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Asymptotic curved interface models in piezoelectric composites

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ABSTRACT

We study the electromechanical behavior of a thin interphase, constituted by a piezoelectric anisotropic shell-like thin layer, embedded between two generic three-dimensional piezoelectric bodies by means of the asymptotic analysis in a general curvilinear framework. After defining a small real dimensionless parameter ε, which will tend to zero, we characterize two different limit models and their associated limit problems, the so-called *weak* and *strong* piezoelectric curved interface models, respectively. Moreover, we identify the non-classical electromechanical transmission conditions at the interface between the two three-dimensional bodies.

1. Introduction

Smart materials have been used over the past few decades in several applications in all fields of aeronautical, mechanical, and civil engineering. For what concerns smart structures, the strain state is constantly under control by means of sensors and actuators, usually made of piezoelectric materials, integrated within the structure. The more and more promising applications of piezoelectric composites have lead researchers to develop new methods and analysis tools for a better understanding of the mechanisms and behaviors of such structures, which are subjected to electromechanical interactions. More often, the piezoelectric actuators are obtained by alternating different thin layers of material with highly contrasted electromechanical properties. This generates different types of complex composites, in which each phase interacts with the others.

The asymptotic methods have been successfully applied for the mathematical justification of thin structure models in both fields of elasticity and piezoelectricity, taking into account also thermal and magnetic effects (see, e.g., [1–3]): this has stimulated researchers to tunnel their efforts toward a formal simplification of the modeling of complex structures obtained by joining elements presenting highly contrasted geometrical and mechanical properties. A thin interphase inserted between two generic media can be considered as the most distinctive bonded joint. The asymptotic expansions method allows one to replace the original problem with a reduced transmission problem, in which the thin interphase is substituted by a two-dimensional material surface, i.e. a so-called *imperfect interface*, between the two three-dimensional bodies with non-classical transmission conditions. Within the theory of elasticity, the asymptotic analysis of a thin elastic interphase between two elastic materials has been deeply investigated through the years, by varying the rigidity ratios between the

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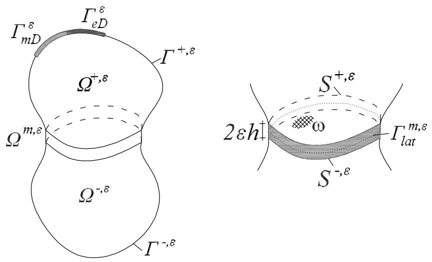

Fig. 1. The geometry of the composite: configuration in the curvilinear coordinates system.

thin inclusion and the surrounding materials and by considering different geometry features (see, e.g., [4–7], within the theory of elasticity, and see [8,9], within the theory of piezoelectricity, including also thermal and magnetic couplings).

This work is conceived as the curvilinear generalization of a previous work [8] on asymptotic planar weak and strong piezoelectric interface models. In the present work, we identify two different interface limit models of a piezoelectric assembly constituted by a thin piezoelectric shell-like layer inserted between two generic piezoelectric bodies by means of an asymptotic analysis in a general curvilinear framework. By defining a small real parameter ε, associated with the thickness and the electromechanical properties of the middle layer, we perform an asymptotic analysis by letting ε tend to zero. We analyze two different situations by varying the electromechanical stiffness ratios between the middle layer and the adherents: namely, the *weak* piezoelectric curved interface, where the electromechanical coefficients of the intermediate domain have order of magnitude ε with respect to those of the surrounding bodies, the *strong* piezoelectric curved interface, where the electromechanical rigidities have order of magnitude $\frac{1}{\varepsilon}$. Within the reduced models, the interphase is replaced by a material surface (*strong* case) or a constraint (*weak* case) whose energy, in both cases, is the limit of the interphase energy. This surface energy is then translated in ad hoc transmission conditions at the interface.

The paper is organized as follows. In Sect. 2, we define the position of the problem and we perform the asymptotic analysis of the problem. In Sect. 3 and Sect. 4, we deduce, respectively, the two limit interface models. Finally, we discuss the results and propose some future developments in the concluding remarks in Sect. 5.

