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Tensorial Polar Decomposition of 2D fourth-order tensors

Décomposition polaire tensorielle des tenseurs 2D d'ordre 4

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A B S T R A C T

One studies the structure of 2D symmetric fourth-order tensors, *i.e.* having both minor and major indicial symmetries. Verchery polar decomposition is rewritten in a tensorial form entitled Tensorial Polar Decomposition. The main result is that any 2D symmetric fourth-order tensor can be written in terms of second-order tensors only in a decomposition that makes explicitly appear invariants and symmetry classes. The link with harmonic decomposition is made thanks to Kelvin decomposition of its harmonic term.

R É S U M É

On étudie la structure des tenseurs 2D symétriques d'ordre 4, c'est-à-dire : ayant aussi bien la symétrie indicielle mineure que la symétrie majeure. La décomposition polaire de Verchery est réécrite sous forme tensorielle nommée décomposition polaire tensorielle. Le résultat principal est que tout tenseur 2D symétrique d'ordre 4 peut s'écrire à l'aide de tenseurs d'ordre 2 uniquement dans une décomposition faisant apparaître explicitement les invariants et les classes de symétrie. Le lien avec la décomposition harmonique est fait en utilisant la décomposition de Kelvin de son terme harmonique.

0. Introduction

The structure of 3D fourth-order elasticity tensor has been intensively studied since 19th-century controversy on the number of independent elasticity constants. Major and minor symmetries reduce to 21 the number of material parameters (symmetric tensors $\mathbb{T}$ referred to as multi-constant tensors), when an elasticity tensor having all Cauchy indicial symmetries $\mathrm{T}_{ijkl} = \mathrm{T}_{ikjl}$ only has 15 material parameters (supersymmetric or rari-constant tensors).

A well-known tool for the study of symmetry classes is the isomorphic harmonic decomposition $2\mathbb{H}^0 \oplus 2\mathbb{H}^2 \oplus \mathbb{H}^4$ of symmetric tensors space [1–4], defining a scalar (real) space as $\mathbb{H}^0$, second-order harmonic tensors $\mathbf{h} \in \mathbb{H}^2$ as traceless (deviatoric, $\sum_k h_{kk} = 0$) symmetric tensors and fourth-order harmonic tensors $\mathbb{H} \in \mathbb{H}^4$ as traceless supersymmetric/rari-

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constant tensors ($\mathrm{H}_{ijkl} = \mathrm{H}_{ikjl}$, $\sum_k \mathrm{H}_{kkij} = \sum_k \mathrm{H}_{kikj} = 0$). In other words, any symmetric tensor $\mathbb{T}$, such as a triclinic elasticity tensor, can be represented by two Lamé isotropic constants $\in \mathbb{H}^0$, by two second-order harmonic tensors $\in \mathbb{H}^2$ and by one fourth-order harmonic tensor $\in \mathbb{H}^4$.

In 2D, some simplifications arise as scalar expressions for the components $\mathrm{T}_{ijkl}(\theta)$ of symmetric tensor $\mathbb{T}$ (having both minor and major symmetries) may be derived by making explicitly appear the dependency upon frame angle θ [5] and upon invariants [6–9]. Theses expressions do not have a complete tensorial counterpart in the literature [10].

In the present note, we therefore propose a tensorial rewriting and an associated interpretation of Verchery polar decomposition for 2D fourth-order tensors with both minor and major indicial symmetries. It is shown in Section 2 that any 2D symmetric tensor $\mathbb{T}$ (resp. any 2D harmonic fourth-order tensor $\mathbb{H} \in \mathbb{H}^{4(2D)}$) can be expressed by means of 2 scalar invariants and of 2 second-order deviatoric tensors $\in \mathbb{H}^{2(2D)}$ (respectively of only one second-order deviatoric tensor $\mathbf{h}_0 \in \mathbb{H}^{2(2D)}$). The link with harmonic decomposition is made in Section 4. The general tensorial expression of harmonic elements $\in \mathbb{H}^{4(2D)}$ is retrieved in Section 5 by the use of the Kelvin decomposition.

