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Asymptotic behavior of the spectrum of an elliptic problem in a domain with aperiodically distributed concentrated masses ☆

Comportement asymptotique du spectre d'un problème elliptique dans un domaine avec des masses concentrées distribuées de manière aléatoire

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ABSTRACT

In this paper, we consider a spectral problem with singular perturbation of density located near the boundary of the domain, depending on a small parameter. We prove the compactness theorem and study the behavior of eigenelements to the given problem, as the small parameter tends to zero.

RÉSUMÉ

Dans cet article, nous considérons un problème spectral avec une perturbation singulière de la densité située près de la limite du domaine, dépendant d'un petit paramètre. Nous prouvons le théorème de la compacité et étudions le comportement des éléments génériques du problème donné, lorsque le petit paramètre tend vers zéro.

1. Introduction

Problems in domains with concentrated masses have attracted the attention of mathematicians because of their nontrivial behavior (see, for instance [1–7]). In these papers, the authors have studied the asymptotic behavior of eigenvalues and eigenfunctions by means of asymptotic methods and methods of the homogenization theory.

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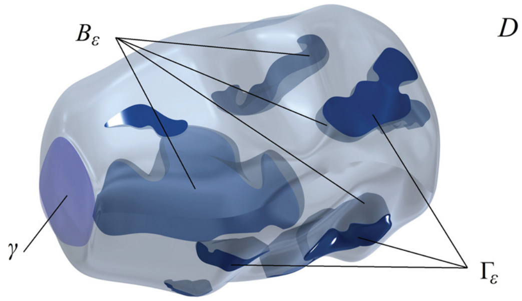

Fig. 1. Domain with nontrivial micro structure near the boundary.

In the present paper, we consider an aperiodic distribution of concentrated masses and give an example of random geometry of the domain and of the masses.

2. Settings and main results

Denote by $D \subset \mathbb{R}^n, n \geq 2$ a domain with sufficiently smooth boundary and inhomogeneous density depending on a small parameter $\varepsilon > 0$. Let us also denote by Γ_ε a part of the boundary ∂D of the domain D, having a cellular structure with ε-scale, and by γ the fixed part of the boundary ∂D, and by ν the outward vector normal to the boundary ∂D. Assume that the density in the domain D has the form

$$\rho_\varepsilon(x) = \begin{cases} \varepsilon^{-m} & \text{in} \quad B_\varepsilon, \\ 1 & \text{in} \quad D \setminus \overline{B_\varepsilon} \end{cases}, \qquad 0 < m < 2 \tag{1}$$

and B_ε is a sufficiently smooth part of the domain D, $\overline{B_\varepsilon} \cap \partial D = \Gamma_\varepsilon$, with thickness of order $\mathcal{O}(\varepsilon)$, i.e. $\mathrm{dist}(x, \partial D) \leq \varkappa\, \varepsilon$, $x \in B_\varepsilon$, $\varkappa = \mathrm{const}$ (see Fig. 1).

Denote by $H_0^1(D, \Gamma_\varepsilon \cup \gamma)$ the closure by the Sobolev norm of $W_2^1(D)$, the set of smooth functions with compact support in $\overline{D} \backslash (\Gamma_\varepsilon \cup \gamma)$. We consider the spectral problems

$$\begin{cases} \Delta u_\varepsilon^k + \lambda_\varepsilon^k \rho_\varepsilon u_\varepsilon^k = 0 & \text{in} \quad D \\ u_\varepsilon^k = 0 & \text{on} \quad \Gamma_\varepsilon \cup \gamma \\ \dfrac{\partial u_\varepsilon^k}{\partial \nu} = 0 & \text{on} \quad \partial D \backslash (\Gamma_\varepsilon \cup \gamma), \qquad k = 1, 2 \ldots \end{cases} \tag{2}$$

and

$$\begin{cases} \Delta u^k + \lambda_0^k u^k = 0 & \text{in} \quad D \\ u^k = 0 & \text{on} \quad \partial D, \qquad k = 1, 2 \ldots \end{cases} \tag{3}$$

Here, $u_\varepsilon^k \in H_0^1(D, \Gamma_\varepsilon \cup \gamma), u^k \in H_0^1(D), k = 1, 2, \ldots$ are orthogonal bases in $L^2(D)$. The sets $\{\lambda_\varepsilon^k\}, \{\lambda_0^k\}, k = 1, 2, \ldots$ are the corresponding eigenvalues such that $0 < \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \cdots \leq \lambda_\varepsilon^k \leq \ldots$, $0 \leq \lambda_0^1 \leq \lambda_0^2 \leq \cdots \leq \lambda_0^k \leq \ldots$, and they are repeated with respect to their multiplicities.

