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# Thin linearly piezoelectric junctions

## *Jonctions minces linéairement piézoélectriques*

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ABSTRACT

Through a rigorous mathematical analysis, we present various asymptotic models for a thin piezoelectric junction between two linearly piezoelectric or elastic bodies. Depending on the relative behavior of a stiffness parameter with respect to its thickness, the joint is replaced by either a(n) (electro)mechanical constraint or a piezoelectric material surface.



RÉSUMÉ

Par une analyse mathématique rigoureuse, nous présentons divers modèles asymptotiques pour une jonction mince piézoélectrique entre deux corps linéairement élastiques ou piézoélectriques. Selon l'ordre de grandeur relatif entre un paramètre de rigidité et l'épaisseur, le joint est remplacé par une liaison (électro)mécanique ou par une surface matérielle piézoélectrique.



## 1. Introduction

We present various asymptotic models, indexed by $p=(p_1,p_2)\in\{1,2,3,4\}^2$, for a thin piezoelectric junction between two linearly piezoelectric ($p_2=1$) or elastic ($p_2>1$) bodies. Index $p_1$ is relative to the magnitude of the piezoelectric coefficients of the adhesive, characterized by a single parameter $\mu$, with respect to that of the constant thickness $2\varepsilon$ of a layer containing the adhesive. More precisely, we assume that $h:=(\varepsilon,\mu)$ takes values in a countable set with a sole cluster point $\bar h\in\{0\}\times[0,+\infty]$ so that:

$$\begin{cases} p_1=1:\ \bar\mu_1:=\lim_{h\to\bar h}(\varepsilon\mu)\in(0,+\infty)\\ p_1=2:\ \bar\mu_1:=\lim_{h\to\bar h}(\varepsilon\mu)=0,\quad \bar\mu_2:=\lim_{h\to\bar h}(\mu/2\varepsilon)=+\infty\\ p_1=3:\ \bar\mu_2:=\lim_{h\to\bar h}(\mu/2\varepsilon)\in(0,+\infty)\\ p_1=4:\ \bar\mu_2:=\lim_{h\to\bar h}(\mu/2\varepsilon)=0.\end{cases} \tag{1}$$

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As previously said, index $p_2$ characterizes the status of the adherents but also that of the interfaces between adherents and adhesive:

$$\begin{cases} p_2 = 1: \text{ the two interfaces are electromechanically perfectly permeable,} \\ p_2 = 2: \text{ the two interfaces are electrically impermeable,} \\ p_2 = 3: \text{ one interface is electrically impermeable while the other is electroded,} \\ p_2 = 4: \text{ the two interfaces are electroded.} \end{cases} \tag{2}$$

The space $\mathbb{R}^3$ is assimilated with the physical Euclidean space with basis $\{e_1, e_2, e_3\}$. Let $\Omega$ be a domain, with Lipschitz-continuous boundary, whose intersection $S$ with $\{x_3 = 0\}$ is a domain of $\mathbb{R}^2$ of positive two-dimensional Hausdorff measure $\mathcal{H}_2(S)$. Let $\Omega_\pm := \Omega \cap \{\pm x_3 > 0\}$ and $\varepsilon$ be a small positive number, then adhesive and adherents occupy $B^\varepsilon := S \times (-\varepsilon, \varepsilon)$, $\Omega^\varepsilon_\pm := \Omega_\pm \pm \varepsilon e_3$, respectively; let $\Omega^\varepsilon = \Omega^\varepsilon_+ \cup \Omega^\varepsilon_-$, $S^\varepsilon_\pm := S \pm \varepsilon e_3$, $\mathcal{O}^\varepsilon := \Omega^\varepsilon \cup B^\varepsilon \cup_\pm S^\varepsilon_\pm$. Let $(\Gamma_{\mathrm{mD}}, \Gamma_{\mathrm{mN}})$, $(\Gamma_{\mathrm{eD}}, \Gamma_{\mathrm{eN}})$ be two partitions of $\partial\Omega$ with $\mathcal{H}_2(\Gamma_{\mathrm{mD}})$, $\mathcal{H}_2(\Gamma_{\mathrm{eD}}) > 0$ and $0 < \delta := \mathrm{dist}(\Gamma_{\mathrm{eD}}, S)$. For all $\Gamma$ in $\{\Gamma_{\mathrm{mD}}, \Gamma_{\mathrm{mN}}, \Gamma_{\mathrm{eD}}, \Gamma_{\mathrm{eN}}\}$, $\Gamma_\pm$, $\Gamma^\varepsilon_\pm$, $\Gamma^\varepsilon$ denotes $\Gamma \cap \{\pm x_3 > 0\}$, $\Gamma_\pm \pm \varepsilon e_3$, $\cup_\pm \Gamma^\varepsilon_\pm$, respectively; if $(\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}})$ is a partition of $\gamma := \partial S$, we denote $\{\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}, \gamma\} \times (-\varepsilon, \varepsilon)$ by $\{\Gamma^\varepsilon_{\mathrm{DI}}, \Gamma^\varepsilon_{\mathrm{NI}}, \Gamma^\varepsilon_{\mathrm{lat}}\}$. The structure made of the adhesive and the two adherents, perfectly stuck together along $S^\varepsilon_\pm$, is clamped on $\Gamma^\varepsilon_{\mathrm{mD}}$, subjected to body forces of density $f^\varepsilon$ and to surface forces of density $F^\varepsilon$ on $\Gamma^\varepsilon_{mN}$ and vanishing on $\Gamma^\varepsilon_{\mathrm{lat}}$. Moreover, a given electric potential $\varphi^h_{p_0}$ is applied on $\Gamma^\varepsilon_{\mathrm{DI}}$ and, when $p_2 = 1$, on $\Gamma^\varepsilon_{\mathrm{eD}}$, while electric charges of density $d^\varepsilon$ appear on $\Gamma^\varepsilon_{\mathrm{NI}}$ and, when $p_2 = 1$, on $\Gamma^\varepsilon_{\mathrm{eN}}$.

