end{array} \right\}, \quad j=1,3 \tag{31}$$

where the Cauchy integral is taken in the counterclockwise direction from the upper tip to the lower tip of the rigid line inclusion, and

$$\chi_j(z) = (z-a)^{-\frac{1}{2}-\mathrm{i}\varepsilon_j}(z-\bar{a})^{-\frac{1}{2}+\mathrm{i}\varepsilon_j}, \qquad \varepsilon_j = -\frac{\ln\lambda_j}{2\pi}, \quad j=1,3 \tag{32}$$

It is seen from the analysis in this section that:

(i) when $(C_{44}^{(1)}\epsilon_{11}^{(2)} - C_{44}^{(2)}\epsilon_{11}^{(1)})^2 + 4(C_{44}^{(1)}e_{15}^{(2)} - C_{44}^{(2)}e_{15}^{(1)})(\epsilon_{11}^{(2)}e_{15}^{(1)} - \epsilon_{11}^{(1)}e_{15}^{(2)}) > 0$, the stresses and electric displacements at the tips of the line inclusion exhibit the power type singularities $r^{-\frac{1}{2}\pm\gamma_1}$ and $r^{-\frac{1}{2}\pm\gamma_2}$ with $\gamma_j = \frac{1}{\pi}\arctan(\sqrt{\rho_j})$, $j=1,2$;

(ii) when $(C_{44}^{(1)}\epsilon_{11}^{(2)} - C_{44}^{(2)}\epsilon_{11}^{(1)})^2 + 4(C_{44}^{(1)}e_{15}^{(2)} - C_{44}^{(2)}e_{15}^{(1)})(\epsilon_{11}^{(2)}e_{15}^{(1)} - \epsilon_{11}^{(1)}e_{15}^{(2)}) < 0$, the stresses and electric displacements at the tips of the line inclusion exhibit the singularities $r^{-\frac{1}{2}-\varepsilon''\pm\mathrm{i}\varepsilon'}$ and $r^{-\frac{1}{2}+\varepsilon''\pm\mathrm{i}\varepsilon'}$ with ε' and ε'' being the real and imaginary parts of $\varepsilon_1^{(1)}(=\bar{\varepsilon}_2^{(2)} = -\varepsilon_2^{(1)} = -\bar{\varepsilon}_1^{(2)})$;

(iii) when $(C_{44}^{(1)}\epsilon_{11}^{(2)} - C_{44}^{(2)}\epsilon_{11}^{(1)})^2 + 4(C_{44}^{(1)}e_{15}^{(2)} - C_{44}^{(2)}e_{15}^{(1)})(\epsilon_{11}^{(2)}e_{15}^{(1)} - \epsilon_{11}^{(1)}e_{15}^{(2)}) = 0$ [or equivalently $2e_{15}^{(2)} = e_{15}^{(1)}(\frac{C_{44}^{(2)}}{C_{44}^{(1)}} + \frac{\epsilon_{11}^{(2)}}{\epsilon_{11}^{(1)}}) \pm \sqrt{C_{44}^{(1)}\epsilon_{11}^{(1)} + (e_{15}^{(1)})^2(\frac{C_{44}^{(2)}}{C_{44}^{(1)}} - \frac{\epsilon_{11}^{(2)}}{\epsilon_{11}^{(1)}})}$] and $\mathbf{C}_1$ is not proportional to $\mathbf{C}_2$, the stresses and electric displacements at the tips of the line inclusion exhibit the singularities $r^{-\frac{1}{2}\pm\gamma_1}$ and $r^{-\frac{1}{2}\pm\gamma_1}\ln r$ with $\gamma_1 = \frac{1}{\pi}\arctan(\sqrt{\rho_1})$. The additional logarithmic term in the singularities is due to the remaining Cauchy integral in Eq. (31).

For example, if the inhomogeneity and the matrix possess identical material properties, but have opposite poling directions (i.e. $C_{44}^{(1)} = C_{44}^{(2)}, e_{15}^{(1)} = -e_{15}^{(2)}, \epsilon_{11}^{(1)} = \epsilon_{11}^{(2)}$), Eq. (11) becomes

$$\rho_1 = \frac{1+\mathrm{i}\kappa}{1-\mathrm{i}\kappa}, \qquad \rho_2 = \frac{1-\mathrm{i}\kappa}{1+\mathrm{i}\kappa} \tag{33}$$

where κ is the electromechanical coupling factor defined by

$$\kappa = \frac{|e_{15}^{(1)}|}{\sqrt{C_{44}^{(1)}\epsilon_{11}^{(1)}}} \tag{34}$$

In this case, the stresses and electric displacements exhibit the singularities $r^{-\frac{1}{4}\pm\mathrm{i}\varepsilon'}$ and $r^{-\frac{3}{4}\pm\mathrm{i}\varepsilon'}$ where the oscillatory index ε' as a function of κ is illustrated in Fig. 2. When $\kappa=1$, the oscillatory index attains its maximum value of $\max\{\varepsilon'\} = \frac{\ln(\sqrt{2}+1)}{2\pi} = 0.1403$.