In what follows, we use the definition from [8].

Definition 2.1. A family of closed sets $\Gamma_\varepsilon \subset \partial D$ is called SELFSIMILAR if there exist constants $C_1 > 0$ and $s, 1 < s \leq 2$ independent of ε, such that for any $\varepsilon, 0 < \varepsilon \leq \varepsilon_0$ and for any smooth function $\varphi \in C^\infty(\overline{D})$ whose support does not intersect Γ_ε, the following inequality

$$\Big(\int_{\partial D} |\varphi|^s \mathrm{d}x\Big)^{\frac{1}{s}} \leq C_1 \Big(\varepsilon \int_D |\nabla \varphi|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \tag{4}$$

holds true.

If the family $\{\Gamma_\varepsilon\}$ is selfsimilar, then the eigenvalues and eigenfunctions of the problems (2) and (3) have the following asymptotic properties as $\varepsilon \to 0$.

Theorem 2.1. *For the eigenvalues $\lambda_\varepsilon^k, \lambda_0^k$ of problems (2) and (3), respectively, the convergence $\lambda_\varepsilon^k \to \lambda_0^k$ is valid as $\varepsilon \to 0$.*

Define $\boldsymbol{R_\varepsilon} : L_2(D) \to L_{2,\rho_\varepsilon}(D)$ as the following operator $\boldsymbol{R_\varepsilon} f = f(1-\chi_\varepsilon)$, where χ_ε is the characteristic function of B_ε, and $L_{2,\rho_\varepsilon}(D)$ is the weighted space with the inner product $(f,g)_{L_{2,\rho_\varepsilon}(D)} = \int\limits_D \rho_\varepsilon(x) f(x) g(x) \mathrm{d}x$.

Theorem 2.2. *Let us consider the same hypothesis as in Theorem 2.1. Suppose that k, l are integers, $k \geq 0, l \geq 1$, and $\lambda_0^k < \lambda_0^{k+1} = \cdots = \lambda_0^{k+l} < \lambda_0^{k+l+1}$. Then, for any eigenfunction w of (3), associated with the eigenvalue λ_0^{k+1}, there exists a linear combination $\overline{u}_\varepsilon$ of the eigenfunctions $u_\varepsilon^{k+1}, \ldots, u_\varepsilon^{k+l}$ of problem (2) such that: $\overline{u}_\varepsilon \to \boldsymbol{R_\varepsilon} w$ as $\varepsilon \to 0$.*

3. Compactness theorem

Consider the boundary value problem associated with the spectral problem (2). We have

$$\begin{cases} -\Delta u_\varepsilon = \rho_\varepsilon f & \text{in} \quad D \\ u_\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \cup \gamma \subset \partial D \\ \varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu} = g & \text{on} \quad \partial D \backslash (\Gamma_\varepsilon \cup \gamma) \end{cases} \tag{5}$$

Along with the property (4) we use the Poincaré and the Friedrichs inequalities in the following form. For any functions in D, the inequality

$$\int\limits_D \varphi^2 \mathrm{d}x \leq C_2 \left(\left(\int\limits_{\partial D} |\varphi| \mathrm{d}x \right)^2 + \int\limits_D |\nabla \varphi|^2 \mathrm{d}x \right) \tag{6}$$

for any function in D, vanishing on γ, the inequality

$$\int\limits_D \varphi^2 \mathrm{d}x \leq C_2 \int\limits_D |\nabla \varphi|^2 \mathrm{d}x \tag{7}$$

where the constant C_2 depends only on the domain, holds true.

We require the boundedness of the domain D, the smoothness of its boundary and the regularity of the set $\Gamma_\varepsilon \subset \partial D$ only to satisfy (4), (6).

If the family $\{\Gamma_\varepsilon\}$ is selfsimilar, then solutions u_ε to the problem (5) have the following asymptotic properties as $\varepsilon \to 0$.