Tensorial products $\underline{\otimes}$, $\overline{\otimes}$, $\underline{\overline{\otimes}}$ will be used. They are defined as follows: $(\mathbf{X}\underline{\otimes}\mathbf{Y})_{ijkl} = X_{ik}Y_{jl}$, $(\mathbf{X}\overline{\otimes}\mathbf{Y})_{ijkl} = X_{il}Y_{jk}$, $\mathbf{X}\underline{\overline{\otimes}}\mathbf{Y} = \frac{1}{2}(\mathbf{X}\underline{\otimes}\mathbf{Y} + \mathbf{X}\overline{\otimes}\mathbf{Y})$.

1. 2D quadratic form using the polar formalism

Let us consider any 2D fourth-order tensor $\mathbb{T}$ with minor and major symmetries. In the polar formalism [6,7], five invariants are defined; four out of them are elastic moduli (t_0, t_1, r_0, r_1), and the last one is the angular difference $\varphi_0 - \varphi_1$ (each φ_n is not an invariant by itself, see the discussion on joint invariant at the end of Section 2). A basic result of the polar formalism is the expression of the Cartesian components of $\mathbb{T}$ in terms of polar parameters, in a frame rotated of an angle θ:

$$
\begin{aligned}
&\mathrm{T}_{1111}(\theta) = t_0 + 2t_1 + r_0 \cos 4(\varphi_0 - \theta) + 4r_1 \cos 2(\varphi_1 - \theta) \\
&\mathrm{T}_{1112}(\theta) = r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta) \\
&\mathrm{T}_{1122}(\theta) = -t_0 + 2t_1 - r_0 \cos 4(\varphi_0 - \theta) \\
&\mathrm{T}_{1212}(\theta) = t_0 - r_0 \cos 4(\varphi_0 - \theta) \\
&\mathrm{T}_{1222}(\theta) = -r_0 \sin 4(\varphi_0 - \theta) + 2r_1 \sin 2(\varphi_1 - \theta) \\
&\mathrm{T}_{2222}(\theta) = t_0 + 2t_1 + r_0 \cos 4(\varphi_0 - \theta) - 4r_1 \cos 2(\varphi_1 - \theta)
\end{aligned} \tag{1}
$$

t_0 and t_1 terms are frame independent (they define the isotropic part of $\mathbb{T}$ from a generalization of Lamé constants to anisotropy), r_1 terms rotates in $\cos 2(\varphi_1 - \theta)$ and $\sin 2(\varphi_1 - \theta)$ as second-order tensors do (Eq. (2)), the r_0 term rotates twice more in $\cos 4(\varphi_0 - \theta)$ and $\sin 4(\varphi_0 - \theta)$. In a given frame θ, the knowledge of the six independent coefficients of any 2D symmetric tensor $\mathbb{T}$ is equivalent to the knowledge of the five invariants $(t_0, t_1, r_0, r_1, \varphi_0 - \varphi_1)$ and of one angle, either $\varphi_0 - \theta$ or $\varphi_1 - \theta$.

Still in 2D, a general expression for any symmetric second-order tensor $\mathbf{s}$, making appear explicitly frame angle θ, is

$$
\mathbf{s} = s_\mathrm{m}\mathbf{1} + \mathbf{s}' = s_\mathrm{m}\mathbf{1} + s_\mathrm{eq}\begin{bmatrix} \cos 2(\varphi - \theta)) & \sin 2(\varphi - \theta) \\ \sin 2(\varphi - \theta) & -\cos 2(\varphi - \theta) \end{bmatrix} \quad \text{with} \quad \begin{cases} s_\mathrm{m} = \frac{1}{2}\operatorname{tr}\mathbf{s} \\ s_\mathrm{eq} = \sqrt{\frac{1}{2}\mathbf{s}' : \mathbf{s}'} \end{cases} \tag{2}
$$

with first (mean) and second (2D von Mises) invariants defined as s_m and s_eq and where φ is the orientation of principal basis of $\mathbf{s}$ (it is not an invariant of $\mathbf{s}$). The expression of the associated quadratic form is:

$$
\frac{1}{2}\mathbf{s} : \mathbb{T} : \mathbf{s} = 2t_0 s_\mathrm{eq}^2 + 4t_1 s_\mathrm{m}^2 + 2r_0 s_\mathrm{eq}^2 \cos 4(\varphi_0 - \varphi) + 8r_1 s_\mathrm{m} s_\mathrm{eq} \cos 2(\varphi_1 - \varphi) \tag{3}
$$

Explicit formulae giving polar invariants as a function of components T_{ijkl} can be found in [7].