4. A rigid line inclusion debonded from the inhomogeneity

In the second configuration (see Section 2), the boundary conditions on the conducting rigid line inclusion (which is debonded from the inhomogeneity) can be expressed in terms of $\mathbf{f}_1(z), \mathbf{f}_2(z), \bar{\mathbf{f}}_1(\frac{R^2}{z}), \bar{\mathbf{f}}_2(\frac{R^2}{z})$ as follows:

$$\mathbf{f}_1^+(z) + \bar{\mathbf{f}}_1^-\left(\frac{R^2}{z}\right) = \mathbf{0}, \qquad \mathbf{f}_2^-(z) - \bar{\mathbf{f}}_2^+\left(\frac{R^2}{z}\right) = \mathbf{0}, \quad z \in L_c \tag{35}$$

Substitution of Eqs. (6) and (7) into the above yields the following vector Riemann–Hilbert problem:

$$\begin{bmatrix} \mathbf{C}_2 & \mathbf{C}_1 \\ \mathbf{C}_1 & -\mathbf{C}_1 \end{bmatrix} \begin{bmatrix} \mathbf{h}_1^+(z) \\ \mathbf{h}_2^+(z) \end{bmatrix} - \begin{bmatrix} -\mathbf{C}_2 & \mathbf{C}_1 \\ \mathbf{C}_1 & \mathbf{C}_1 \end{bmatrix} \begin{bmatrix} \mathbf{h}_1^-(z) \\ \mathbf{h}_2^-(z) \end{bmatrix} = 2\begin{bmatrix} -\mathbf{C}_2 \\ \mathbf{C}_2 \end{bmatrix} \mathbf{k}z - 2\begin{bmatrix} \mathbf{C}_2 \\ \mathbf{C}_2 \end{bmatrix} \bar{\mathbf{k}} R^2 z^{-1}, \quad z \in L_c \tag{36}$$

If we introduce the transform in Eq. (14) for the case of distinct eigenvalues in Eq. (11), the above can be rewritten as:

$$\begin{aligned} &\begin{bmatrix} \rho_1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1^+(z) \\ q_1^+(z) \end{bmatrix} - \begin{bmatrix} -\rho_1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1^-(z) \\ q_1^-(z) \end{bmatrix} = 2zY_1(\rho_1, \delta_1, k_1, k_2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 2R^2 z^{-1} Y_1(\rho_1, \delta_1, \bar{k}_1, \bar{k}_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &\begin{bmatrix} \rho_2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_2^+(z) \\ q_2^+(z) \end{bmatrix} - \begin{bmatrix} -\rho_2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_2^-(z) \\ q_2^-(z) \end{bmatrix} = 2zY_1(\rho_2, \delta_2, k_1, k_2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 2R^2 z^{-1} Y_1(\rho_2, \delta_2, \bar{k}_1, \bar{k}_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &z \in L_c \end{aligned} \tag{37}$$

where $Y_1(\rho, \delta, k_1, k_2)$ has been defined in Eq. (16)$_1$. We can see that the vector R–H problem for $\begin{bmatrix} p_1(z) & q_1(z) \end{bmatrix}^{\mathrm{T}}$ is decoupled from that for $\begin{bmatrix} p_2(z) & q_2(z) \end{bmatrix}^{\mathrm{T}}$.

After the introduction of the transform in Eq. (21) and utilization of Eq. (20), Eq. (37) can be decoupled into

$$\begin{aligned} &\xi_j^+(z) - \lambda_1^{(j)} \xi_j^-(z) = z\frac{\mathrm{i}Y_1(\rho_j, \delta_j, k_1, k_2)}{\sqrt{\rho_j}} + z^{-1}\frac{\mathrm{i}R^2\lambda_1^{(j)} Y_1(\rho_j, \delta_j, \bar{k}_1, \bar{k}_2)}{\sqrt{\rho_j}} \\ &\eta_j^+(z) - \lambda_2^{(j)} \eta_j^-(z) = -z\frac{\mathrm{i}Y_1(\rho_j, \delta_j, k_1, k_2)}{\sqrt{\rho_j}} - z^{-1}\frac{\mathrm{i}R^2\lambda_2^{(j)} Y_1(\rho_j, \delta_j, \bar{k}_1, \bar{k}_2)}{\sqrt{\rho_j}}, \quad z \in L_c, \ j = 1, 2 \end{aligned} \tag{38}$$