2. Position of the problem and asymptotic expansions

Let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega := \left\{\partial\Omega^+ \cap \partial\Omega^-\right\}^\circ$ be the interior of the common part of the boundaries which is assumed to be a non-empty domain in $\mathbb{R}^2$ having a positive two-dimensional measure. Let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ be an immersion such that the two vectors $\mathbf{a}_\alpha(\tilde{x}) := \partial_\alpha \boldsymbol{\theta}(\tilde{x})$ form the covariant basis of the tangent plane to the surface $\boldsymbol{\theta}(\omega)$ at each point $\boldsymbol{\theta}(\tilde{x})$, with $\tilde{x} = (x_\alpha) \in \omega$; the two vectors $\mathbf{a}^\alpha(\tilde{x})$, defined by the relation $\mathbf{a}_\alpha \cdot \mathbf{a}^\beta = \delta^\beta_\alpha$, form the contravariant basis of the tangent plane. Also let $\mathbf{a}_3(\tilde{x}) = \mathbf{a}^3(\tilde{x}) := \frac{\mathbf{a}_1(\tilde{x}) \wedge \mathbf{a}_2(\tilde{x})}{|\mathbf{a}_1(\tilde{x}) \wedge \mathbf{a}_2(\tilde{x})|}$ be the unit normal vector to $\boldsymbol{\theta}(\omega)$. The covariant and contravariant components $a_{\alpha\beta}$ and $a^{\alpha\beta}$ of the first fundamental form, the covariant and mixed components of the second fundamental form, and the Christoffel symbols of the surface are respectively defined by: $a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, $a^{\alpha\beta} := \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$, $b_{\alpha\beta} := \mathbf{a}^3 \cdot \partial_\beta \mathbf{a}_\alpha$, $b^\beta_\alpha := a^{\beta\sigma} b_{\alpha\sigma}$ and $\Gamma^\sigma_{\alpha\beta} := \mathbf{a}^\sigma \cdot \partial_\beta \mathbf{a}_\alpha$. The covariant derivative of $T^{\alpha\beta}$ are defined by $T^{\alpha\beta}|_\tau := \partial_\tau T^{\alpha\beta} + \Gamma^\alpha_{\beta\sigma} T^{\tau\sigma} + \Gamma^\beta_{\tau\sigma} T^{\alpha\sigma}$.

Let $0 < \varepsilon < 1$ be a dimensionless *small* real parameter. Let us consider $\Omega^{m,\varepsilon} := \omega \times (-\varepsilon h, \varepsilon h)$, $S^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon h\}$ and $\Gamma^{m,\varepsilon}_{lat} := \partial\omega \times (-\varepsilon h, \varepsilon h)$. Let x^ε denote the generic point in the set $\overline{\Omega}^{m,\varepsilon}$ with $x^\varepsilon_\alpha = x_\alpha$. We consider a *shell-like* domain with middle surface $\boldsymbol{\theta}(\overline{\omega})$ and thickness $2\varepsilon h$, whose reference configuration is the image $\boldsymbol{\Theta}^{m,\varepsilon}(\overline{\Omega}^{m,\varepsilon}) \subset \mathbb{R}^3$ of the set $\overline{\Omega}^{m,\varepsilon}$ through the mapping given by $\boldsymbol{\Theta}^{m,\varepsilon}(x^\varepsilon) := \boldsymbol{\theta}(\tilde{x}) + x^\varepsilon_3 \mathbf{a}_3(\tilde{x})$, for all $x^\varepsilon = (\tilde{x}, x^\varepsilon_3) \in \overline{\Omega}^{m,\varepsilon}$.

Moreover, we suppose that there exists an immersion $\boldsymbol{\Theta}^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ defined as follows:

$$\boldsymbol{\Theta}^\varepsilon := \begin{cases} \boldsymbol{\Theta}^{\pm,\varepsilon} & \text{on } \overline{\Omega}^{\pm,\varepsilon} \\ \boldsymbol{\Theta}^{m,\varepsilon} & \text{on } \overline{\Omega}^{m,\varepsilon} \end{cases}, \quad \boldsymbol{\Theta}^{\pm,\varepsilon}(S^{\pm,\varepsilon}) = \boldsymbol{\Theta}^{m,\varepsilon}(S^{\pm,\varepsilon}),$$