If $\sigma^h_p$, $u^h_p$, $e(u^h_p)$, $D^h_p$, $\varphi^h_p$ stand for the fields of stress, displacement, strain, electric displacement and electric potential, respectively, the constitutive equations of the structure, for all $p_1$ in $\{1, 2, 3, 4\}$, read as:

$$\begin{cases} (\sigma^h_p, D^h_p) = \mu M_{\mathrm{I}}\big(e(u^h_p), \nabla\varphi^h_p\big) & \text{in } B^\varepsilon \ \ \forall p_2 \in \{1,2,3,4\}, \\ \begin{cases} (\sigma^h_p, D^h_p) = M^\varepsilon_{\mathrm{E}}\big(e(u^h_p), \nabla\varphi^h_p\big) & \text{in } \Omega^\varepsilon \ \text{ if } p_2 = 1, \\ \sigma^h_p = a^\varepsilon_{\mathrm{E}} e(u^h_p) & \text{in } \Omega^\varepsilon \ \text{ if } p_2 > 1 \end{cases} \end{cases} \tag{3}$$

where

$$(M^\varepsilon_{\mathrm{E}}, a^\varepsilon_{\mathrm{E}})(x) = (M_{\mathrm{E}}, a_{\mathrm{E}})(x \mp \varepsilon e_3) \quad \forall x \in \Omega^\varepsilon_\pm \tag{4}$$

$$\begin{cases} (M_{\mathrm{I}}, M_{\mathrm{E}}) \in L^\infty\big(S \times \Omega;\, \mathrm{Lin}(\mathbb{K})\big) \text{ such that} \\ M_{\mathrm{P}} = \begin{bmatrix} a_{\mathrm{P}} & -b_{\mathrm{P}} \\ b^T_{\mathrm{P}} & c_{\mathrm{P}} \end{bmatrix};\ \exists\kappa > 0, \quad \kappa|k|^2 \le M_{\mathrm{P}}(x)k \cdot k, \quad \forall k \in \mathbb{K} := \mathbb{S} \times \mathbb{R}^3, \ \text{a.e. } x \in \Omega,\ \forall \mathrm{P} \in \{\mathrm{I}, \mathrm{E}\} \end{cases} \tag{5}$$

and $\mathrm{Lin}(\mathbb{S}^3)$ is the space of linear operators on the space $\mathbb{S}^N$ of $N \times N$ symmetric matrices whose inner product and norm are noted $\cdot$ and $|\cdot|$ as in $\mathbb{R}^3$ (the same notations for the norm and inner product stand also for $\mathbb{K}$).

Lastly we have to add the following conditions on $S^\varepsilon_\pm$:

$$\begin{cases} p_2 = 2 \quad D^h_p \cdot e_3 = 0 \quad \text{on } S^\varepsilon_\pm, \\ p_2 = 3 \quad D^h_p \cdot e_3 = 0 \quad \text{on } S^\varepsilon_+, \quad \varphi^h_p = \varphi^h_{p_0} \text{ on } S^\varepsilon_-, \\ p_2 = 4 \quad \varphi^h_p = \varphi^h_{p_0} \qquad \text{on } S^\varepsilon_\pm, \end{cases} \tag{6}$$

the electric potential $\varphi^h_{p_0}$ being given on $S^\varepsilon_+$ or $S^\varepsilon_\pm$.

It will be convenient to use the following notations:

$$\begin{cases} \hat{k} := (\hat{e}, \hat{g}) \quad \hat{e} := e_{\alpha\beta},\ 1 \le \alpha, \beta \le 2, \quad \hat{g} := (g_1, g_2), \quad \forall k = (e, g) \in \mathbb{K} \\ \tilde{e} \in \mathbb{S}^3; \quad \tilde{e}_{\alpha\beta} = e_{\alpha\beta},\ 1 \le \alpha, \beta \le 2, \quad \tilde{e}_{i3} = 0,\ 1 \le i \le 3, \quad \forall e \in \mathbb{S}^3 \\ k(r) = k(v, \psi) := \big(e(v), \nabla\psi\big) \quad \forall r \in H^1(\mathcal{O}; \mathbb{R}^3 \times \mathbb{R}) \\ e(v) \in \mathcal{D}'(S; \mathbb{S}^2); \quad \big(e(v)\big)_{\alpha\beta} = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha),\ 1 \le \alpha, \beta \le 2, \quad \forall v \in \mathcal{D}'(S; \mathbb{R}^3) \end{cases} \tag{7}$$

and the *same* symbol $e(\cdot)$ shall *also* stand for the symmetrized gradient in the sense of distributions of $\mathcal{D}'(\mathcal{O}; \mathbb{R}^3)$, $\mathcal{O} \in \{\mathcal{O}^\varepsilon, \Omega, \Omega \setminus S, B^\varepsilon, \Omega^\varepsilon\}$ or $\mathcal{D}'(S; \mathbb{R}^2)$. An electromechanical state with vanishing electric potential on $\Gamma^\varepsilon_{\mathrm{DI}}$ and on $\Gamma^\varepsilon_{\mathrm{eD}}$ when $p_2 = 1$ will belong to $V^\varepsilon_p := H^1_{\Gamma^\varepsilon_{\mathrm{mD}}}(\mathcal{O}^\varepsilon; \mathbb{R}^3) \times \Phi^\varepsilon_{p_2}$, with