Theorem 3.1. *Assume that $g \in L_{s'}(\partial D)$, where s' is the mutual number to the number s from Definition 2.1, i.e. $\frac{1}{s} + \frac{1}{s'} = 1$. Then*

- *(i) the sequence u_ε is bounded in the space $L_s(\partial D)$ as $\varepsilon \to 0$;*
- *(ii) there exists a measurable function $\mathcal{C} : \partial D \to [0, +\infty)$ and a subsequence $\varepsilon_k \to 0$ independent of the function $g \in L_{s'}(\partial D)$, such that u_{ε_k} weakly converges to $\mathcal{C}(x) g(x)$ in $L_s(\partial D)$ as $\varepsilon_k \to 0$;*
- *(iii) the sequence u_ε is compact in $L_p(D)$, where $p < \frac{ns}{n-1}$, and the subsequence u_{ε_k} strongly converges in $L_p(D)$ to the function u_0, which satisfies the problem*

$$\begin{cases} -\Delta u_0 = f & \text{in} \quad D \\ u_0 = \mathcal{C} g & \text{on} \quad \partial D \end{cases} \tag{8}$$

Proof. The statement *(iii)* of the theorem follows from *(i)*, *(ii)* and the following Lemma from [8].

Lemma 3.2. *Let D be a domain with smooth boundary. If the sequence of solutions v_ε to the Poisson equation with sufficiently smooth right-hand side in D is weakly compact in $L_s(\partial D)$, $s > 1$, then it is strongly compact in $L_p(D)$, $p < \frac{ns}{n-1}$.*

Remark 1. In [8] this statement is proved for a sequence of harmonic functions, but the proof can be generalized step by step for the sequence of solutions to the Poisson equation in D.

To prove *(i)* we use the integral identity of the problem (5). We have

$$\varepsilon \int\limits_D \nabla u_\varepsilon \nabla v \, \mathrm{d}x = \varepsilon \int\limits_D \rho_\varepsilon f v \, \mathrm{d}x + \int\limits_{\partial D} g v \, \mathrm{d}x \tag{9}$$

for any smooth v with compact support in $\overline{D} \backslash \Gamma_\varepsilon$. From (4) and (6), it follows that the functionals in the left- and right-hand sides of (9) on $H_0^1(D, \Gamma_\varepsilon \cup \gamma)$ satisfy the Lax–Milgram Lemma (see, for instance, [9]). Hence the solution $u_\varepsilon \in H_0^1(D, \Gamma_\varepsilon \cup \gamma)$ does exist and is unique. Besides, due to continuity, the inequalities (4), (6) and (9) hold true for functions from $H_0^1(D, \Gamma_\varepsilon \cup \gamma)$. Moreover, there is a continuous trace operator from $H_0^1(D, \Gamma_\varepsilon \cup \gamma)$ to $L_s(\partial D)$.

Substituting $v = u_\varepsilon$ in (9), we get

$$\varepsilon \int\limits_D |\nabla u_\varepsilon|^2 dx \le \Big(\int\limits_{\partial D} |g|^{s'} dl\Big)^{\frac{1}{s'}} \Big(\int\limits_{\partial D} |u_\varepsilon|^s dl\Big)^{\frac{1}{s}} + \\ + \sqrt{\varepsilon \int\limits_D f^2\, dx}\sqrt{\varepsilon \int\limits_D u_\varepsilon^2\, dx} + \varepsilon^{1-m}\sqrt{\int\limits_{B_\varepsilon} f^2\, dx}\sqrt{\int\limits_{B_\varepsilon} u_\varepsilon^2\, dx} \tag{10}$$

Using (4), the Friedrichs type inequalities (7) and $\int\limits_{B_\varepsilon} u_\varepsilon^2\, dx \le K\varepsilon^2 \int\limits_D |\nabla u_\varepsilon|^2\, dx$, and keeping in mind that $\int\limits_{B_\varepsilon} f^2\, dx = \mathcal{O}(\varepsilon)$, we derive the following estimates:

$$\varepsilon \int\limits_D |\nabla u_\varepsilon|^2 dx \le C_3, \qquad \int\limits_{\partial D} |u_\varepsilon|^s dl \le C_3 \tag{11}$$

with the constant C_3 independent of ε. Thus, we proved *(i)*.

