C. R. Mecanique 344 (2016) 753–758

Contents lists available at ScienceDirect

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Homogenization of random attractors for reaction–diffusion systems ☆

Homogénéisation des attracteurs aléatoires pour les systèmes d'équations de réaction–diffusion

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ARTICLE INFO

Article history:
Received 10 October 2016
Accepted 24 October 2016
Available online 4 November 2016

Keywords:
Reaction–diffusion systems
Attractors
Homogenization
Random functions

Mots-clés :
Systèmes de réaction–diffusion
Attracteurs
Homogénéisation
Fonctions aléatoires

ABSTRACT

We consider reaction–diffusion systems with randomly oscillating terms. We construct the deterministic homogenized reaction–diffusion system and prove that the trajectory attractors of the initial systems converge to the trajectory attractors of the homogenized systems.

RÉSUMÉ

Nous considérons les systèmes d'équations de réaction–diffusion avec termes aléatoirement oscillants. Nous construisons le système homogénéisé déterministe d'équations et prouvons que les attracteurs trajectoires des systèmes initiaux convergent vers les attracteurs trajectoires des systèmes d'équations homogénéisées.

1. Introduction

In this paper, we study an asymptotic behavior of attractors of the reaction–diffusion systems with randomly oscillating terms. To study such a phenomenon, we apply the homogenization method (cf., for example, [1–7], for the random case

☆ GAC and VVC were partially supported by RFBR grants 15-01-07920 and 14-01-00346. KAB was partially supported by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (CS MES RK) no. 0816/GF4.

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http://dx.doi.org/10.1016/j.crme.2016.10.015

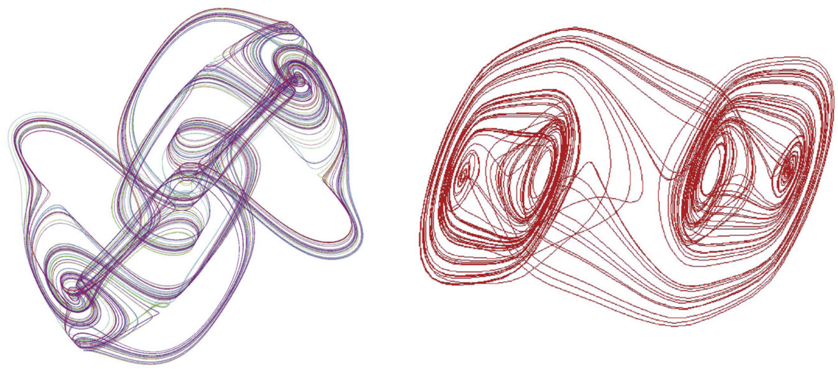

Fig. 1. Thomas' cyclically symmetric attractor (Model: Clint Sprott) and a 4-spiral strange attractor exhibited by the modified Chua's circuit (Model: M.A. Aziz Alaoui).

cf., for instance, [8–11]), as well as a delicate analysis of trajectory and global attractors (see, for example, [12–14] and references therein), see Fig. 1.

In this paper, we prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the autonomous reaction–diffusion system with a randomly oscillating term converges almost surely as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized reaction–diffusion system in an appropriate functional space.

2. Homogenization

Assume that $(\Omega, \mathcal{B}, \mu)$ is a probability space, i.e. the set Ω is endowed with a σ-algebra $\mathcal{B}$ of its subsets and a σ-additive nonnegative measure μ on $\mathcal{B}$ such that $\mu(\Omega) = 1$.

We consider the system of reaction–diffusion equations with randomly oscillating terms of the form

$$\partial_t u = a\Delta u - b\left(x, \frac{x}{\varepsilon}, \omega\right) f(u) + g\left(x, \frac{x}{\varepsilon}, \omega\right), \; u|_{\partial D} = 0 \tag{1}$$

where $x \in D \Subset \mathbb{R}^n$, $u = (u^1, \dots, u^N)$, $f = (f^1, \dots, f^N)$, and $g = (g^1, \dots, g^N)$. Here a is an $N \times N$ matrix with positive symmetric part and $b(x, z, \omega) \in C(D \times \mathbb{R}^N \times \Omega)$ is a real positive function. The Laplace operator $\Delta := \partial^2_{x_1} + \dots + \partial^2_{x_n}$ acts in x-space.