2. Proposed Tensorial Polar Decomposition

Introducing the two second-order deviatoric tensors $\mathbf{R}_0$, $\mathbf{R}_1$,

$$
\mathbf{R}_0 = \mathbf{R}_0' = \begin{bmatrix} \cos 2(\varphi_0 - \theta) & \sin 2(\varphi_0 - \theta) \\ \sin 2(\varphi_0 - \theta) & -\cos 2(\varphi_0 - \theta) \end{bmatrix} \qquad \mathbf{R}_1 = \mathbf{R}_1' = \begin{bmatrix} \cos 2(\varphi_1 - \theta) & \sin 2(\varphi_1 - \theta) \\ \sin 2(\varphi_1 - \theta) & -\cos 2(\varphi_1 - \theta) \end{bmatrix} \tag{4}
$$

of 2D von Mises equivalent norm $R_{0\,\mathrm{eq}} = R_{1\,\mathrm{eq}} = 1$, and of principal direction φ_0, φ_1, of course possibly different from principal direction φ of tensor $\mathbf{s}$. One has first equalities concerning r_1-term,

$$
(\mathbf{s} : \mathbf{R}_1')\operatorname{tr}\mathbf{s} = \operatorname{tr}(\mathbf{s} \cdot \mathbf{R}_1' \cdot \mathbf{s}) = 4 s_\mathrm{m} s_\mathrm{eq} \cos 2(\varphi_1 - \varphi) \tag{5}
$$

From $(\mathbf{s}:\mathbf{R}_0')^2 = 4s_{\text{eq}}^2\cos^2 2(\varphi_0-\varphi) = 2s_{\text{eq}}^2\,(1+\cos 4(\varphi_0-\varphi))$ and $2s_{\text{eq}}^2 = \mathbf{s}':\mathbf{s}'$ second equality concerning r_0-term is:

$$2r_0 s_{\text{eq}}^2 \cos 4\,(\varphi_0-\varphi) = r_0\left[\left(\mathbf{s}:\mathbf{R}_0'\right)^2 - \mathbf{s}':\mathbf{s}'\right] \tag{6}$$

The quadratic form (3) can therefore be rewritten into the following intrinsic form

$$\frac{1}{2}\mathbf{s}:\mathbb{T}:\mathbf{s} = t_0\,\mathbf{s}':\mathbf{s}' + t_1\,(\operatorname{tr}\mathbf{s})^2 + r_0\left[\left(\mathbf{s}:\mathbf{R}_0'\right)^2 - \mathbf{s}':\mathbf{s}'\right] + 2r_1\operatorname{tr}\left(\mathbf{s}\cdot\mathbf{R}_1'\cdot\mathbf{s}\right) \tag{7}$$

From the last equation, the intrinsic form of the polar decomposition of a symmetric fourth-order tensor $\mathbb{T}$ is obtained in terms of polar invariants t_0, t_1, r_0 and r_1 and of the two second-order deviatoric tensors $\mathbf{R}_0'$ and $\mathbf{R}_1'$,

$$\mathbb{T} = 2t_0\mathbb{J} + 2t_1\mathbf{1}\otimes\mathbf{1} + 2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0' - \mathbb{J}\right] + 2r_1\left(\mathbf{1}\,\overline{\underline{\otimes}}\,\mathbf{R}_1' + \mathbf{R}_1'\,\overline{\underline{\otimes}}\,\mathbf{1}\right) \tag{8}$$

It is equivalent in the present 2D case to

$$\mathbb{T} = 2t_0\mathbb{J} + 2t_1\mathbf{1}\otimes\mathbf{1} + 2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0' - \mathbb{J}\right] + 2r_1\left(\mathbf{1}\otimes\mathbf{R}_1' + \mathbf{R}_1'\otimes\mathbf{1}\right) \tag{9}$$

thanks to the mathematical property (5) valid $\forall\mathbf{s}$, which implies:

$$\mathbf{1}\,\overline{\underline{\otimes}}\,\mathbf{R}_1' + \mathbf{R}_1'\,\overline{\underline{\otimes}}\,\mathbf{1} = \mathbf{1}\otimes\mathbf{R}_1' + \mathbf{R}_1'\otimes\mathbf{1} \tag{10}$$

Eq. (10) is not intrinsic to tensorial products, it stands only in 2D. The tensor $\mathbb{J} = \mathbb{I} - \frac{1}{2}\mathbf{1}\otimes\mathbf{1}$ (defined here in 2D) takes the deviatoric part of any second-order tensor $\mathbf{X}$ (*i.e.* $\mathbb{J}:\mathbf{X} = \mathbf{X}'$).