Let us consider an auxiliary problem

$$\begin{cases} -\Delta w_\varepsilon = \rho_\varepsilon f \quad \text{in} \quad D \\ w_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon \\ \varepsilon \partial w_\varepsilon \nu = 1 \quad \text{on} \quad \partial D \setminus \Gamma_\varepsilon \end{cases} \tag{12}$$

which correspond to the initial problem with $g = 1$. The corresponding identity in $H_0^1(D, \Gamma_\varepsilon \cup \gamma)$ has the form

$$\varepsilon \int\limits_D \nabla w_\varepsilon \nabla v\, dx = \varepsilon \int\limits_D \rho_\varepsilon f v\, dx + \int\limits_{\partial D} v\, dx \tag{13}$$

The solution $w_\varepsilon \in H_0^1(D, \Gamma_\varepsilon \cup \gamma)$ satisfies the bounds analogous to (11), i.e.

$$\varepsilon \int\limits_D |\nabla w_\varepsilon|^2 dx \le C_4, \qquad \int\limits_{\partial D} |w_\varepsilon|^s dx \le C_4 \tag{14}$$

with the constant C_4 independent of ε. From (14), we conclude that it is possible to choose a subsequence $\varepsilon = \varepsilon_k$, such that the traces of w_{ε_k} weakly converge in $L_s(\partial D)$. We denote the limit function on ∂D by $\mathcal{C}(x)$. Obviously, $\mathcal{C} \in L_s(\partial D)$. By the maximum principle for solutions to elliptic equations, we also have $\mathcal{C}(x) \ge 0$. Taking an arbitrary function $\theta \in C^\infty(\overline{D})$, we substitute in the identities (9), (13) $v = \theta w_\varepsilon$ and $v = \theta u_\varepsilon$, respectively. Subtracting these identities from each other, we get

$$\varepsilon \int\limits_D (w_\varepsilon \nabla u_\varepsilon - u_\varepsilon \nabla w_\varepsilon)\nabla\theta\, dx = \varepsilon \int\limits_D \rho_\varepsilon f \theta\, (w_\varepsilon - u_\varepsilon)\, dx + \int\limits_{\partial D} (g\, w_\varepsilon - u_\varepsilon)\,\theta\, dx \tag{15}$$

Show that the left-hand side and the first term in the right-hand side of (15) converge to zero as $\varepsilon \to 0$. In fact, the estimates (11), (14) and the Poincaré inequality give the boundedness of $\sqrt{\varepsilon} u_\varepsilon$ and $\sqrt{\varepsilon} w_\varepsilon$ in $W_2^1(D)$. By the Rellich theorem (see, for instance, [9]), the sequences of these functions are strongly compact in $L_2(D)$, and converge to zero in the norm of the space $L_p(D)$, $p < \frac{ns}{n-1}$. Hence, the sequence converges to zero in $L_2(D)$. Thus, in the products under the integrals in the left-hand side of (15), one multiplier is bounded in $L_2(D)$ as $\varepsilon \to 0$, and another tends to zero. And the first term in the right-hand side also converges to zero, since $m < 2$ and the sequences $\sqrt{\varepsilon} u_\varepsilon$ and $\sqrt{\varepsilon} w_\varepsilon$ converge to zero.

In the second term of the right-hand side of (15), we pass to the limit as $\varepsilon_k \to 0$. The function w_{ε_k} weakly converges to $\mathcal{C}(x)$ in $L_s(\partial D)$. The functions u_{ε_k} are bounded in $L_s(\partial D)$. Taking a subsequence from the subsequence ε_k such that u_{ε_k} weakly converges in $L_s(\partial D)$ to some limit function u_0 on ∂D, and pass to the limit on this subsubsequence. We deduce

$$\int\limits_{\partial D} \Big(g(x)\,\mathcal{C}(x) - u_0\Big)\,\theta\, dx = 0$$

Because of the arbitrariness of the choice of $\theta \in C_\infty(\overline{D})$ on ∂D, the function $u_0 = g\,C$, i.e. the function u_0, is independent of the choice of the subsubsequence. Hence the whole subsequence u_{ε_k} has a unique limit. Theorem 3.1 is proved. □

Remark 2. We finally prove that the solution u_ε to the problem (5) converges to the solution u_0 to the problem (8) as $\varepsilon \to 0$.

4. Proof of the main theorems

We use the approach from [10] to the spectral problem (2). Applying Theorems 1.4 and 1.7 from [10, Section III.1], we finalize the proof of Theorems 2.1 and 2.2.

5. An example of random geometry

In this section, we describe in general and then in particular the structure of micro inhomogeneous sets on the boundary. To describe the family $\{\Gamma_\varepsilon\}$ in detail, we use an approach from [11] and [9].