$$\begin{cases} \Phi^\varepsilon_1 = H^1_{\Gamma^\varepsilon_{\mathrm{DI}} \cup \Gamma^\varepsilon_{\mathrm{eD}}}(\mathcal{O}^\varepsilon) \\ \Phi^\varepsilon_2 = H^1_{\Gamma^\varepsilon_{\mathrm{DI}}}(B^\varepsilon) \text{ if } \mathcal{H}_2(\Gamma^\varepsilon_{\mathrm{DI}}) > 0,\ H^1_m(B^\varepsilon) \text{ if } \mathcal{H}_2(\Gamma^\varepsilon_{\mathrm{DI}}) = 0 \\ \Phi^\varepsilon_3 = H^1_{\Gamma^\varepsilon_{\mathrm{DI}} \cup S^\varepsilon_-}(B^\varepsilon) \\ \Phi^\varepsilon_4 = H^1_{\Gamma^\varepsilon_{\mathrm{DI}} \cup_\pm S^\varepsilon_\pm}(B^\varepsilon) \end{cases} \tag{8}$$

where, for any domain $\mathcal{O}$ of $\mathbb{R}^N$, $N=2,3$, $H^1_\Sigma(\mathcal{O};\mathbb{R}^M)$ denotes the subspace of $H^1(\mathcal{O};\mathbb{R}^M)$, $M=1$ or 3, of all elements with vanishing traces on a part $\Sigma$ of the boundary of $\mathcal{O}$, while $H^1_m(\mathcal{O};\mathbb{R}^M)$ denotes the subspace of all elements with vanishing average.

We make the following assumptions on the data:

$$\begin{cases} \text{Given } (f,F,d_\mathrm{E},d_\mathrm{I}) \text{ in } L^2(\Omega;\mathbb{R}^3)\times L^2(\Gamma_\mathrm{mN};\mathbb{R}^3)\times L^2(\Gamma_\mathrm{eN})\times L^2\big(\gamma_\mathrm{N}\times(-1,1)\big) \\ \text{with } \int_{\Gamma_\mathrm{lat}} d_\mathrm{I}=0 \text{ when } p_2=2 \text{ and } \mathcal{H}_2\big(\gamma_\mathrm{D}\times(-1,1)\big)=0, \\ \varphi_{o\mathrm{I}} \text{ in } H^{3/2}(\mathbb{R}) \text{ vanishing in } \{|x_3|>1+\delta/2\}, \text{ and } \varphi_{o\mathrm{E}} \text{ in } H^1(\Omega) \text{ vanishing on } S, \text{ then:} \\ f^\varepsilon(x)=f(x\mp\varepsilon e_3) \text{ a.e. } x\in\Omega^\varepsilon_\pm, \quad f^\varepsilon(x)=0 \text{ a.e. } x\in B^\varepsilon, \\ F^\varepsilon(x)=F(x\mp\varepsilon e_3) \text{ a.e. } x\in\Gamma^\varepsilon_{\mathrm{mN}\pm}, \\ d^\varepsilon(x)=(\mu/\varepsilon)^{1/2}d_\mathrm{I}(\hat{x},x_3/\varepsilon) \text{ a.e. } x\in B^\varepsilon, \\ d^\varepsilon(x)=d_\mathrm{E}(x\mp\varepsilon e_3) \text{ a.e. } x\in\Gamma^\varepsilon_{\mathrm{eN}\pm} \quad \text{if } p_2=1, \\ \varphi^h_{p_o}(x)=\begin{cases}\varphi_{o\mathrm{E}}(x\mp\varepsilon e_3)+\varepsilon^{p_\mathrm{DI}}\varphi_{o\mathrm{I}}(x\pm(1-\varepsilon)e_3) \text{ a.e. } x\in\Omega^\varepsilon_\pm \\ \varepsilon^{p_\mathrm{DI}}\varphi_{o\mathrm{I}}(\hat{x},x_3/\varepsilon) \text{ a.e. } x\in B^\varepsilon\end{cases} \end{cases} \tag{9}$$

where $p_\mathrm{DI}$ is such that $p_\mathrm{DI}=0$ if $\partial_3\varphi_{o\mathrm{I}}=0$ in $S\times(-1,1)$, $p_\mathrm{DI}=1$ if $\partial_3\varphi_{o\mathrm{I}}\neq 0$ in $S\times(-1,1)$. We also introduce the element $\varphi_o$ of $H^{1,1}(\Omega,S):=\{\psi\in H^1(\Omega)$ whose trace $\gamma_o(\psi)$ on $S$ belongs to $H^1(S)\}$ defined by $\varphi_o(x)=\varphi_{o\mathrm{E}}(x)+(1-p_{DI})\varphi_{o\mathrm{I}}(x\pm e_3)$ a.e. $x\in\Omega_\pm$. We note $\overline{\varphi}_o$ the trace on $\gamma_\mathrm{D}$ of $\varphi_o$ and set $\triangle\varphi_{o\mathrm{I}}=\frac{1}{2}\big(\varphi_{o\mathrm{I}}(\cdot,1)-\varphi_{o\mathrm{I}}(\cdot,-1)\big)$.