Equations (8)–(9) define the Tensorial Polar Decomposition of any 2D tensor $\mathbb{T}$ having both minor and major symmetries. As both r_0- and r_1-terms are found rari-constant, the rari-constancy $\mathrm{T}_{ijkl} = \mathrm{T}_{ikjl}$ amounts to $t_0 = t_1$.

Note that joint invariant $\mathbf{R}_0':\mathbf{R}_1'$ reads

$$\mathbf{R}_0':\mathbf{R}_1' = 2\cos 2\,(\varphi_0 - \varphi_1) \tag{11}$$

It is an invariant of tensor $\mathbb{T}$, as is the polar angular invariant $\varphi_0 - \varphi_1$.

The intrinsic form of the polar decomposition makes explicitly appear polar moduli and angles, therefore the material symmetries, including ordinary orthotropies $\varphi_0 - \varphi_1 = k\,\frac{\pi}{4}$, $k\in\{0,1\}$ [7]. For instance, if $\mathbb{T}$ is a 2D elasticity tensor, isotropy is $r_0 = r_1 = 0$, the square symmetry is $r_1 = 0$, the r_0-orthotropy is $r_0 = 0$, the ordinary orthotropy, with $k = 0$ is $\mathbf{R}_0' = \mathbf{R}_1'$ and the ordinary orthotropy with $k = 1$ is $\mathbf{R}_0':\mathbf{R}_1' = 0$.

3. Orthogonality of generators

Tensorial Polar Decomposition (9) can be recast as the sum of polar moduli $2g_n$ times generators $\mathbb{G}^{(n)}$, which are fourth-order tensors (factors 2 appear for consistency with original Verchery work, polar moduli g_n standing either for t_n or for r_n),

$$\mathbb{T} = \sum_0^3 2g_n\mathbb{G}^{(n)} = \sum_1^2 2t_n\mathbb{G}_t^{(n)} + \sum_1^2 2r_n\mathbb{G}_r^{(n)} \tag{12}$$

Fourth-order generator tensors $\mathbb{G}^{(n)}$ are of two kinds: the $\mathbb{G}_t^{(n)}$ are definite positive and do not depend upon frame orientation θ, while the $\mathbb{G}_r^{(n)} = \mathbb{G}_r^{(n)}(\theta)$ are frame dependent:

$$\mathbb{G}_t^{(0)} = \mathbb{J}\,, \qquad \mathbb{G}_t^{(1)} = \mathbf{1}\otimes\mathbf{1}\,, \qquad \mathbb{G}_r^{(0)} = \mathbf{R}_0'\otimes\mathbf{R}_0' - \mathbb{J}\,, \qquad \mathbb{G}_r^{(1)} = \mathbf{1}\otimes\mathbf{R}_1' + \mathbf{R}_1'\otimes\mathbf{1} \tag{13}$$

The generators are orthogonal with respect to the scalar product :: as

$$\mathbb{G}^{(n)}::\mathbb{G}^{(m)} = \sum_{i=1}^2\sum_{j=1}^2\sum_{k=1}^2\sum_{l=1}^2 G_{ijkl}^{(n)}\,G_{ijkl}^{(m)} = 0 \quad \forall\, m\neq n \tag{14}$$

They all have a constant norm, frame independent, as

$$\mathbb{G}_t^{(0)}::\mathbb{G}_t^{(0)} = 2 \qquad \mathbb{G}_t^{(1)}::\mathbb{G}_t^{(1)} = 4 \qquad \mathbb{G}_r^{(0)}::\mathbb{G}_r^{(0)} = 2 \qquad \mathbb{G}_r^{(1)}::\mathbb{G}_r^{(1)} = 8 \tag{15}$$