The solutions to the above four decoupled scalar R–H problems can be conveniently derived as:

$$\begin{aligned} \xi_j'(z) = {}& \frac{\mathrm{i}Y_1(\rho_j, \delta_j, k_1, k_2)}{\sqrt{\rho_j}(1 - \lambda_1^{(j)})} \left\{1 - \tilde{\chi}_1^{(j)}(z)\left[z - \mathrm{Re}\{a\} + 2\tilde{\varepsilon}_1^{(j)} \mathrm{Im}\{a\}\right]\right\} \\ & - \frac{\mathrm{i}R^2\lambda_1^{(j)} Y_1(\rho_j, \delta_j, \bar{k}_1, \bar{k}_2)}{\sqrt{\rho_j}(1 - \lambda_1^{(j)})} \left[\frac{1}{z^2} - \frac{\tilde{\chi}_1^{(j)}(z)}{\tilde{\chi}_1^{(j)}(0) z^2} + \frac{\tilde{\chi}_1^{(j)}(z)\tilde{\chi}_1'^{(j)}(0)}{[\tilde{\chi}_1^{(j)}(0)]^2 z}\right] \\ \eta_j'(z) = {}& -\frac{\mathrm{i}Y_1(\rho_j, \delta_j, k_1, k_2)}{\sqrt{\rho_j}(1 - \lambda_2^{(j)})} \left\{1 - \tilde{\chi}_2^{(j)}(z)\left[z - \mathrm{Re}\{a\} + 2\tilde{\varepsilon}_2^{(j)} \mathrm{Im}\{a\}\right]\right\} \\ & + \frac{\mathrm{i}R^2\lambda_2^{(j)} Y_1(\rho_j, \delta_j, \bar{k}_1, \bar{k}_2)}{\sqrt{\rho_j}(1 - \lambda_2^{(j)})} \left[\frac{1}{z^2} - \frac{\tilde{\chi}_2^{(j)}(z)}{\tilde{\chi}_2^{(j)}(0) z^2} + \frac{\tilde{\chi}_2^{(j)}(z)\tilde{\chi}_2'^{(j)}(0)}{[\tilde{\chi}_2^{(j)}(0)]^2 z}\right], \quad j = 1, 2 \end{aligned} \tag{39}$$

where

$$\tilde{\chi}_k^{(j)}(z) = (z - a)^{-\frac{1}{2} - \mathrm{i}\tilde{\varepsilon}_k^{(j)}} (z - \bar{a})^{-\frac{1}{2} + \mathrm{i}\tilde{\varepsilon}_k^{(j)}}, \qquad \tilde{\varepsilon}_k^{(j)} = -\frac{\ln(-\lambda_k^{(j)})}{2\pi}, \quad j, k = 1, 2 \tag{40}$$

The branch cuts for $\tilde{\chi}_k^{(j)}(z)$ are taken along $L_c(z)$ such that $\tilde{\chi}_k^{(j)}(z) \cong z^{-1}$ as $|z| \to \infty$.

When $\rho_1 = \rho_2$ in Eq. (11) and $\mathbf{C}_1$ is not proportional to $\mathbf{C}_2$, we can introduce the transform in Eq. (25). By applying Eq. (28), Eq. (36) becomes

$$\begin{aligned} &\begin{bmatrix} g_1^+(z) \\ g_2^+(z) \end{bmatrix} - \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} g_1^-(z) \\ g_2^-(z) \end{bmatrix} = -2\begin{bmatrix} \frac{1}{d_1}\mathbf{y}_2^T - \frac{d_2}{d_1^2}\mathbf{y}_1^{\mathrm{T}} \\ \frac{1}{d_1}\mathbf{y}_1^{\mathrm{T}} \end{bmatrix} \left\{\begin{bmatrix} \mathbf{C}_2 \\ -\mathbf{C}_2 \end{bmatrix} \mathbf{k}z + \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{C}_2 \end{bmatrix} \bar{\mathbf{k}} R^2 z^{-1}\right\} \\ &\begin{bmatrix} g_3^+(z) \\ g_4^+(z) \end{bmatrix} - \begin{bmatrix} \lambda_3 & 1 \\ 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} g_3^-(z) \\ g_4^-(z) \end{bmatrix} = -2\begin{bmatrix} \frac{1}{d_3}\mathbf{y}_4^T - \frac{d_4}{d_3^2}\mathbf{y}_3^{\mathrm{T}} \\ \frac{1}{d_3}\mathbf{y}_3^{\mathrm{T}} \end{bmatrix} \left\{\begin{bmatrix} \mathbf{C}_2 \\ -\mathbf{C}_2 \end{bmatrix} \mathbf{k}z + \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{C}_2 \end{bmatrix} \bar{\mathbf{k}} R^2 z^{-1}\right\}, \quad z \in L_c \end{aligned} \tag{41}$$