We note that all the results can be extended to the systems with nonlinear terms of the form $\sum_{j=1}^m b_j\left(x, \frac{x}{\varepsilon}, \omega\right) f_j(u)$, where b_j are positively defined matrices and $f_j(u)$ are vector functions. For brevity, we consider the case $m = 1$ and $b_1\left(x, \frac{x}{\varepsilon}, \omega\right) = b\left(x, \frac{x}{\varepsilon}, \omega\right) I$, where I is the identity matrix and b is a real function.

For the sake of simplicity, we assume that the vector function $f(v) \in C(\mathbb{R}^N; \mathbb{R}^N)$ satisfies the following inequalities:

$$f(v) \cdot v \geq \gamma |v|^p - C, \; |f(v)| \leq C_1\left(|v|^{p-1} + 1\right), \; p \geq 2 \tag{2}$$

Notice that we *do not assume* that the function $f(v)$ satisfies the Lipschitz condition with respect to v.

Assume that T_ξ, $\xi \in \mathbb{R}^n$, is an ergodic dynamical system. The function $b(x, \frac{x}{\varepsilon}, \omega)$ and the vector function $g(x, \frac{x}{\varepsilon}, \omega)$ are statistically homogeneous, i.e. $b(x, \xi, \omega) = \mathbf{B}(x, T_\xi\omega)$ and $g(x, \xi, \omega) = \mathbf{G}(x, T_\xi\omega)$, where $\mathbf{B} : D \times \Omega \to \mathbb{R}$ and $\mathbf{G} : D \times \Omega \to \mathbb{R}^N$ are measurable.

We also assume that $b(x, z, \omega) \in C_b(\overline{D} \times \mathbb{R} \times \Omega)$ and

$$\beta_1 \geq b(x, z, \omega) \geq \beta_0 > 0, \; \forall x \in D, \; z \in \mathbb{R}^n, \; \omega \in \Omega \tag{3}$$

the function $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ has the average $b^{\mathrm{hom}}(x) = \mathbb{E}(\mathbf{B})(x)$ as $\varepsilon \to 0+$ in $L_{\infty,*w}(D)$, that is, almost surely

$$\int_D b\left(x, \frac{x}{\varepsilon}, \omega\right) \varphi(x)\, \mathrm{d}x \to \int_D b^{\mathrm{hom}}(x)\, \varphi(x)\, \mathrm{d}x \; (\varepsilon \to 0+) \tag{4}$$

for any function $\varphi \in L_1(D)$. For the vector function $g\left(x, \frac{x}{\varepsilon}, \omega\right)$, we assume that it has the average $g^{\mathrm{hom}}(x) = \mathbb{E}(\mathbf{G})(x)$ in the space $V' = \left(H^{-1}(D)\right)^N$:

$$g\left(x, \frac{x}{\varepsilon}, \omega\right) \rightharpoonup g^{\mathrm{hom}}(x) \; (\varepsilon \to 0+) \text{ weakly in } V'$$

that is, almost surely

$$\left\langle g\left(x, \frac{x}{\varepsilon}, \omega\right), \varphi(x)\right\rangle \to \left\langle g^{\mathrm{hom}}(x), \varphi(x)\right\rangle \; (\varepsilon \to 0+) \tag{5}$$

for any $\varphi \in V = \left(H_0^1(D)\right)^N$. In particular, the following functions are available:

$$g\left(x, \frac{x}{\varepsilon}, \omega\right) = g_0\left(x, \frac{x}{\varepsilon}, \omega\right) + \sum_{i=1}^{n} \partial_{x_i} g_i\left(x, \frac{x}{\varepsilon}, \omega\right)$$

where the functions $g_i\left(x, \frac{x}{\varepsilon}, \omega\right)$ have the averages $g_i^{\hom}(x) \in (L_2(D))^N$ in $H = (L_2(D))^N$ and almost surely

$$\left\langle g_i\left(x, \frac{x}{\varepsilon}, \omega\right), \varphi(x)\right\rangle \to \left\langle g_i^{\hom}(x), \varphi(x)\right\rangle \ (\varepsilon \to 0+)\ \forall \varphi \in H,\ i = 1, \ldots, n$$

We note that the H-norms of the functions $\partial_{x_i} g_i\left(x, \frac{x}{\varepsilon}, \omega\right) = g_{ix_i}\left(x, \frac{x}{\varepsilon}, \omega\right) + \frac{1}{\varepsilon} g_{iz_i}\left(x, \frac{x}{\varepsilon}, \omega\right)$ can tend to infinity as $\varepsilon \to 0+$. These functions are bounded in the space V' only.