with $\boldsymbol{\Theta}^{\pm,\varepsilon} : \overline{\Omega}^{\pm,\varepsilon} \to \mathbb{R}^3$ immersions over $\overline{\Omega}^{\pm,\varepsilon}$ defining the curvilinear coordinates on $\overline{\Omega}^{\pm,\varepsilon}$, see Fig. 1. We will note by $g^\varepsilon_{ij} := (\partial^\varepsilon_i \boldsymbol{\Theta}^\varepsilon \cdot \partial^\varepsilon_j \boldsymbol{\Theta}^\varepsilon)$, the covariant components of the metric tensor, with $g^\varepsilon := \det(g^\varepsilon_{ij})$, $\Gamma^{p,\varepsilon}_{ij}$, the Christoffel symbols of the second kind induced by the metric g^ε_{ij} and $T^{ij}\|_k := \partial_k T^{ij} + \Gamma^i_{\ell j} T^{\ell k} + \Gamma^j_{\ell k} T^{\ell i}$, the covariant derivatives of T^{ij}.

Let $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$ and $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ be two suitable partitions of $\partial\Omega^\varepsilon := \Gamma^{\pm,\varepsilon} \cup \Gamma^{m,\varepsilon}_{lat}$. The composite is, on the one hand, clamped along Γ^ε_{mD} and at an electrical potential $\varphi^\varepsilon_0 = 0$ on Γ^ε_{eD} and, on the other hand, subject to surface forces $g^{i,\varepsilon}$ on Γ^ε_{mN} and surface electrical charges d^ε on Γ^ε_{eN}. The assembly is also subject to body forces $f^{i,\varepsilon}$ and electrical loadings ρ^ε_e acting in $\Omega^{\pm,\varepsilon}$. The work of the external electromechanical loadings in curvilinear coordinates takes the following form:

$$L^\varepsilon(r^\varepsilon) := \int_{\Omega^{\pm,\varepsilon}} (f^{i,\varepsilon} v_i^\varepsilon + \rho_e^\varepsilon \psi^\varepsilon)\sqrt{g_\pm^\varepsilon}\,\mathrm{d}x^\varepsilon + \int_{\Gamma_{mN}^\varepsilon} g^{i,\varepsilon} v_i^\varepsilon \sqrt{g_\pm^\varepsilon}\,\mathrm{d}\Gamma^\varepsilon - \int_{\Gamma_{eN}^\varepsilon} d^\varepsilon \psi^\varepsilon \sqrt{g_\pm^\varepsilon}\,\mathrm{d}\Gamma^\varepsilon.$$

We suppose that $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$ are constituted by three homogeneous anisotropic piezoelectric materials, whose constitutive laws are defined as follows:

$$\begin{cases} \sigma^{ij,\varepsilon}(\mathbf{u}^\varepsilon,\varphi^\varepsilon) = C^{ijk\ell,\varepsilon} e^\varepsilon_{k\|\ell}(\mathbf{u}^\varepsilon) - P^{kij,\varepsilon} E_k^\varepsilon(\varphi^\varepsilon), \\ D^{i,\varepsilon}(\mathbf{u}^\varepsilon,\varphi^\varepsilon) = P^{ijk,\varepsilon} e^\varepsilon_{j\|k}(\mathbf{u}^\varepsilon) + H^{ij,\varepsilon} E_j^\varepsilon(\varphi^\varepsilon), \end{cases}$$

where $\sigma^{ij,\varepsilon}$ and $D^{i,\varepsilon}$ represent, respectively, the contravariant components of the Cauchy stress tensor and the contravariant components of the electrical displacement vector, $e^\varepsilon_{i\|j}(\mathbf{v}^\varepsilon) := \frac{1}{2}(\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon$ represent the covariant components of the linearized change of metric tensor, φ^ε is the electrical potential and $E_i^\varepsilon(\varphi^\varepsilon) := -\partial_i^\varepsilon \varphi^\varepsilon$ its associated electrical field. $C^{ijk\ell,\varepsilon}$, $P^{ijk,\varepsilon}$ and $H^{ij,\varepsilon}$ represent, respectively, the contravariant components of the fourth-order elasticity tensor, the third-order piezoelectric coupling tensor, and the second-order dielectric tensor related to $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$.