Then, if $\mathcal{M}_p$ and $\mathcal{L}_p$ are defined by:

$$\begin{cases} \mathcal{M}_p(s,r):=\begin{cases}\int_{\Omega^\varepsilon} M^\varepsilon_\mathrm{E}k(s)\cdot k(r)\,\mathrm{d}x+\mu\int_{B^\varepsilon}M_\mathrm{I}k(s)\cdot k(r)\,\mathrm{d}x, & \text{if } p_2=1 \\ \int_{\Omega^\varepsilon} a^\varepsilon_\mathrm{E}e(u)\cdot e(v)\,\mathrm{d}x+\mu\int_{B^\varepsilon}M_\mathrm{I}k(s)\cdot k(r)\,\mathrm{d}x, & \text{if } p_2>1\end{cases} \\ \mathcal{L}_p(r):=\int_\Omega f^\varepsilon\cdot v\,\mathrm{d}x+\int_{\Gamma^\varepsilon_\mathrm{mN}}F^\varepsilon\cdot v\,\mathrm{d}\mathcal{H}_2+\int_{\Gamma^\varepsilon_\mathrm{DI}\cup\Gamma^\varepsilon}d^\varepsilon\psi\,\mathrm{d}\mathcal{H}_2 \quad \Gamma^\varepsilon=\Gamma^\varepsilon_\mathrm{eN} \text{ if } p_2=1, \Gamma^\varepsilon=\varnothing \text{ if } p_2>1. \end{cases} \tag{10}$$

Seeking an equilibrium state leads to the problem

$$(\mathcal{P}^h_p):\quad \text{Find } s^h_p \text{ in } (0,\varphi^h_{p_0})+V^\varepsilon_p \text{ such that} \quad \mathcal{M}_p(s^h_p,r)=\mathcal{L}_p(r),\quad \forall r\in V^\varepsilon_p$$

which, by Stampacchia theorem, has a unique solution.

## 2. The asymptotic models

In the following $C$, $C'$ will denote various constants independent of $h$ that may vary from line to line. It will be convenient in the cases $p_2>1$ to use the same symbol $s^h_p$ for $(u^h_p,\tilde{\varphi}^h_p)$ where $\tilde{\varphi}^h_p$ denotes the extension into $\Omega^\varepsilon$ of $\varphi^h_p$ by 0. Without loss of generality, we suppose $\mathcal{H}_2(\Gamma_{\mathrm{mD}+})>0$; moreover, when $p_1=4$, we assume $\mathcal{H}_2(\Gamma_{\mathrm{mD}\pm})>0$, $\mathcal{H}_2(\Gamma_{\mathrm{eD}\pm})>0$. We recall that if the same symbol $u^h_p$ denotes a continuous extension from $H^1(\Omega^\varepsilon_+;\mathbb{R}^3)$ into $H^1(\{x_3>\varepsilon\};\mathbb{R}^3)$ and $\eta$ is a $C^\infty_o(\mathbb{R})$ cut-off function such that

$$\eta=1 \text{ on } [-\frac{\delta}{3},\frac{\delta}{3}],\quad 0\le\eta\le1 \text{ and } 0<\left|\frac{\mathrm{d}\eta}{\mathrm{d}x_3}\right|\le\frac{4}{\delta} \text{ on } \frac{\delta}{3}\le|x_3|\le\frac{2\delta}{3},\quad \eta=0 \text{ on } \left\{|x_3|\ge\frac{2\delta}{3}\right\} \tag{11}$$

then Korn inequality imply

$$\begin{aligned}\int_{B^\varepsilon}|\nabla u^h_p|^2\,\mathrm{d}x &\le \int_{S\times(-\varepsilon,\delta-\varepsilon)}|\nabla\eta u^h_p|^2\,\mathrm{d}x\le C\int_{S\times(-\varepsilon,\delta-\varepsilon)}|e(\eta u^h_p)|^2\,\mathrm{d}x \\ &\le C'\Big(|e(u^h_p)|^2_{L^2(B^\varepsilon;\mathbb{S}^3)}+|e(u^h_p)|_{L^2(\Omega^\varepsilon_+;\mathbb{S}^3)}\Big)\end{aligned} \tag{12}$$

We will propose our models by studying the asymptotic behavior of $s^h_p$ when $h$ goes to $\bar{h}$ in three steps.

As announced in [1], we may proceed, when $p_1=1$, similarly but by taking due account of the realistic nonvanishing electric loading on $\Gamma^\varepsilon_\mathrm{lat}$ easily handled though the standard inequalities:

$$\begin{cases}\int_{\Gamma^\varepsilon_\mathrm{NI}}\psi^2\,\mathrm{d}\mathcal{H}_2\le C\int_{B^\varepsilon}|\nabla\psi|^2\,\mathrm{d}x \quad \forall\psi\in H^1_{\Gamma^\varepsilon_\mathrm{NI}}(B^\varepsilon) \text{ if } \mathcal{H}_2(\Gamma^\varepsilon_\mathrm{DI})>0 \\ \int_{\Gamma^\varepsilon_\mathrm{lat}}\psi^2\,\mathrm{d}\mathcal{H}_2\le C\int_{B^\varepsilon}|\nabla\psi|^2\,\mathrm{d}x \quad \forall\psi\in H^1_m(B^\varepsilon) \text{ if } \mathcal{H}_2(\Gamma^\varepsilon_\mathrm{DI})>0\end{cases} \tag{13}$$

When $p_1>1$, we proceed in the spirit of [2–4].