As in [15,13] we study weak solutions (trajectories) of the system (1), that is, the functions $u(x,t) \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_p^{\mathrm{loc}}\left(\mathbb{R}_+; (L_p(D))^N\right)$ that satisfy (1) in the sense of distributions of the space $D'\left(\mathbb{R}_+; (H^{-r}(D))^N\right)$, where $r = \max\{1, n(1/2 - 1/p)\}$ (the number r is defined by the corresponding Sobolev embedding theorem). For every $u_0 \in H$, there exists at least one weak solution $u(x,t)$ of the system (1) such that $u(0) = u_0$ (see [12,15,13]). This solution is not necessarily unique because we do not assume the Lipschitz condition for $f(v)$ with respect to v. We denote by $\mathcal{K}_\varepsilon^+$ the set of all weak solutions to the system (1).

Consider the translation semigroup $\{T(h)\}$ acting on the trajectory space $\mathcal{K}_\varepsilon^+$ by the formula $T(h)u(x,t) = u(x, t+h)$ for $h \geq 0$.

We study the trajectory attractor $\mathfrak{A}_\varepsilon$ of the system (1), which, by definition, coincides with the global $\left(\mathcal{F}_+^b, \Theta_+^{\mathrm{loc}}\right)$-attractor of the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_\varepsilon^+$ (see [12–14]). Here we denote

$$\Theta_+^{\mathrm{loc}} = L_{\infty,*w}^{\mathrm{loc}}(\mathbb{R}_+; H) \cap L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_{p,w}^{\mathrm{loc}}\left(\mathbb{R}_+; (L_p(D))^N\right) \cap \left\{v \mid \partial_t v \in L_{q,w}^{\mathrm{loc}}\left(\mathbb{R}_+; (H^{-r}(D))^N\right)\right\}$$

$$\mathcal{F}_+^b = L_\infty^b(\mathbb{R}_+; H) \cap L_2^b(\mathbb{R}_+; V) \cap L_p^b\left(\mathbb{R}_+; (L_p(D))^N\right) \cap \left\{v \mid \partial_t v \in L_q^b\left(\mathbb{R}_+; (H^{-r}(D))^N\right)\right\}$$

Recall that Θ_+^{loc} is the local weak topology, which is determined by the weak and $*$-weak convergence of sequences $\{v_m\}$ and $\{\partial_t v_m\}$ in the corresponding spaces. The trajectory space $\mathcal{K}_\varepsilon^+$ is supplied with topology Θ_+^{loc}. The Banach space $\mathcal{F}_+^b$ is used to define bounded sets in $\mathcal{K}_\varepsilon^+$.

By $\mathcal{K}_\varepsilon$, we denote the kernel of the system (1) that is the set of all complete solutions (complete trajectories) $u(x,t)$ defined for all $t \in \mathbb{R}$ that are bounded in the space $\mathcal{F}^b$, where

$$\mathcal{F}^b = L_\infty^b(\mathbb{R}; H) \cap L_2^b(\mathbb{R}; V) \cap L_p^b\left(\mathbb{R}; (L_p(D))^N\right) \cap \cap \left\{v \mid \partial_t v \in L_q^b\left(\mathbb{R}; (H^{-r}(D))^N\right)\right\}$$

Proposition 2.1. *Under conditions (2), (4), and (5), the system (1) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topology Θ_+^{loc}. The set $\mathfrak{A}_\varepsilon$ is almost surely uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,*

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathcal{K}_\varepsilon \tag{6}$$

the kernel $\mathcal{K}_\varepsilon$ is non-empty, uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$ and compact in the topology Θ^{loc}, where

$$\Theta^{\mathrm{loc}} = L_{\infty,*w}^{\mathrm{loc}}(\mathbb{R}; H) \cap L_{2,w}^{\mathrm{loc}}(\mathbb{R}; V) \cap L_{p,w}^{\mathrm{loc}}\left(\mathbb{R}; (L_p(D))^N\right) \cap \cap \left\{v \mid \partial_t v \in L_{q,w}^{\mathrm{loc}}\left(\mathbb{R}; (H^{-r}(D))^N\right)\right\}$$

The proof of this proposition almost coincides with the proof given in [13] for a deterministic case.