Let $\Sigma^\varepsilon \subset \partial\Omega^\varepsilon$, we introduce the functional spaces

$$V(\Omega^\varepsilon,\Sigma^\varepsilon) := \{v^\varepsilon \in H^1(\Omega^\varepsilon);\ v^\varepsilon = 0 \text{ on } \Sigma^\varepsilon\},\ \ \mathbf{V}(\Omega^\varepsilon,\Sigma^\varepsilon) := [V(\Omega^\varepsilon,\Sigma^\varepsilon)]^3.$$

The electromechanical state at the equilibrium is determined by the pair $s^\varepsilon := (\mathbf{u}^\varepsilon,\varphi^\varepsilon)$. The physical variational problem in curvilinear coordinates defined over the variable domain Ω^ε can be written as

$$\begin{cases} \text{Find } s^\varepsilon \in \mathbf{V}(\Omega^\varepsilon,\Gamma^\varepsilon_{mD}) \times V(\Omega^\varepsilon,\Gamma^\varepsilon_{eD}) \text{ such that} \\ A^{-,\varepsilon}(s^\varepsilon,r^\varepsilon) + A^{+,\varepsilon}(s^\varepsilon,r^\varepsilon) + A^{m,\varepsilon}(s^\varepsilon,r^\varepsilon) = L^\varepsilon(r^\varepsilon), \end{cases} \tag{1}$$

for all $r^\varepsilon \in \mathbf{V}(\Omega^\varepsilon,\Gamma^\varepsilon_{mD}) \times V(\Omega^\varepsilon,\Gamma^\varepsilon_{eD})$, where $A^{\pm,\varepsilon}(\cdot,\cdot)$ and $A^{m,\varepsilon}(\cdot,\cdot)$ are defined by

$$\begin{aligned} A^{\pm,m,\varepsilon}(s^\varepsilon,r^\varepsilon) := \int_{\Omega^{\pm,m,\varepsilon}} &\Big\{ C^{ijk\ell,\varepsilon}_{\pm,m} e^\varepsilon_{k\|\ell}(\mathbf{u}^\varepsilon) e^\varepsilon_{i\|j}(\mathbf{v}^\varepsilon) + H^{ij,\varepsilon}_{\pm,m} E_j^\varepsilon(\varphi^\varepsilon) E_i^\varepsilon(\psi^\varepsilon) + \\ &+ P^{ihk,\varepsilon}_{\pm,m}(E_i^\varepsilon(\psi^\varepsilon) e^\varepsilon_{h\|k}(\mathbf{u}^\varepsilon) - E_i^\varepsilon(\varphi^\varepsilon) e^\varepsilon_{h\|k}(\mathbf{v}^\varepsilon)) \Big\} \sqrt{g^\varepsilon_{\pm,m}}\,\mathrm{d}x^\varepsilon. \end{aligned}$$

In order to study the asymptotic behavior of the solution to problem (1) when ε tends to zero, we rewrite the problem on a fixed domain $\Omega := \Omega^\pm \cup \Omega^m$, independent of ε, by using the classical change of variables as in [1,7], where $\Omega^\pm := \{x \pm h\mathbf{e}_3,\ x \in \Omega^\pm\}$, $\Omega^m := \omega \times (-h,h)$ and $S^\pm := \omega \times \{\pm h\}$.

We suppose that the electromechanical parameters of $\Omega^{\pm,\varepsilon}$ are independent of ε, so that $C^{ijk\ell,\varepsilon}_\pm := C^{ijk\ell}_\pm$, $P^{ijk,\varepsilon}_\pm := P^{ijk}_\pm$ and $H^{ij,\varepsilon}_\pm := H^{ij}_\pm$. While the constitutive coefficients associated with $\Omega^{m,\varepsilon}$ admit the following asymptotic behavior with respect to ε: $C^{ijk\ell}_m(\varepsilon)\sqrt{g^m(\varepsilon)} = \varepsilon^p C^{ijk\ell}_m(0)\sqrt{a} + O(\varepsilon^p)$, $P^{ijk}_m(\varepsilon)\sqrt{g^m(\varepsilon)} = \varepsilon^p P^{ijk}_m(0)\sqrt{a} + O(\varepsilon^p)$ and $H^{ij}_m(\varepsilon)\sqrt{g^m(\varepsilon)} = \varepsilon^p H^{ij}_m(0)\sqrt{a} + O(\varepsilon^p)$, with $a := \det(a_{\alpha\beta})$ and $p \in \{-1,1\}$. For an extensive treatment of the asymptotic analysis for shells and shell-like elastic inclusions, see [1,7].