**Step 1 (a priori estimates)**: By taking $r=s^h_p-(0,\varphi^h_{p_0})$ in the variational formulation of $(\mathcal{P}^\varepsilon_p)$, (12), (13) and

$$\int_S|s^h_p(\cdot,\varepsilon)-s^h_p(\cdot,-\varepsilon)|^2\,\mathrm{d}\hat{x}\le 2\varepsilon\int_{B^\varepsilon}|\nabla s^h_p|^2\,\mathrm{d}x \tag{14}$$

imply:

$$\mu|k(s_p^h)|^2_{L^2(B^\varepsilon;\mathbb{K})} + |k(s_p^h)|^2_{L^2(\Omega^\varepsilon;\mathbb{K})} \le C. \tag{15}$$

**Step 2 (convergence of $(s_p^h)$):** The following two tools are well suitable to describe the asymptotic behavior of the electromechanical state in the adherents and adhesive, respectively. First, let $T^\varepsilon$ be the mapping from $H^1(\Omega^\varepsilon;\mathbb{R}^3\times\mathbb{R})$ into $H^1(\Omega\setminus S;\mathbb{R}^3\times\mathbb{R})$ defined by:

$$(T^\varepsilon r)(x) = \big(T^\varepsilon(v,\psi)\big)(x) = (T_1^\varepsilon v, T_2^\varepsilon \psi)(x) := (v,\psi)(x\pm\varepsilon e_3) \quad \forall x\in\Omega_\pm. \tag{16}$$

Note that $T^\varepsilon s_p^h = (T_1 u_p^h, 0)$ if $p_2 > 1$! For any $w$ in $H^1(\Omega\setminus S;\mathbb{R}^N)$, $N\in\{1,3\}$, if $\gamma_o^\pm(w^\pm)$ denotes the trace on $S$ of its restriction $w^\pm$ to $\Omega_\pm$, $[\![w]\!]$ stands for $\gamma_o^+(w^+) - \gamma_o^-(w^-)$.

Next, as for all $r$ of $H^1(B^\varepsilon;\mathbb{R}^3\times\mathbb{R})$, one has

$$\int_{B^\varepsilon} \Big| r(x) - \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} r(\hat{x},x_3)\,\mathrm{d}x_3\Big|^2 \mathrm{d}x \le C\varepsilon^2 \int_{B^\varepsilon} |\nabla r|^2\,\mathrm{d}x \tag{17}$$

we introduce the following element of $L^2(S;\mathbb{K})$:

$$k_p(\varepsilon,r) := \frac{1}{(2\varepsilon)^q}\int_{-\varepsilon}^{\varepsilon} k(r)(\cdot,x_3)\,\mathrm{d}x_3, \quad q=\max(2-p_1,0) \tag{18}$$

and, obviously, there holds

$$\widehat{k_p(\varepsilon,r)} = (e(\widehat{U_p}),\nabla\Phi_p), \quad S_p^h := (U_p^h,\Phi_p^h) := \frac{1}{(2\varepsilon)^q}\int_{-\varepsilon}^{\varepsilon} s_p^h(\cdot,x_3)\,\mathrm{d}x_3. \tag{19}$$

So (12), (14) and (15) imply:

$$\begin{cases} |k(T^\varepsilon s_p^h)|_{L^2(\Omega\setminus S;\mathbb{K})} \le C, \quad |[\![T^\varepsilon s_p^h]\!]|^2_{L^2(S;\mathbb{R}^3\times\mathbb{R})} \le C\Big(\varepsilon + \dfrac{\varepsilon}{\mu}\Big) \\ |k_p(\varepsilon,s_p^h)|^2_{L^2(S;\mathbb{K})} \le C\varepsilon^{-2q}\cdot\dfrac{\varepsilon}{\mu} \\ |U_p^h|^2_{L^2(S;\mathbb{R}^3)} \le C\varepsilon^{2(1-q)}\Big(1+\frac{\varepsilon}{\mu}\Big) \\ \varepsilon^{2(q-1)}|\Phi_p^h|^2_{L^2(S)} \le Cc^*(h) \quad c^*(h) = \Big(1+\dfrac{\varepsilon}{\mu}\Big) \text{ if } p_2=1,\ \dfrac{1}{\varepsilon\mu} \text{ if } p_2=2,\ \varepsilon^2+\dfrac{\varepsilon}{\mu} \text{ if } p_2>2 \\ |e(\widehat{U}_p^h),\nabla\Phi_p^h|^2_{L^2(S;\mathbb{S}^2\times\mathbb{R}^2)} \le C\dfrac{1}{\varepsilon^{2q}}\cdot\dfrac{\varepsilon}{\mu} \\ |S_p^h - \gamma_o^\pm((T^\varepsilon s_p^h)^\pm)|^2_{L^2(S;\mathbb{R}^3\times\mathbb{R})} \le C\Big(\varepsilon+\dfrac{\varepsilon}{\mu}\Big) \quad \text{if } p_1=p_2=1 \\ |U_p^h - \gamma_o^\pm((T_1^\varepsilon u_p^h)^\pm)|^2_{L^2(S;\mathbb{R}^3)} \le C\Big(\varepsilon+\dfrac{\varepsilon}{\mu}\Big) \quad \text{if } p_1=1,\ p_2>1. \end{cases} \tag{20}$$

Thus, if $a\otimes_S b$ denotes the symmetrized tensor product of $a$ and $b$ in $\mathbb{R}^3$, we deduce:

**Proposition 2.1.** *There exists $\bar{s}_p = (\bar{u}_p,\bar{\varphi}_p)$ in $H^1_{\Gamma_{\mathrm{mD}}}(\Omega\setminus S;\mathbb{R}^3)\times H^1_{\Gamma_{\mathrm{eD}}}(\Omega\setminus S)$ such that $T^\varepsilon s_p^h$ weakly converges in $H^1(\Omega\setminus S;\mathbb{R}^3\times\mathbb{R})$ toward some $\bar{s}_p$; $\bar{\varphi}_p = 0$ when $p_2>1$ and $\bar{s}_p$ belongs to $H^1(\Omega;\mathbb{R}^3\times\mathbb{R})$ when $p_1\le 2$.*