Recall that $\Theta_+^{\mathrm{loc}} \subset L_2^{\mathrm{loc}}\left(\mathbb{R}_+; (H^{1-\delta}(D))^N\right)$, $0 < \delta \leq 1$, and therefore the trajectory attractor $\mathfrak{A}_\varepsilon$ attracts bounded sets of trajectories of the system (1) in the local strong topology of the space $L_2^{\mathrm{loc}}\left(\mathbb{R}_+; (H^{1-\delta}(D))^N\right)$.

Along with the random system (1), we consider the averaged deterministic system

$$\partial_t \bar{u} = a\Delta\bar{u} - b^{\hom}(x) f(\bar{u}) + g^{\hom}(x),\ \bar{u}|_{\partial D} = 0 \tag{7}$$

Clearly system (7) also has a trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$ corresponding to the system (7) and

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}$$

where $\overline{\mathcal{K}}$ is the kernel of system (7) in $\mathcal{F}^b$. The set $\overline{\mathfrak{A}}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.

The following statement holds true.

Theorem 2.1. *The following limit holds almost surely in the topology Θ_+^{loc}*

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \text{ as } \varepsilon \to 0+ \tag{8}$$

Moreover, almost surely

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \text{ in } \Theta^{\text{loc}} \tag{9}$$

Proof. It is clear that (9) implies (8). Therefore it is sufficient to prove (9), that is, for every neighborhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc}, there exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that almost surely

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \text{ for } \varepsilon < \varepsilon_1 \tag{10}$$

Suppose that (10) is not true. Consider the corresponding subset $\Omega' \subset \Omega$ with $\mu(\Omega') > 0$ and (10) does not hold for all $\omega \in \Omega'$. Then, for each $\omega \in \Omega'$, there exists a neighborhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_n \to 0+$ $(n \to \infty)$, and a sequence $u_{\varepsilon_n}(\cdot) = u_{\varepsilon_n}(\omega, t) \in \mathcal{K}_{\varepsilon_n}$ such that

$$u_{\varepsilon_n} \notin \mathcal{O}'(\overline{\mathcal{K}}) \text{ for all } n \in \mathbb{N}, \omega \in \Omega' \tag{11}$$

For each $\omega \in \Omega'$, the function $u_{\varepsilon_n}(t)$, $t \in \mathbb{R}$ is the solution to the system

$$\partial_t u_{\varepsilon_n} = a\Delta u_{\varepsilon_n} - b\left(x, \frac{x}{\varepsilon_n}, \omega\right) f(u_{\varepsilon_n}) + g\left(x, \frac{x}{\varepsilon_n}, \omega\right), \; u_{\varepsilon_n}|_{\partial D} = 0 \tag{12}$$

on the entire time axis $t \in \mathbb{R}$. Moreover, the sequence $\{u_{\varepsilon_n}(t)\}$ is bounded in $\mathcal{F}^b$ for each $\omega \in \Omega'$, that is,

$$\begin{gathered} \|u_{\varepsilon_n}\|_{\mathcal{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\varepsilon_n}(t)\|_H + \\ \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_n}(s)\|_V^2 \, \mathrm{d}s \right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_n}(s)\|_{L_p}^p \, \mathrm{d}s \right)^{1/p} + \\ \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|\partial_t u_{\varepsilon_n}(s)\|_{H^{-r}}^q \, \mathrm{d}s \right)^{1/q} \le C \; \text{ for all } n \in \mathbb{N} \end{gathered} \tag{13}$$

Here, the constant C is independent of n. Hence there exists a subsequence $\{u_{\varepsilon_n'}(t)\} \subset \{u_{\varepsilon_n}(t)\}$ that we label the same, such that

$$u_{\varepsilon_n}(t) \to \bar{u}(t) \text{ as } n \to \infty \; \text{ in } \Theta^{\text{loc}} \tag{14}$$

where $\bar{u}(\cdot) \in \mathcal{F}^b$ and $\bar{u}(t)$ satisfies (13) with the same constant C. In detail we have that $u_{\varepsilon_n}(t) \rightharpoonup \bar{u}(t)$ $(n \to \infty)$ weakly in $L_{2,w}^{\text{loc}}(\mathbb{R}; V)$, weakly in $L_{p,w}^{\text{loc}}\left(\mathbb{R}; \left(L_p(D)\right)^N\right)$, $*$-weakly in $L_{\infty,*w}^{\text{loc}}(\mathbb{R}_+; H)$ and $\partial_t u_{\varepsilon_n}(t) \rightharpoonup \partial_t \bar{u}(t)$ $(n \to \infty)$ weakly in $L_{q,w}^{\text{loc}}\left(\mathbb{R}; \left(H^{-r}(D)\right)^N\right)$. We claim that $\bar{u}(\cdot) \in \overline{\mathcal{K}}$. We have already proved that $\|\bar{u}\|_{\mathcal{F}^b} \le C$. So we have to establish that $\bar{u}(t)$ is a weak solution to (7). Using (13) and (5), we obtain that