Finally, we assume that $L^\varepsilon(r^\varepsilon) = L(r)$. According to the previous assumptions, problem (1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following scaled problem in curvilinear coordinates:

$$\begin{cases} \text{Find } s(\varepsilon) \in \mathbf{V}(\Omega,\Gamma_{mD}) \times V(\Omega,\Gamma_{eD}) \text{ such that} \\ A^-(s(\varepsilon),r) + A^+(s(\varepsilon),r) + A^m(s(\varepsilon),r) = L(r), \end{cases} \tag{2}$$

for all $r \in \mathbf{V}(\Omega,\Gamma_{mD}) \times V(\Omega,\Gamma_{eD})$, $p \in \{-1,1\}$, where

$$\begin{aligned} A^m(s(\varepsilon),r) := \int_{\Omega^m} &\Big\{ C^{ijk\ell}_m(\varepsilon) e_{k\|\ell}(\varepsilon;\mathbf{u}(\varepsilon)) e_{i\|j}(\varepsilon;\mathbf{v}) + \frac{1}{\varepsilon^2} H^{33}_m(\varepsilon)\partial_3\varphi(\varepsilon)\,\partial_3\psi \\ &+ \frac{1}{\varepsilon} H^{\alpha 3}_m(\varepsilon)(\partial_\alpha\varphi(\varepsilon)\partial_3\psi + \partial_3\varphi(\varepsilon)\partial_\alpha\psi) + H^{\alpha\beta}_m(\varepsilon)\partial_\alpha\varphi(\varepsilon)\,\partial_\beta\psi + \\ &+ P^{\alpha hk}_m(\varepsilon)(\partial_\alpha\varphi(\varepsilon) e_{h\|k}(\varepsilon;\mathbf{v}) - \partial_\alpha\psi e_{h\|k}(\varepsilon;\mathbf{u}(\varepsilon))) + \\ &+ \frac{1}{\varepsilon} P^{3hk}_m(\varepsilon)(\partial_3\varphi(\varepsilon) e_{h\|k}(\varepsilon;\mathbf{v}) - \partial_3\psi e_{h\|k}(\varepsilon;\mathbf{u}(\varepsilon))) \Big\} \sqrt{g_m(\varepsilon)}\,\mathrm{d}x, \end{aligned}$$

where $e_{\alpha\|\beta}(\varepsilon;\mathbf{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p$, $e_{\alpha\|3}(\varepsilon;\mathbf{v}) := \frac{1}{2}(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3) - \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma$ and $e_{3\|3}(\varepsilon;\mathbf{v}) := \frac{1}{\varepsilon}\partial_3 v_3$. We can now perform an asymptotic analysis of the rescaled problem (2) and distinguish the two cases of weak and strong piezoelectric interfaces. Since the rescaled problem (2) has a polynomial structure with respect to the small parameter ε, we can look for the solution to the problem as a series of powers of ε: $s(\varepsilon) = s^0 + \varepsilon s^1 + \varepsilon^2 s^2 + \ldots$, which implies that

$\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \varepsilon^2\mathbf{u}^2 + \ldots$ and $\varphi(\varepsilon) = \varphi^0 + \varepsilon\varphi^1 + \varepsilon^2\varphi^2 + \ldots$. By substituting the expression of the asymptotic development in (2) and by identifying the terms with identical power of ε, we can finally characterize the limit problems for $p = -1$ and $p = 1$.