4. Link with harmonic decomposition

In 3D there are only two independent traces $\mathbf{d} = \operatorname{tr}_{12}\mathbb{T} = \operatorname{tr}_{34}\mathbb{T}$ (of components $\sum_{k=1}^3 \mathrm{T}_{kkij}$) and $\mathbf{v} = \operatorname{tr}_{13}\mathbb{T} = \operatorname{tr}_{23}\mathbb{T} = \operatorname{tr}_{14}\mathbb{T} = \operatorname{tr}_{24}\mathbb{T}$ (of components $\sum_{k=1}^3 \mathrm{T}_{kikj}$) for the symmetric tensor $\mathbb{T}$. The symmetric second-order tensor $\mathbf{d}$ is dilatation tensor, of deviatoric part $\mathbf{d}'$, the symmetric second-order tensor $\mathbf{v}$ is the Voigt tensor, of deviatoric part $\mathbf{v}'$. The 3D harmonic decomposition $2\mathbb{H}^0\oplus 2\mathbb{H}^2\oplus\mathbb{H}^4$ of fourth-order tensor vector space reads then [2–4]:

$$\mathbb{T} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\,\mathbf{1}\,\overline{\underline{\otimes}}\,\mathbf{1} + \mathbf{1}\otimes\mathbf{h}_1 + \mathbf{h}_1\otimes\mathbf{1} + \mathbf{1}\,\underline{\otimes}\,\mathbf{h}_2 + \mathbf{h}_2\,\underline{\otimes}\,\mathbf{1} + \mathbf{1}\,\overline{\otimes}\,\mathbf{h}_2 + \mathbf{h}_2\,\overline{\otimes}\,\mathbf{1} + \mathbb{H} \tag{16}$$

or in an equivalent manner

$$\mathbb{T}=\lambda\mathbf{1}\otimes\mathbf{1}+2\mu\,\mathbb{I}+\mathbf{1}\otimes\mathbf{h}_1+\mathbf{h}_1\otimes\mathbf{1}+2\left(\mathbf{1}\overline{\underline{\otimes}}\mathbf{h}_2+\mathbf{h}_2\overline{\underline{\otimes}}\mathbf{1}\right)+\mathbb{H} \tag{17}$$

with as constants $\lambda=\frac{1}{30}(4\operatorname{tr}\mathbf{d}-2\operatorname{tr}\mathbf{v})$ and $\mu=\frac{1}{30}(3\operatorname{tr}\mathbf{v}-\operatorname{tr}\mathbf{d})$, as traceless symmetric second-order tensor $\mathbf{h}_1=\mathbf{h}_1'=\frac{1}{7}(5\mathbf{d}'-4\mathbf{v}')\in\mathbb{H}^2$ and $\mathbf{h}_2=\mathbf{h}_2'=\frac{1}{7}(3\mathbf{v}'-\mathbf{d}')\in\mathbb{H}^2$ and as traceless rari-constant tensor $\mathbb{H}\in\mathbb{H}^4$.

In 2D (see [10]), in a consistent manner with mathematical property (10) and 2D equality $\mathbf{v}'=\mathbf{d}'$, if one still sets $\mathbf{d}=\operatorname{tr}_{12}\mathbb{T}$, $\mathbf{v}=\operatorname{tr}_{13}\mathbb{T}$ of components $d_{ij}=\sum_{k=1}^{2}\mathrm{T}_{kkij}$ and $v_{ij}=\sum_{k=1}^{2}\mathrm{T}_{kikj}$, the harmonic decomposition of fourth-order tensor vector space reads $2\mathbb{H}^0\oplus\mathbb{H}^{2(2D)}\oplus\mathbb{H}^{4(2D)}$ or

$$\mathbb{T}=\lambda\mathbf{1}\otimes\mathbf{1}+2\mu\,\mathbb{I}+\mathbf{1}\otimes\mathbf{h}+\mathbf{h}\otimes\mathbf{1}+\mathbb{H} \tag{18}$$