The solutions to the above are finally found to be

$$g'_j(z) = \frac{2}{d_j(1-\lambda_j)}\left[\left(\frac{d_{j+1}}{d_j} + \frac{1}{1-\lambda_j}\right)\mathbf{y}_j^{\mathrm{T}} - \mathbf{y}_{j+1}^{\mathrm{T}}\right]\left\{\begin{array}{l}\begin{bmatrix}\mathbf{C}_2\\-\mathbf{C}_2\end{bmatrix}\mathbf{k}\left\{1-\tilde{\chi}_j(z)\left[z-\mathrm{Re}\{a\}+2\tilde{\varepsilon}_j\,\mathrm{Im}\{a\}\right]\right\}\\ -\begin{bmatrix}\mathbf{C}_2\\\mathbf{C}_2\end{bmatrix}\bar{\mathbf{k}}R^2\left[\frac{1}{z^2}-\frac{\tilde{\chi}_j(z)}{\tilde{\chi}_j(0)z^2}+\frac{\tilde{\chi}_j(z)\tilde{\chi}'_j(0)}{[\tilde{\chi}_j(0)]^2 z}\right]\end{array}\right\}$$

$$-\frac{\tilde{\chi}_j(z)}{\mathrm{i}\pi d_j\lambda_j(1-\lambda_j)}\int_{L_c}\frac{1}{t-z}\mathbf{y}_j^{\mathrm{T}}\left\{\begin{bmatrix}\mathbf{C}_2\\-\mathbf{C}_2\end{bmatrix}\mathbf{k}\left[t-\mathrm{Re}\{a\}+2\tilde{\varepsilon}_j\,\mathrm{Im}\{a\}\right]-\begin{bmatrix}\mathbf{C}_2\\\mathbf{C}_2\end{bmatrix}\bar{\mathbf{k}}R^2\left[\frac{1}{\tilde{\chi}_j(0)t^2}-\frac{\tilde{\chi}'_j(0)}{[\tilde{\chi}_j(0)]^2 t}\right]\right\}\mathrm{d}t$$

$$g'_{j+1}(z) = -\frac{2}{d_j(1-\lambda_j)}\mathbf{y}_j^{\mathrm{T}}\left\{\begin{array}{l}\begin{bmatrix}\mathbf{C}_2\\-\mathbf{C}_2\end{bmatrix}\mathbf{k}\left\{1-\tilde{\chi}_j(z)\left[z-\mathrm{Re}\{a\}+2\tilde{\varepsilon}_j\,\mathrm{Im}\{a\}\right]\right\}\\ -\begin{bmatrix}\mathbf{C}_2\\\mathbf{C}_2\end{bmatrix}\bar{\mathbf{k}}R^2\left[\frac{1}{z^2}-\frac{\tilde{\chi}_j(z)}{\tilde{\chi}_j(0)z^2}+\frac{\tilde{\chi}_j(z)\tilde{\chi}'_j(0)}{[\tilde{\chi}_j(0)]^2 z}\right]\end{array}\right\},\quad j=1,3 \tag{42}$$

where

$$\tilde{\chi}_j(z) = (z-a)^{-\frac{1}{2}-\mathrm{i}\tilde{\varepsilon}_j}(z-\bar{a})^{-\frac{1}{2}+\mathrm{i}\tilde{\varepsilon}_j},\qquad \tilde{\varepsilon}_j = -\frac{\ln(-\lambda_j)}{2\pi},\quad j=1,3 \tag{43}$$

The singularities in electric displacements and stresses at the tips of the rigid line inclusion debonded from the inhomogeneity are identical to those discussed in Sec. 3.

5. Conclusions

We have solved the mixed boundary value problem associated with a debonded arc-shaped conducting rigid line inclusion at the interface between a circular piezoelectric inhomogeneity and an infinite piezoelectric matrix. We address two configurations: (i) the inner side of the line inclusion is bonded to the inhomogeneity whilst its outer side is debonded from the matrix; (ii) the inner side of the line inclusion is debonded from the inhomogeneity, whilst its outer side is bonded to the matrix. Our analysis indicates that the nature of the singularities in stresses and electric displacements at the tips of the debonded conducting rigid line inclusion depends on the sign of the term in the square root in Eq. (11). In summary, three types of singularity are possible at the tips.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 11272121) and through a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (Grant No. RGPIN – 2017-03716115112).

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