3. The *strong* piezoelectric curved interface: the case $p = -1$

In this Section, we characterize the limit model for a strong piezoelectric interface. Let us define the following functionals spaces:

$$X(\tilde{\Omega}, \Sigma) := \{v \in H^1(\tilde{\Omega}),\ v|_\omega \in H^1(\omega)\ v = 0 \text{ on } \Sigma\},\ \mathbf{X}(\tilde{\Omega}, \Sigma) := [X(\tilde{\Omega}, \Sigma)]^3,\ \ \tilde{\Omega} :=:= \Omega^+ \cup \omega \cup \Omega^-.$$

If we choose $p = -1$ in (2), the formulation of the limit problem is stated in the following theorem.

Theorem 3.1. *The leading term $s^0 = (\mathbf{u}^0, \varphi^0)$ of the asymptotic expansion satisfies the following variational problem:*

$$\begin{cases} \text{Find } s^0 \in \mathbf{X}(\tilde{\Omega}, \Gamma_{mD}) \times X(\tilde{\Omega}, \Gamma_{eD}) \ \text{ such that} \\ A^-(s^0, r) + A^+(s^0, r) + \mathcal{A}^m(s^0, r) = L(r), \end{cases} \tag{3}$$

for all $r \in \mathbf{X}(\tilde{\Omega}, \Gamma_{mD}) \times X(\tilde{\Omega}, \Gamma_{eD})$, with

$$\mathcal{A}^m(s^0, r) := 2h \int_\omega \left\{ \left(\tilde{C}^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\mathbf{u}^0) + \tilde{P}^{\tau\alpha\beta} \partial_\tau \varphi^0 \right) \gamma_{\alpha\beta}(\mathbf{v}) + \left(-\tilde{P}^{\alpha\sigma\tau} \gamma_{\sigma\tau}(\mathbf{u}^0) + \tilde{H}^{\alpha\tau} \partial_\tau \varphi^0 \right) \partial_\alpha \psi \right\} \sqrt{a}\, \mathrm{d}\tilde{x},$$

where $\gamma_{\alpha\beta}(\mathbf{v}) := \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\tau_{\alpha\beta} v_\tau - b_{\alpha\beta} v_3$ denote the covariant components of the linearized change of metric tensor and $\tilde{C}^{\alpha\beta\sigma\tau}$, $\tilde{P}^{\tau\alpha\beta}$ and $\tilde{H}^{\alpha\tau}$ represent the reduced interface electromechanical moduli.

The variational limit problem results into a non-classical transmission problem between Ω^+ and Ω^- with ad hoc transmission conditions at the interface ω. This problem represents a piezoelectric curvilinear generalization of the Ventcel-type transmission conditions obtained for strong elastic and piezoelectric interfaces in [7,8]. After an integration by parts, we can rewrite problem (3) in its differential form:

Electrostatic problems	**Elasticity problems**	**Transmission conditions on ω**
$\partial_i D^i_\pm + \Gamma^p_{pi} D^i_\pm = \rho_e$ in $\Omega^\pm$,	$-\sigma^{ij}_\pm \Vert_j = f^i$ in $\Omega^\pm$,	$[\![\mathbf{u}^0]\!] = \mathbf{0}$, $[\![\varphi^0]\!] = 0$,
$D^i_\pm n_i = d$ on Γ_{eN},	$\sigma^{ij}_\pm n_j = g^i$ on Γ_{mN},	$[\![\sigma^{\alpha 3}]\!] = -\tilde{\sigma}^{\alpha\beta}\vert_\beta$,
$\varphi^0 = 0$ on Γ_{eD},	$\mathbf{u}^0 = \mathbf{0}$ on Γ_{mD},	$[\![\sigma^{33}]\!] = -b_{\alpha\beta}\tilde{\sigma}^{\alpha\beta}$,
$\tilde{D}^\alpha \nu_\alpha = 0$ on γ_{eN},	$\tilde{\sigma}^{\alpha i} \nu_\alpha = 0$ on γ_{mN},	$[\![D^3]\!] = -(\partial_\alpha \tilde{D}^\alpha + \Gamma^\tau_{\tau\alpha} \tilde{D}^\alpha)$,