with as 2D constants $\lambda=\frac{1}{2}(\operatorname{tr}\mathbf{d}-\operatorname{tr}\mathbf{v})$ and $\mu=\frac{1}{2}(2\operatorname{tr}\mathbf{v}-\operatorname{tr}\mathbf{d})$, as 2D the harmonic tensors $\mathbf{h}=\mathbf{d}'/2=\mathbf{v}'/2\in\mathbb{H}^{2(2D)}$ and $\mathbb{H}\in\mathbb{H}^{4(2D)}$. One easily recognizes the constant and linear terms of Tensorial Polar Decomposition (9), using $\mathbb{I}=\mathbb{J}+\frac{1}{2}\mathbf{1}\otimes\mathbf{1}$, with

$$t_0=\mu\,,\qquad t_1=\frac{\lambda+\mu}{2}\,,\qquad 2r_1=h_{\mathrm{eq}}\,,\qquad 2r_1\mathbf{R}_1'=\mathbf{h} \tag{19}$$

The harmonic $\mathbb{H}^{4(2D)}$-term is given in explicit form in Section 2 thanks to polar decomposition by means of an extra traceless second-order tensor $\mathbf{h}_0=\mathbf{h}_0'=\sqrt{2r_0}\,\mathbf{R}_0'\in\mathbb{H}^{2(2D)}$ as

$$\mathbb{H}=2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0'-\mathbb{J}\right]=\mathbf{h}_0\otimes\mathbf{h}_0-\frac{1}{2}\mathbf{h}_0:\mathbf{h}_0\,\mathbb{J}\qquad \operatorname{tr}_{12}\mathbb{H}=\operatorname{tr}_{13}\mathbb{H}=0 \tag{20}$$

This shows that the Tensorial Polar Decomposition of 2D symmetric fourth-order tensors is the direct sum $2\mathbb{H}^0\oplus 2\mathbb{H}^{2(2D)}$.

We propose in next section to use the Kelvin decomposition in order to derive the explicit r_0-form of $\mathbb{H}$ and to prove that $r_0\geq 0$, as needed.

5. Retrieving the explicit r_0-form of $\mathbb{H}\in\mathbb{H}^{4(2D)}$

The harmonic fourth-order tensor $\mathbb{H}$ introduced in the previous section is

$$\mathbb{H}=\mathbb{T}-\lambda\mathbf{1}\otimes\mathbf{1}-2\mu\,\mathbb{I}-\mathbf{1}\otimes\mathbf{h}-\mathbf{h}\otimes\mathbf{1} \tag{21}$$

Let us use its harmonic properties $\operatorname{tr}_{12}\mathbb{H}=\operatorname{tr}_{13}\mathbb{H}=0$ and the remark that they correspond to the orthogonality of generator $\mathbb{G}_r^{(0)}$ with respect to both constant generators $\mathbb{G}_t^{(0)}=\mathbb{J}$ and $\mathbb{G}_t^{(1)}=\mathbf{1}\otimes\mathbf{1}$.

The Kelvin (spectral) decomposition of $\mathbb{H}$ [11–14,8], gives, here in 2D,

$$\mathbb{H}=\sum_{I=0}^{2}\Lambda_I\mathbf{e}^I\otimes\mathbf{e}^I\qquad \mathbf{e}^I:\mathbf{e}^J=\delta_{IJ} \tag{22}$$

with $\mathbb{H}:\mathbf{e}^I=\Lambda_I\mathbf{e}^I$ (no sum) defining Kelvin's moduli Λ_I and modes $\mathbf{e}^I$. The traceless condition $\mathbb{H}:\mathbf{1}=\mathbf{1}:\mathbb{H}=\operatorname{tr}_{12}\mathbb{H}=0$ implies that $\mathbf{1}$ is an eigentensor (a Kelvin mode) of $\mathbb{H}$, associated with Kelvin's modulus $\Lambda_2=0$, and that the first two eigentensors $\mathbf{e}^I$ are deviatoric, $\mathbf{e}^I=\mathbf{e}^{I\prime}$. The mathematical property of Kelvin's projectors to give a partition of the unit tensor reads then

$$\mathbf{e}^{1\prime}\otimes\mathbf{e}^{1\prime}=\mathbb{I}-\frac{1}{2}\mathbf{1}\otimes\mathbf{1}-\mathbf{e}^{0\prime}\otimes\mathbf{e}^{0\prime}=\mathbb{J}-\mathbf{e}^{0\prime}\otimes\mathbf{e}^{0\prime} \tag{23}$$

so that Kelvin's decomposition (22) becomes

$$\mathbb{H}=(\Lambda_0-\Lambda_1)\,\mathbf{e}^{0\prime}\otimes\mathbf{e}^{0\prime}+\Lambda_1\mathbb{J} \tag{24}$$