*When $p_1\neq 4$, $k_p(\varepsilon,s_p^h)$ weakly converges in $L^2(S;\mathbb{K})$ toward some $\bar{k}_p = (\bar{e}_p,\bar{g}_p)$, and there exists $(\overline{U}_p,\overline{\Phi}_p)$ in $H^1(S;\mathbb{R}^3\times\mathbb{R})$ such that $(\widehat{U_p^h},\Phi_p^h)$ converges weakly in $H^1(S;\mathbb{R}^2\times\mathbb{R})$ toward $(\widehat{\overline{U}}_p,\overline{\Phi}_p)$, $(U_p^h)_3$ converges strongly in $L^2(S)$ toward $(\overline{U}_p)_3$. Moreover*

i) *when $p_1=1$, $\overline{U}_p=\gamma_0(\bar{u}_p)$ for all $p_2$, while $\bar{\Phi}_p$ is equal to $\gamma_0(\bar{\varphi}_p)$ when $p_2=1$ or to $\gamma_0(\varphi_0)$ when $p_2\ge 3$. Furthermore, the trace on $\gamma_{\mathrm{D}}$ of $\bar{\Phi}_p$ is equal to $\bar{\varphi}_o$ while $\widehat{\bar{k}_p} = \big(e(\widehat{\bar{u}_p}),\nabla\bar{\Phi}_p\big)$, $(\bar{g}_p)_3 = \triangle\varphi_{o\mathrm{I}}$ when $p_2=4$;*
ii) *when $p_1=2$, $\overline{U}_p=0$, $\bar{\Phi}_p=0$ and $\bar{k}_p=0$;*
iii) *when $p_1=3$, $\overline{U}_p=0$ and $\bar{e}_p=[\![\bar{u}_p]\!]\otimes_S e_3$ for all $p_2$, while $\bar{\Phi}_p$ and $\widehat{\bar{g}_p}$ vanish only when $p_2\neq 2$, $(\bar{g}_{(3,1)})_3=[\![\bar{\varphi}_p]\!]$, $(\bar{g}_{(3,4)})_3=0$.*

Actually, in the next step, we will show that $(\bar{u}_p,\bar{\varphi}_p)$ is necessarily the unique solution to a variational problem so that the whole sequences converge. To identify $\bar{k}_p$, when $p_1>1$, it suffices to go to the limit in the identity

$$\int_{B^\varepsilon}\Big(k(s_p^h)(x)\cdot\tau^i(\hat{x})+s_p^h(x)\cdot\operatorname{div}\tau^i(\hat{x})\Big)\,\mathrm{d}x=\int_S\Big([\![T_1^\varepsilon u_p^h]\!]_i\,\theta_1\Big)+\Big([\![T_2^\varepsilon\varphi_p^h]\!]\,\theta_2\Big)\,\mathrm{d}\hat{x} \tag{21}$$

with $\tau^i=(\theta_1 e_3\otimes_S e_i,\theta_2 e_3)$, $\theta_j\in C_o^\infty(S)$, $j=1,2$, $i=1,2,3$ and to use the convergence to 0 in the sense of distributions of $\widehat{k_p(\varepsilon,s_p^h)}$ by due account of (20)!

**Step 3 (identification of $(\bar{s}_p,\bar{k}_p)$):**

When $p_1>1$, we simply go to the limit in the variational formulation of $(\mathcal{P}_p^h)$ by using suitable test-functions $r_p^\varepsilon$. For all $w^1$ in $H^1_{\Gamma_{\mathrm{mD}}}(\Omega;\mathbb{R}^3)$ and all $\zeta^1$ in $H^1_{\Gamma_{\mathrm{eD}}}(\Omega)$ vanishing in a neighborhood of $\gamma_{\mathrm{D}}$, let $(w^{1,\varepsilon},\zeta^{1,\varepsilon})$ be defined by

$$\big(w^{1,\varepsilon},\zeta^{1,\varepsilon}\big)(x)=\begin{cases}(w^1,\zeta^1)(x\mp\varepsilon e_3) & \text{a.e. } x\in\Omega_\pm^\varepsilon\\ (w^1,\zeta^1)(\hat{x},0) & \text{a.e. } x\in B^\varepsilon\end{cases}$$

For all $w^2$ in $H^1_{\Gamma_{\mathrm{mD}}}(\Omega\setminus S;\mathbb{R}^3)$ and all $\zeta^2$ in $H^1_{\Gamma_{\mathrm{eD}}}(\Omega\setminus S)$ vanishing in a neighborhood of $\gamma_{\mathrm{D}}$, let $(w^{2,\varepsilon},\zeta^{2,\varepsilon})$ be defined by

$$(w^{2,\varepsilon},\zeta^{2,\varepsilon})(x)=\begin{cases}(w^2,\zeta^2)(x\mp\varepsilon e_3) & \text{a.e. } x\in\Omega_\pm^\varepsilon\\ (w^a,\zeta^a)(\hat{x},x_3/\varepsilon)+\frac{|x_3|}{\varepsilon}(w^s,\zeta^s)(\hat{x},x_3/\varepsilon) & \text{a.e. } x\in B^\varepsilon\end{cases}$$

with

$$(w^a,\zeta^a)(x)=\frac{1}{2}[(w^2,\zeta^2)(\hat{x},x_3)-(w^2,\zeta^2)(\hat{x},-x_3)]$$

$$(w^s,\zeta^s)(x)=\frac{1}{2}[(w^2,\zeta^2)(\hat{x},x_3)+(w^2,\zeta^2)(\hat{x},-x_3)]$$