5.1. Notation

Let $(\Omega,\ \mathfrak{B},\ \mu)$ be a probability space with a semigroup of mappings $T_\xi\ :\ \Omega \to \Omega$, measurable in $\omega \in \Omega$, $\xi \in \mathbb{R}^{n-1}$ and preserving the measure μ on Ω. We assume the following group property to be satisfied: for any $\xi, \eta \in \mathbb{R}^{n-1}$ and any $\omega \in \Omega$ we have $T_\xi \circ T_\eta\ \omega = T_{\xi+\eta}\ \omega$, $T_0\ \omega = \omega$.

Definition 5.1. We call the measurable function $\varphi : \Omega \times \mathbb{R}^{n-1} \to \mathbb{R}$ a RANDOM STATISTICALLY HOMOGENEOUS FUNCTION if it has the form $\varphi = \phi(T_\xi\ \omega)$.

Definition 5.2. We call the random set HOMOGENEOUS if its characteristic function is statistically homogeneous.

The family T on Ω forms an $(n-1)$-dimensional dynamical system. In the further analysis, we assume T to be ERGODIC, i.e. any μ-measurable function on Ω, invariant with respect to this semigroup T is almost everywhere a constant. Under this assumption, the following Birkhoff theorem holds true (see, for instance, [11] and [9]).

Theorem 5.1 *(The Birkhoff theorem). For any function $\phi \in L_\alpha(\Omega)$ $(\alpha \geq 1)$ and any bounded domain $D \subset \mathbb{R}^{n-1}$, we almost surely have*

$$\lim_{\varepsilon \to 0} \frac{1}{|D|} \int\limits_D \phi\big(T_{\frac{x}{\varepsilon}}\omega\big)\,\mathrm{d}x = \int\limits_\Omega \phi(\omega)\ \mu(\mathrm{d}\omega) \equiv \langle \phi \rangle$$

Here we denoted by $\langle\cdot\rangle$ the mathematical expectation and by $|\cdot|$ the volume of a domain. From the Birkhoff theorem, one can deduce that functions $\phi\big(T_{\frac{x}{\varepsilon}}\omega\big)$ weakly converge almost surely to $\langle\phi\rangle$ in $L^{\mathrm{loc}}_\alpha(\mathbb{R}^{n-1})$ as $\varepsilon \to 0$.

5.2. Structure of Γ_ε

To simplify the exposition, we assume that $D = \{(x, z),\ \ 0 < x_i < 1,\ \ 0 < z < 1\}$, $\Gamma_\varepsilon = Q \cap \varepsilon V(\omega)$, where $Q = \{(x, z),\ \ 0 < x_i < 1,\ \ z = 0\}$ is the lower face of the cube and $V(\omega) \in \mathbb{R}^{n-1}$ is statistically homogeneous. Also we denote by γ the other faces of the cube $\partial D \backslash Q$. Here x are local coordinates on ∂D, and z is a coordinate along the normal to ∂D.

So that the family $\{\Gamma_\varepsilon\}$ be selfsimilar in the sense of Definition 2.1, we demand that the statistically homogeneous set $V(\omega)$ satisfy an additional property, which we call nondegeneracy.

Definition 5.3. A random statistically homogeneous closed set $V(\omega) \subset \mathbb{R}^{n-1}$ is called NONDEGENERATE if there exists a positive statistically homogeneous function $h = h(\omega)$ such that, for almost all ω and for any function $\varphi \in C_0^\infty\big(\mathbb{R}^n \backslash V(\omega)\big)$ with a compact support not containing $V(\omega)$, the following inequality:

$$\int\limits_{\mathbb{R}^{n-1}} h\big(T_\xi \omega\big)\varphi^2(x, 0)\mathrm{d}x \leq \int\limits_0^\infty \int\limits_{\mathbb{R}^{n-1}} |\nabla \varphi(x, z)|^2 \mathrm{d}x\,\mathrm{d}z \tag{16}$$

holds true, wherein

$$\langle h^{-1-\delta} \rangle < +\infty \tag{17}$$

with some $\delta,\ \ 0 < \delta \leq +\infty$.