$$\partial_t u_{\varepsilon_n} - a\Delta u_{\varepsilon_n} - g\left(x, \frac{x}{\varepsilon_n}, \omega\right) \to \partial_t \bar{u} - a\Delta \bar{u} - g^{\text{hom}}(x) \text{ as } n \to \infty \tag{15}$$

in the space $D'\left(\mathbb{R}; \left(H^{-r}(D)\right)^N\right)$ because the derivative operators ∂_t and Δ are continuous in the space of distributions. Let us prove that

$$b\left(x, \frac{x}{\varepsilon_n}\right) f(u_{\varepsilon_n}) \rightharpoonup b^{\text{hom}}(x)\, f(\bar{u}) \text{ as } n \to \infty \tag{16}$$

weakly in $L_{q,w}^{\text{loc}}\left(\mathbb{R}; \left(L_q(D)\right)^N\right)$. We fix an arbitrary number $M > 0$. The sequence $\{u_{\varepsilon_n}(t)\}$ is bounded in $L_p\left(]-M, M[; \left(L_p(D)\right)^N\right)$ (see (13)). Hence by (2), the sequence $\{f(u_{\varepsilon_n}(t))\}$ is bounded in $L_q\left(]-M, M[; \left(L_q(D)\right)^N\right)$. Since $\{u_{\varepsilon_n}(t)\}$ is bounded in $L_2\left(]-M, M[; \left(H_0^1(D)\right)^N\right)$ and $\{\partial_t u_{\varepsilon_n}(t)\}$ is bounded in $L_q\left(]-M, M[; \left(H^{-r}(D)\right)^N\right)$, we can assume that $u_{\varepsilon_n}(t) \to \bar{u}(t)$ as $n \to \infty$ strongly in $L_2\left(]-M, M[; (L_2(D))^N\right) = L_2\left(D \times]-M, M[\right)^N$ and therefore

$$u_{\varepsilon_n}(x,t) \to \bar{u}(x,t) \text{ as } n \to \infty \text{ a.e. in } (x,t) \in D \times]-M,M[$$

Since the function $f(v)$ is continuous with respect to $v \in \mathbb{R}^N$, we conclude that

$$f(u_{\varepsilon_n}(x,t)) \to f(\bar{u}(x,t)) \text{ as } n \to \infty \text{ a.e. in } (x,t) \in D \times]-M,M[\tag{17}$$

We have

$$b\left(x, \frac{x}{\varepsilon_n}, \omega\right) f(u_{\varepsilon_n}) - b^{\text{hom}}(x) f(\bar{u}) =$$
$$b\left(x, \frac{x}{\varepsilon_n}, \omega\right)\left(f(u_{\varepsilon_n}) - f(\bar{u})\right) + \left(b\left(x, \frac{x}{\varepsilon_n}, \omega\right) - b^{\text{hom}}(x)\right) f(\bar{u}) \tag{18}$$

Let us show that both summands in the right-hand side of (18) converge to zero as $n \to \infty$ weakly in $L_q\left(]-M,M[;\left(L_q(D)\right)^N\right) = \left(L_q(D \times]-M,M[)\right)^N$. The sequence $b\left(x, \frac{x}{\varepsilon_n}, \omega\right)\left(f(u_{\varepsilon_n}) - f(\bar{u})\right)$ tends to zero as $n \to \infty$ almost everywhere in $(x,t) \in D \times]-M,M[$ (see (17)) and is bounded in the space $\left(L_q(D \times]-M,M[)\right)^N$ (see (3)). Therefore Lemma 1.3 from [16, Chapter 1, Section 1] implies that

$$b\left(x, \frac{x}{\varepsilon_n}, \omega\right)\left(f(u_{\varepsilon_n}) - f(\bar{u})\right) \rightharpoonup 0 \text{ as } n \to \infty$$