By construction, $\mathbb{H}$ is orthogonal to generator $\mathbb{G}_t^{(0)}=\mathbb{J}=\mathbb{I}-\frac{1}{2}\mathbf{1}\otimes\mathbf{1}$. This gives:

$$\begin{aligned}\mathbb{H}::\mathbb{J}&=(\Lambda_0-\Lambda_1)\,\mathbf{e}^{0\prime}\otimes\mathbf{e}^{0\prime}::\mathbb{J}+\Lambda_1\mathbb{J}::\mathbb{J}=0\\&=(\Lambda_0-\Lambda_1)\,\mathbf{e}^{0\prime}:\mathbf{e}^{0\prime}+2\Lambda_1=(\Lambda_0-\Lambda_1)+2\Lambda_1=0\end{aligned} \tag{25}$$

This shows that $\Lambda_1=-\Lambda_0$ so that one just has proven that any $\mathbb{H}\in\mathbb{H}^{4(2D)}$ has for expression

$$\mathbb{H}=\Lambda_0\left[2\mathbf{e}^{0\prime}\otimes\mathbf{e}^{0\prime}-\mathbb{J}\right] \tag{26}$$

Setting $\mathbf{R}_0'=\sqrt{2}\,\mathbf{e}^{0\prime}$ as deviatoric second-order tensor of equivalent norm $R_{0\,\mathrm{eq}}=1$, ends up to

$$\mathbb{H}=2r_0\left[\mathbf{R}_0'\otimes\mathbf{R}_0'-\mathbb{J}\right]\qquad r_0=\frac{\Lambda_0}{2} \tag{27}$$

There are two possibilities for the definition of tensor $\mathbf{R}'_0$ and of modulus r_0 as there are two Kelvin modes $I = 0$ and $I = 1$ orthogonal to Kelvin's mode $\mathbf{e}^2 = \mathbf{1}/\sqrt{2}$. Only the one at positive eigenvalue, set as $I = 0$, $\Lambda_0 \geq 0$ (leaving then $\Lambda_1 \leq 0$ for $I = 1$) gives a positive r_0 as retained in standard polar decomposition of 2D symmetric tensors and as needed at the end of previous section. The polar modulus $r_0 = \Lambda_0/2 \geq 0$ is therefore shown to be a half-positive eigenvalue of the harmonic fourth-order tensor $\mathbb{H}$ and $\mathbf{R}'_0$ is the associated Kelvin mode multiplied by $\sqrt{2}$.

Altogether with expression (21) due to harmonic decomposition, the present derivations (and key Eq. (27)) are an alternate proof of Verchery polar decomposition, using tensorial mathematical tools instead of a complex variable method in case of original proof.

6. Conclusion

We have proposed a tensorial intrinsic form for Verchery polar decomposition of any 2D fourth-order symmetric tensor $\mathbb{T}$. Two proofs are given, a first one from the rewriting of quadratic form (3) associated with tensor $\mathbb{T}$, a second one combining both harmonic and Kelvin decompositions.

Compared to harmonic decomposition, the main results are:

- the generators obtained are found orthogonal to each other (in sense of scalar product :: for fourth-order tensors) and of constant norm, independent from frame angle,
- the polar invariants of tensor $\mathbb{T}$ explicitly appear, making easy the study of symmetry classes and sub-classes,
- the structure of the harmonic fourth-order tensor $\mathbb{H} \in \mathbb{H}^{4(2D)}$ is given: any traceless rari-constant (harmonic) tensor $\mathbb{H}$ is shown to be expressed thanks to a single deviatoric (harmonic) second-order tensor $\mathbf{h}_0$ or in an equivalent manner in the polar formalism thanks to the polar invariant r_0 and to the deviatoric tensor $\mathbf{R}'_0$ of the unit 2D von Mises norm.

As a conclusion, any 2D symmetric fourth-order tensor can be expressed thanks to two scalars and to two symmetric second-order deviatoric tensors in a decomposition that makes explicitly appear invariants and symmetry classes.

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