Assume that $V(\omega)$ is a union in $\mathbb{R}^{n-1}$ of balls with radii $\rho_i > 0$ centered at the isolated points y_i. Let respectively $B(\omega)$ be a union in $\mathbb{R}^n$ of semiballs $(z > 0)$ with radii $\rho_i > 0$ centered in the same isolated points y_i. The balls are allowed to intersect (see the left Fig. 2). Denote by $\boldsymbol{r} = r_\omega(y)$ the distance from $y \in \mathbb{R}^{n-1}$ to the nearest center y_i, $\boldsymbol{\rho} = \rho_\omega(y)$ is the radius of the ball centered in y_i, nearest to y. If $V(\omega)$ is statistically homogeneous domain, then the function $\boldsymbol{r}$ è $\boldsymbol{\rho}$ is also statistically homogeneous.

The following statement can be proved in a similar way as in [8].

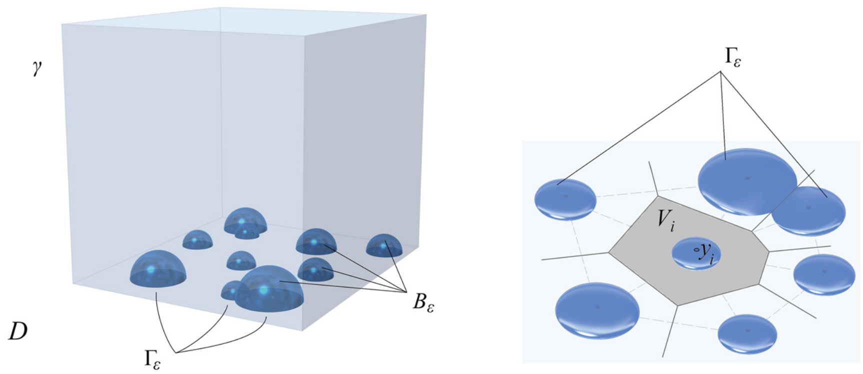

Fig. 2. Cube D with concentrated masses near the boundary and the Voronoy diagram on Q.

Lemma 5.2. *The inequality (16) holds true, if*

$$h = \frac{1}{\boldsymbol{\rho}} H\Big(\frac{\boldsymbol{r}}{\boldsymbol{\rho}}\Big), \qquad H(t) = \begin{cases} \frac{1}{2t^3}, & n=2 \\ \frac{1}{8t^3\log(t+1)}, & n=3 \\ \frac{n-3}{4t^{2n-3}}, & n>3 \end{cases} \tag{18}$$

Proof. We split $\mathbb{R}^{n-1}$ into measurable subsets V_i, consisting of points for which y_i is the nearest center (see the right Fig. 2). According to our assumption, the set $\{y_i\}$ has no accumulation points, hence V_i are polyhedra. In each of them, we set the polar system of coordinates (r, θ), where $r = |y - y_i|$ and θ are polar angles. Obviously, the polyhedra are star-shaped with respect to the center; hence, their boundaries are defined in polar coordinates by unique functions $r = R(\theta)$. Inside the polyhedra, the functions $\rho_\omega(y)$ are equal to the respective constants $\boldsymbol{\rho} > 0$.

For any point $M \in V_i$ with coordinates $(\boldsymbol{r}, \theta)$, $\boldsymbol{r} > \boldsymbol{\rho}$, we set $\boldsymbol{a} = \frac{\boldsymbol{\rho}^2}{\boldsymbol{r}}$ and construct a point $\boldsymbol{M} \in V_i$ with coordinates $(\boldsymbol{a}, \theta)$, $0 < \boldsymbol{a} < \boldsymbol{\rho}$. Connect the points M and $\boldsymbol{M}$ by the curve l in the cylinder $V_i \times [0, \infty)$, which is defined in the cylindrical coordinates (r, θ, z) by the equation

$$z = \frac{(\boldsymbol{r} - r)(r - \boldsymbol{a})}{\boldsymbol{r} - \boldsymbol{a}}, \quad \theta = \text{const}, \quad \boldsymbol{a} \le r \le \boldsymbol{r} \tag{19}$$

Consider an arbitrary function $\varphi \in C_0^\infty\Big(\mathbb{R}^n \backslash V(\omega)\Big)$ with compact support, for which we verify the inequality (16). In each cylinder $V_i \times [0, \infty)$ we have $\varphi(r, \theta, 0) \equiv 0$ if $r < \boldsymbol{\rho}$. We represent the value of φ in the point $(\boldsymbol{r}, \theta, 0)$, $\boldsymbol{r} > \boldsymbol{\rho}$, in the form of the integral over the curve l, i.e.