where $\tilde{\sigma}^{\alpha\beta} := \tilde{C}^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\mathbf{u}^0) + \tilde{P}^{\tau\alpha\beta}\partial_\tau\varphi^0$ and $\tilde{D}^\alpha := \tilde{P}^{\alpha\sigma\tau}\gamma_{\sigma\tau}(\mathbf{u}^0) - \tilde{H}^{\alpha\tau}\partial_\tau\varphi^0$ represent, respectively, the two-dimensional membrane stresses and electric displacement, $[\![f]\!] := f|_{S^+} - f|_{S^-} = f_+ - f_-$ represents the jump function of f at the interface between Ω^+ and Ω^- and (ν_α) denotes the unit normal vector to the uncharged electromechanical boundaries γ_{eN}, $\gamma_{mN} \subset \partial\omega$. The limit model for a strong piezoelectric interface implies the continuity of the limit state $s^0 = (\mathbf{u}^0, \varphi^0)$ across the interface. At the same time, it provides that the normal and shear stresses and the normal electric displacement with respect to the tangent plane of the interface are discontinuous: their jumps depend on the two-dimensional curvilinear surface divergence of the membrane stresses and electric displacements. This is a classical feature for what concerns with *mechanically hard* and *highly conducting* interface models, see, e.g., [7,10].

It is interesting to notice that, by considering some particular material symmetry groups for the interphase material, the elastic and electric behaviors of the interface are completely decoupled and, hence, the interface limit model does not show any piezoelectric coupling. For instance, in the case of a monoclinic material of class 2 with unique poling direction in x_3, which is compatible with the material symmetry group of a shell-like structure, the electromechanical coefficients $C^{\alpha\beta\sigma 3}_m(0) = C^{\alpha 333}_m(0) = P^{\alpha\beta\sigma}_m(0) = P^{\alpha 33}_m(0) = P^{3\alpha 3}_m(0) = H^{\alpha 3}_m(0)$ vanish and, hence, the reduced constitutive parameters $\mathcal{P}^{\tau\alpha\beta} = 0$. The bilinear form takes the following simple form

$$\mathcal{A}^m(s^0, r) := 2h \int_\omega \left\{ \tilde{C}^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\mathbf{u}^0) \gamma_{\alpha\beta}(\mathbf{v}) + \tilde{H}^{\alpha\tau} \partial_\tau \varphi^0 \partial_\alpha \psi \right\} \sqrt{a}\, \mathrm{d}\tilde{x},$$

which is associated with a two-dimensional membrane elastic energy and two-dimensional electric surface energy, with no piezoelectric coupling.

4. The *weak* piezoelectric curved interface problem: the case $p = 1$

In this Section, we characterize the limit model for a weak piezoelectric interface. We define the following functional spaces:

$$W(\Omega,\Sigma) := \{v \in L^2(\Omega);\ v^{\pm} \in H^1(\Omega^{\pm}),\ v = 0 \text{ on } \Sigma\},\quad \mathbf{W}(\Omega,\Sigma) := [W(\Omega,\Sigma)]^3.$$

Let us choose $p = 1$ in (2), the formulation of the limit problem is stated in the following theorem.

Theorem 4.1. *The leading term $s^0 = (\mathbf{u}^0, \varphi^0)$ of the asymptotic expansion satisfies the following variational problem:*

$$\begin{cases} \text{Find } s^0 \in \mathbf{W}(\Omega,\Gamma_{mD}) \times W(\Omega,\Gamma_{eD}) \text{ such that} \\ A^-(s^0,r) + A^+(s^0,r) + a^m(s^0,r) = L(r), \end{cases} \tag{4}$$

for all $r \in \mathbf{W}(\Omega,\Gamma_{mD}) \times W(\Omega,\Gamma_{eD})$, with

$$a^m(s^0,r) := \frac{1}{2h}\int_\omega \left\{ \left(C_m^{i3j3}(0)[\![u_j^0]\!] + P_m^{3i3}(0)[\![\varphi^0]\!] \right)[\![v_i]\!] + \left(H_m^{33}(0)[\![\varphi^0]\!] - P_m^{3i3}(0)[\![u_i^0]\!] \right)[\![\psi]\!] \right\} \sqrt{a}\,\mathrm{d}\tilde{x}.$$