weakly in $\left(L_q(D \times]-M,M[)\right)^N$. The sequence $\left(b\left(x, \frac{x}{\varepsilon_n}, \omega\right) - b^{\text{hom}}(x)\right) f(\bar{u})$ also approaches zero as $n \to \infty$ weakly in $\left(L_q(D \times]-M,M[)\right)^N$ because, by our assumption, $b\left(x, \frac{x}{\varepsilon_n}, \omega\right) \rightharpoonup b^{\text{hom}}(x)$ as $n \to \infty$ $*$-weakly in the space $L_{\infty,*w}(]-M,M[; L_2(D))$ and $f(\bar{u}) \in \left(L_q(D \times]-M,M[)\right)^N$. We have proved (16). Using (15) and (16) we pass to the limit in the equation (12) as $n \to \infty$ in the space $D'\left(\mathbb{R}_+; \left(H^{-r}(D)\right)^N\right)$ and we obtain that the function $\bar{u}(x,t)$ satisfies the equation

$$\partial_t \bar{u} = a\Delta\bar{u} - b^{\text{hom}}(x) f(\bar{u}) + g^{\text{hom}}(x), \ \bar{u}|_{\partial D} = 0, \ t \in \mathbb{R}$$

Consequently, $\bar{u} \in \overline{\mathcal{K}}$. We have proved above that $u_{\varepsilon_n}(t) \to \bar{u}(t)$ as $n \to \infty$ in Θ^{loc} for each $\omega \in \Omega'$. The hypotheses $u_{\varepsilon_n}(t) \notin \mathcal{O}'(\overline{\mathcal{K}})$ implies that $\bar{u} \notin \mathcal{O}'(\overline{\mathcal{K}})$ and moreover $\bar{u} \notin \overline{\mathcal{K}}$ for all $\omega \in \Omega'$. We have arrived to the contradiction. The theorem is proved. □

Corollary 2.2. *For every $0 < \delta \le 1$ and for any $M > 0$ almost surely*

$$\text{dist}_{L_2([0,M];H^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \ (\varepsilon \to 0+)$$

Here $\text{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X} \text{dist}_{\mathcal{M}}(x,Y)$ denotes the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$.

Remark 2.1. The analogous theorem holds for random non-autonomous reaction–diffusion systems of the form (1) that contain the terms $b\left(x, \frac{t}{\varepsilon}, t, \omega\right)$ and $g\left(x, \frac{t}{\varepsilon}, t, \omega\right)$ having the uniform averages in time as $\varepsilon \to 0+$.

In conclusion, we briefly consider the reaction–diffusion systems for which the uniqueness theorem of the Cauchy problem takes place. It is sufficient to assume that the nonlinear term $f(u)$ in the equation (1) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \ge -C_2|v_1 - v_2|^2 \ \text{ for all } v_1, v_2 \in \mathbb{R}^N \tag{19}$$

where $C_2 \ge 0$ (see [13]). In this case, the limit in (8) holds in a stronger topology

$$\Theta_+^{\text{loc},1} = L^{\text{loc}}_{\infty,*w}\left(\mathbb{R}_+; \left(H_0^1(D)\right)^N\right) \cap L^{\text{loc}}_{2,w}\left(\mathbb{R}_+; \left(H^2(D)\right)^N\right) \cap L^{\text{loc}}_{p,w}\left(\mathbb{R}_+; \left(L_p(D)\right)^N\right)$$
$$\cap \left\{v \mid \partial_t v \in L^{\text{loc}}_{q,w}\left(\mathbb{R}_+; \left(L_q(D)\right)^N\right)\right\}$$

In particular,

$$\text{dist}_{L_2([0,M];H^{2-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \ (\varepsilon \to 0+)$$
$$\text{dist}_{C([0,M];H^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \ (\varepsilon \to 0+) \ \forall M > 0 \ (0 < \delta \le 1) \tag{20}$$

In [15] and [13] it was proved that if (19) holds, then equations (1) and (7) generate the semigroups $\{S(t)\}$ and $\{\overline{S}(t)\}$ in $H = (L_2(D))^N$, which have the global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ bounded in the space $\left(H_0^1(D)\right)^N$ (see also [12,14]). We clearly have:

$$\mathcal{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\}, \ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}$$

Convergence (20) implies the following assertion.

Corollary 2.3. *The following limit almost surely holds:*

$$\mathrm{dist}_{H^{1-\delta}}\left(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+) \ \forall \delta \in \,]0, 1] \tag{21}$$

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