$$\varphi(\boldsymbol{r}, \theta, 0) = \int_{\boldsymbol{a}}^{\boldsymbol{r}} \frac{\mathrm{d}\varphi}{\mathrm{d}r} \mathrm{d}r \tag{20}$$

where $\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{\partial\varphi}{\partial r} + \frac{\partial\varphi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}r}$ is the derivative along the curve l. Obviously, $\left|\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right| \le |\nabla\varphi|\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}$. Using the Cauchy–Schwartz–Bunjakovskii inequality, we derive

$$\varphi^2(\boldsymbol{r}, \theta, 0) \le \int_{\boldsymbol{a}}^{\boldsymbol{r}} |\nabla\varphi|^2\Big[1 + \Big(\frac{\mathrm{d}z}{\mathrm{d}r}\Big)^2\Big] r^{n-2}\mathrm{d}r \int_{\boldsymbol{a}}^{\boldsymbol{r}} \frac{\mathrm{d}t}{t^{n-2}} \tag{21}$$

Denote

$$I = I(\theta) = \int_{\boldsymbol{\rho}}^{R(\theta)} \varphi^2(\boldsymbol{r}, \theta, 0)\frac{1}{\boldsymbol{\rho}} H\Big(\frac{\boldsymbol{r}}{\boldsymbol{\rho}}\Big)\boldsymbol{r}^{n-2}\mathrm{d}\boldsymbol{r}$$

Integrating $I(\theta)$ with respect to the polar angles and summarizing over all polyhedra V_i, we get the left hand side of the inequality (16). Due to (21) we deduce for I the estimate

$$I \le \int_{\boldsymbol{\rho}}^{R(\theta)} \Bigg(\int_{\boldsymbol{a}}^{\boldsymbol{r}} \Bigg(|\nabla\varphi|^2\Big[1 + \Big(\frac{\mathrm{d}z}{\mathrm{d}r}\Big)^2\Big] r^{n-2}\frac{1}{\boldsymbol{\rho}} H\Big(\frac{\boldsymbol{r}}{\boldsymbol{\rho}}\Big)\boldsymbol{r}^{n-2}\Big(\int_{\boldsymbol{a}}^{\boldsymbol{r}} \frac{\mathrm{d}t}{t^{n-2}}\Big)\Bigg)\mathrm{d}r\Bigg)\mathrm{d}\boldsymbol{r}$$

In this estimate we change variables $(\boldsymbol{r}, r)$ by the variables (z, r). The Jacobian has the form

$$\frac{\mathrm{d}z}{\mathrm{d}\boldsymbol{r}} = \frac{\boldsymbol{r}(r-\boldsymbol{a})^2 + \boldsymbol{a}(\boldsymbol{r}-r)^2}{\boldsymbol{r}(\boldsymbol{r}-\boldsymbol{a})^2} > 0$$

By direct calculations we prove the following inequalities:

$$1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\Big|_{\boldsymbol{a}\le r\le \boldsymbol{r}} \le 2, \qquad \frac{\mathrm{d}z}{\mathrm{d}\boldsymbol{r}}\Big|_{\boldsymbol{a}\le r\le \boldsymbol{r}} \ge \frac{\boldsymbol{a}}{\boldsymbol{r}+\boldsymbol{a}}$$

Thus,

$$I \le 2\int\limits_{\boldsymbol{\rho}}^{R}\Bigg(\int\limits_{\boldsymbol{a}}^{\boldsymbol{r}}\Bigg(|\nabla\varphi|^2 r^{n-2}\frac{\mathrm{d}z}{\mathrm{d}\boldsymbol{r}} \max_{\boldsymbol{r},\ \boldsymbol{\rho}\le\boldsymbol{r}\le R}\Big[\frac{1}{\boldsymbol{\rho}}H\Big(\frac{\boldsymbol{r}}{\boldsymbol{\rho}}\Big)\boldsymbol{r}^{n-2}\Big(\frac{\boldsymbol{a}}{\boldsymbol{r}+\boldsymbol{a}}\Big)^{-1}\int\limits_{\boldsymbol{a}}^{\boldsymbol{r}}\frac{\mathrm{d}t}{t^{n-2}}\Big]\Bigg)\mathrm{d}r\Bigg)\mathrm{d}\boldsymbol{r}$$