Thanks to the asymptotic analysis, we transform the limit problem onto a coupled electromechanical interface problem between Ω^+ and Ω^-, with non-classical transmission conditions at the interface ω. This problem represents a piezoelectric curvilinear generalization of the weak linear elastic and piezoelectric interfaces obtained in [4,8]. We rewrite problem (4) in its differential form and we obtain:

Electrostatic problems	**Elasticity problems**	**Transmission conditions on ω**
$\begin{cases} \partial_i D^i_\pm + \Gamma^p_{pi} D^i_\pm = \rho_e & \text{in } \Omega^\pm, \\ D^i_\pm n_i = d & \text{on } \Gamma_{eN}, \\ \varphi^0 = 0 & \text{on } \Gamma_{eD}, \end{cases}$	$\begin{cases} -\sigma^{ij}_\pm \Vert_j = f^i & \text{in } \Omega^\pm, \\ \sigma^{ij}_\pm n_j = g^i & \text{on } \Gamma_{mN}, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_{mD}, \end{cases}$	$\begin{cases} \sigma^{i3}_\pm = -\frac{1}{2h}\left(C_m^{i3j3}(0)[\![u_j^0]\!] + P_m^{3i3}(0)[\![\varphi^0]\!] \right), \\ D^3_\pm = \frac{1}{2h}\left(H_m^{33}(0)[\![\varphi^0]\!] - P_m^{3i3}(0)[\![u_i^0]\!] \right). \end{cases}$

The limit model for a piezoelectric curved interface provides a discontinuity of the limit state $s^0 = (\mathbf{u}^0, \varphi^0)$ at the interface between Ω^+ and Ω^-. Besides, subtracting two by two the transmission conditions above, we obtain that the jump of the normal and shear stresses and the jump of the normal electric displacement, relatively to the plane of the interface ω, vanish, so that $[\![\sigma^{i3}]\!] = [\![D^3]\!] = 0$. This feature is common for what concerns *mechanically soft* and *lowly conducting* interface models, see, e.g., [4,5,10].

In the case of a middle layer constituted by a monoclinic class-2 material with a unique poling direction in x_3, since the electromechanical coefficients $C_m^{\alpha\beta\sigma3}(0) = C_m^{\alpha333}(0) = P_m^{\alpha\beta\sigma}(0) = P_m^{\alpha33}(0) = P_m^{3\alpha3}(0) = H_m^{\alpha3}(0) = 0$, the bilinear form related to the interface problem reduces to

$$\begin{aligned} a^m(s^0,r) := \frac{1}{2h}\int_\omega & \Big\{ C_m^{\alpha3\beta3}(0)[\![u_\alpha^0]\!][\![v_\beta]\!] + \left(C_m^{3333}(0)[\![u_3^0]\!] + P_m^{333}(0)[\![\varphi^0]\!] \right)[\![v_3]\!] + \\ & + \left(-P_m^{333}(0)[\![u_3^0]\!] + H_m^{33}(0)[\![\varphi^0]\!] \right)[\![\psi]\!] \Big\} \sqrt{a}\,\mathrm{d}\tilde{x}. \end{aligned}$$

In this particular case, the elastic membrane behavior of the interface is completely decoupled from the piezoelectric transversal behavior: from a mechanical point of view, the interphase layer is replaced, respectively, by a distribution of elastic shear springs reacting to the gap of the membrane displacements and by a distribution of piezoelectric transversal springs reacting to the jump of the transversal displacements and electric potential between the top and bottom faces.

5. Concluding remarks

In the present work, we derive two limit interface models corresponding to a generic piezoelectric composite with a piezoelectric curved interphase through an asymptotic analysis. We analyze two particular cases: the first case, for $p = 1$, corresponding from a mechanical point of view to a soft weakly conducting piezoelectric interphase, leads to a *weak* curved interface model; the latter, for $p = -1$, corresponding to a rigid highly conducting interphase into two piezoelectric media, leads to an *strong* curved interface model. In both cases, the interphase is replaced by a particular surface energy, which is associated with ad hoc transmission conditions at the interface of the two bodies.

As future developments, we would like to study more complex curved interface problems taking into account thermo-electromagnetoelastic couplings and time-dependent phenomena as in [9].

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