The choice of $H(t)$ leads to

$$2\max_{\boldsymbol{r},\ \boldsymbol{\rho}\le\boldsymbol{r}\le R}\Big[\frac{1}{\boldsymbol{\rho}}H\Big(\frac{\boldsymbol{r}}{\boldsymbol{\rho}}\Big)\boldsymbol{r}^{n-2}\Big(\frac{\boldsymbol{a}}{\boldsymbol{r}+\boldsymbol{a}}\Big)^{-1}\int\limits_{\boldsymbol{a}}^{\boldsymbol{r}}\frac{\mathrm{d}t}{t^{n-2}}\Big] \le 1 \tag{22}$$

for any $\boldsymbol{\rho}$ and R. Keeping in mind (22), changing variables $(\boldsymbol{r}, r)$ by (z, r) and increasing the domain of integration, we derive

$$I \le \int\limits_{\boldsymbol{\rho}}^{R(\theta)}\Big(\int\limits_{\boldsymbol{a}}^{\boldsymbol{r}}\Big(|\nabla\varphi|^2 r^{n-2}\frac{\mathrm{d}z}{\mathrm{d}\boldsymbol{r}}\Big)\mathrm{d}r\Big)\mathrm{d}\boldsymbol{r} \le \int\limits_{0}^{\infty}\Big(\int\limits_{0}^{R(\theta)}|\nabla\varphi|^2 r^{n-2}\mathrm{d}r\Big)\mathrm{d}z$$

Finally, integrating over the polar angles and summarizing on i, we obtain (16). The lemma is proved. □

5.3. Convergence result

Using the integral identities and integral inequalities, we prove the following statement.

Theorem 5.3. *Suppose that $V(\omega)$ is a nondegenerate closed set with $\delta > 0$ in Definition 5.3, then $\{\Gamma_\varepsilon\}$ is a selfsimilar family with $s = 1 + \frac{\delta}{(2+\delta)}$ and the solutions u_ε to the problem (5) satisfy the conditions of Theorem 3.1. In addition, the limit function u_0 is unique and deterministic (non-random). The boundary function $C(x)$ does not depend on the choice of a subsequence, equals zero on $\partial D\backslash Q$, and on Q it is equal to a positive constant.*

Remark 3. Analogous results can be proved for the limit spectral problem.

References

[1] A.N. Krylov, Some differential equations of mathematical physics having applications in technical questions, Rep. Nikolaev Naval Acad. 2 (1913) 325–348 (in Russian).
[2] É. Sánchez-Palencia, Perturbation of eigenvalues in thermoelasticity and vibration of systems with concentrated masses, in: Trends and Applications of Pure Mathematics to Mechanics, in: Lecture Notes in Phys., vol. 195, Springer-Verlag, Berlin, 1984, pp. 346–368.
[3] Yu.D. Golovaty, S.A. Nazarov, O.A. Oleinik, T.S. Soboleva, On eigenoscillations of a string with an attached mass, Sib. Math. J. 29 (1989) 744–760.
[4] J. Sanchez-Hubert, É. Sánchez-Palencia, Vibration and Coupling of Continuous Systems. Asymptotic Methods, Springer-Verlag, Berlin, 1989.
[5] D. Gómez, M. Lobo, E. Pérez, On the eigenfunctions associated with the high frequencies in systems with a concentrated mass, J. Math. Pures Appl. 78 (9) (1999) 841–865.
[6] G.A. Chechkin, E. Pérez, E.I. Yablokova, Non-periodic boundary homogenization and "light" concentrated masses, Indiana Univ. Math. J. 54 (2005) 321–348.
[7] T.A. Mel'nyk, G.A. Chechkin, Eigenvibrations of thick cascade junctions with "Super Heavy" concentrated masses, Izv. Math. 79 (3) (2015) 467–511.
[8] B.A. Yu, C.G.A. Averaging, Operators with boundary conditions of fine-scaled structure, Math. Notes 65 (4) (1999) 418–429.
[9] G.A. Chechkin, A.L. Piatnitski, S.A.S. Homogenization, Methods and Applications, American Mathematical Society, Providence, RI, USA, 2007.
[10] O.A. Oleinik, A.S. Shamaev, G.A. Yosifian, Mathematical Problems in Elasticity and Homogenization, North-Holland, Amsterdam, 1992.
[11] V.V. Jikov, S.M. Kozlov, O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer–Verlag, Berlin, 1994.