Note (see the proof of Lemma 4.1 of [3]) that

$$\left|\left(e(w^{2,\varepsilon})-\frac{[\![w^2]\!]\otimes_S e_3}{2\varepsilon},\nabla\zeta^{2,\varepsilon}-\frac{[\![\zeta^2]\!]e_3}{2\varepsilon}\right)\right|_{L^2(B^\varepsilon;\mathbb{K})}\le C\,|(w^2,\zeta^2)|_{H^1(B^\varepsilon\setminus S;\mathbb{R}^3\times\mathbb{R})}.$$

So $r_p^\varepsilon=(v_p^\varepsilon,\psi_p^\varepsilon)$ reads as:

$$\begin{cases}v_p^\varepsilon=w^{\min(p_1-1,2),\varepsilon}, \quad 1\le p_2\le 4\\ \psi_p^\varepsilon=\begin{cases}\zeta^{\min(p_1-1,2),\varepsilon} & p_2=1\\ (\theta_1+x_3\theta_2)/\varepsilon,\quad \theta_1,\theta_2\in C_0^\infty(S) & p_1=3,4,\ p_2=2\\ (1+x_3/\varepsilon)\theta,\quad \theta\in C_0^\infty(S) & p_1=3,4,\ p_2=3\\ 0 & \text{if } (2\le p_1\le 4,\ p_2=4) \text{ or } (p_1=2,\ p_2=2,3)\end{cases}\end{cases} \tag{22}$$

When $p_1=1$, one proceeds in two steps. First we prove

$$(M\bar{k}_p)_p^2=0 \tag{23}$$

where $k_p^i$ denotes the projection on $\mathbb{K}_p^i$ of any element $k$ of $\mathbb{K}$ with:

$$\begin{cases}\mathbb{K}=\mathbb{K}_p^1\oplus\mathbb{K}_p^2\oplus\mathbb{K}_p^3,\\ p_2\le 2:\ \mathbb{K}_p^1:=\{(e,g)\in\mathbb{K};\ e_{i3}=0,g_3=0\},\ \mathbb{K}_p^2:=\big\{(e,g)\in\mathbb{K};\ \hat{e}=0,\hat{g}=0\big\},\mathbb{K}_p^3:=\{0\},\\ p_2>2:\ \mathbb{K}_p^1=\big\{(e,g)\in\mathbb{K};\ e_{i3}=0,\hat{g}=0\big\},\ \mathbb{K}_p^2=\big\{(e,g)\in\mathbb{K};\ \hat{e}=0,g=0\big\},\\ \qquad\quad \mathbb{K}_p^3=\{(e,g)\in\mathbb{K};\ e=0,g_3=0\}\end{cases} \tag{24}$$

For that, we simply use test functions $\rho_p^\varepsilon$ built as follows: given $(w,\psi)$ in $C_o^\infty(S;\mathbb{R}^3\times\mathbb{R})$,

$$\rho_p^\varepsilon(x)=\begin{cases}(x_3+\varepsilon)(w,I_{p_2}\psi)(\hat{x}) & \text{a.e. } x\in B^\varepsilon\\ 2\varepsilon(w^+,I_{p_2}\psi^+)(x-\varepsilon e_3) & \text{in } \Omega_+^\varepsilon,\ 0 \text{ in } \Omega_-^\varepsilon\end{cases} \tag{25}$$

where $(w^+,\psi^+)$ is an extension into $H^1_{\Gamma_{\mathrm{mD}+}}(\Omega_+;\mathbb{R}^3)\times H^1_{\Gamma_{\mathrm{eD}+}}(\Omega_+)$ and $I_{p_2}=1$ if $p_2\le 2$, $I_{p_2}=0$ if $p_2>2$. Hence, as Proposition 2.1 yields $(\bar{k}_p)_p^3=0$, we deduce

$$(M_I\bar{k}_p)^1=\widetilde{M}_{Ip}(\bar{k}_p)^1;\quad \widetilde{M}_{Ip}:=M_{Ip}^{11}-M_{Ip}^{12}(M_{Ip}^{22})^{-1}M_{Ip}^{21} \tag{26}$$

with $M_{Ip}^{ij}$, $1\le i,j\le 3$ being the decomposition of $M_I$ in linear operators mapping $\mathbb{K}_p^i$ into $\mathbb{K}_p^j$.

Next, given $(v,\psi)$ in $\big(H^1_{\Gamma_{\mathrm{mD}}}(\Omega;\mathbb{R}^3)\times H^1_{\Gamma_{\mathrm{eD}}}(\Omega)\big)\cap H^2(\Omega;\mathbb{R}^3\times\mathbb{R})$ $\psi$ vanishing in a neighborhood of $\gamma_{\mathrm{D}}$, we define $r_p^\varepsilon=(v_p^\varepsilon,\psi_p^\varepsilon)$ by:

$$\begin{cases} \widehat{v_p^\varepsilon}(x) = \hat{v}(\hat{x},0) - x_3 \nabla v_3(\hat{x},0), \quad \big(v_3^\varepsilon(x) = v_3(\hat{x},0)\big) & \text{a.e. } x \in B^\varepsilon \\ v_p^\varepsilon(x) = v(x \mp \varepsilon e_3) \mp \varepsilon R^{\pm}\big(\nabla v_3(\cdot,0),0\big)(x \mp \varepsilon e_3) & \text{a.e. } x \in \Omega_\pm^\varepsilon \\ \psi_p^\varepsilon(x) = \psi(x \mp \varepsilon e_3) \text{ in } \Omega_\pm^\varepsilon, \quad \psi(\hat{x},0) \text{ in } B^\varepsilon & \text{if } p_2 = 1 \\ \psi_p^\varepsilon(x) = \psi(\hat{x},0) \text{ in } B^\varepsilon & \text{if } p_2 = 2 \\ \psi_p^\varepsilon(x) = 0 \text{ in } B^\varepsilon & \text{if } p_2 \geq 3 \end{cases} \tag{27}$$

where $R^{\pm}$ is a continuous lifting operator from $H^{1/2}(S;\mathbb{R}^3)$ into $H^1_{\Gamma_{\mathrm{mD}\pm}}(\Omega_\pm;\mathbb{R}^3)$. As $(\widetilde{e(\hat{v})},\nabla\psi)$ belongs to $\mathbb{K}_p^1$ almost everywhere in $S$, (26) yields

$$\lim_{h\to\bar{h}} \int_{B^\varepsilon} \mu M_{\mathrm{I}} k(s_p^h) k(r_p^\varepsilon)\,\mathrm{d}x = \int_S M_I \bar{k}_p \cdot (\widetilde{e(\hat{v})},\nabla\psi)\,\mathrm{d}\hat{x} = \int_S \widetilde{M}_{Ip}(\bar{k}_p^1)\cdot(\widetilde{e(\hat{v})},\nabla\psi)\,\mathrm{d}\hat{x} \tag{28}$$

while, obviously, we have:

$$\begin{cases} \lim_{h\to\bar{h}} \int_{\Omega^\varepsilon} M_{\mathrm{E}}^\varepsilon k(s_p^h)\cdot k(r_p^\varepsilon)\,\mathrm{d}x = \int_\Omega M_{\mathrm{E}} k(\bar{s}_p)\cdot k(v,\psi)\,\mathrm{d}x \quad \text{if } p_2 = 1 \\ \lim_{h\to\bar{h}} \int_{\Omega^\varepsilon} a_{\mathrm{E}}^\varepsilon e(u_p^h)\cdot e(v_p^\varepsilon)\,\mathrm{d}x = \int_\Omega a_{\mathrm{E}} e(\bar{u}_p)\cdot e(v)\,\mathrm{d}x \quad \text{if } p_2 \geq 2 \\ \lim_{h\to\bar{h}} \mathcal{L}_{(1,p_2)}(r^\varepsilon_{(1,p_2)}) \\ \quad := \begin{cases} \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{mN}}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + \int_{\Gamma_{\mathrm{eN}}} d_{\mathrm{E}}\,\psi\,\mathrm{d}\mathcal{H}_2 + (\bar{\mu}_1)^{1/2}\int_{\gamma_N}\left(\int_{-1}^1 d_{\mathrm{I}}(\cdot,x_3)\,\mathrm{d}x_3\right)\psi\,\mathrm{d}\mathcal{H}_2 & p_2 = 1 \\ \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{mN}}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + (\bar{\mu}_1)^{1/2}\int_{\gamma_N}\left(\int_{-1}^1 d_{\mathrm{I}}(\cdot,x_3)\,\mathrm{d}x_3\right)\psi\,\mathrm{d}\mathcal{H}_2 & p_2 \geq 2 \end{cases} \end{cases} \tag{29}$$

Lastly, Jensen inequality and the previously established weak convergences achieve the proof of the following convergence result which supports our asymptotic models in the form of variational problems $\left(\overline{\mathcal{P}}_p\right)$ where the convention $\infty \times 0 = 0$ is understood.

**Theorem 2.1.** *If $p_2 = 1$, when h goes to $\bar{h}$, $T^\varepsilon s_p^h$ converges strongly in $H^1(\Omega\setminus S;\mathbb{R}^3\times\mathbb{R})$ toward $\bar{s}_p$ the unique solution to*

$$\left(\overline{\mathcal{P}}_{(p_1,1)}\right): \quad \begin{cases} \text{Find } (u,\varphi) \text{ in } (0,\varphi_o) + V_{p_1}\times\Psi_{p_1} \text{ such that} \\ \overline{\mathcal{M}}_{(p_1,1)}\big((u,\varphi),(v,\psi)\big) = \overline{\mathcal{L}}_{(p_1,1)}(v,\psi) \quad \forall (v,\psi)\in V_{p_1}\times\Psi_{p_1} \end{cases}$$

*where*

$$\overline{\mathcal{M}}_{(p_1,1)}\big((u,\varphi),(v,\psi)\big) := \begin{cases} \int_\Omega M_{\mathrm{E}} k(u,\varphi)\cdot k(v,\psi)\,\mathrm{d}x + \bar{\mu}_1\int_S \widetilde{M}_{Ip}\big(e(\hat{u}),\nabla\varphi\big)\cdot\big(e(\hat{u}),\nabla\varphi\big)\,\mathrm{d}\hat{x} \\ \quad p_1 = 1 \\ \int_\Omega M_{\mathrm{E}} k(u,\varphi)\cdot k(v,\psi)\,\mathrm{d}x + \bar{\mu}_2\int_S M_{\mathrm{I}}\big([\![u]\!]\otimes_S e_3,[\![\varphi]\!]e_3\big)\cdot\big([\![v]\!]\otimes_S e_3,[\![\psi]\!]e_3\big)\,\mathrm{d}\hat{x} \\ \quad p_1 \geq 2 \end{cases}$$

$$\overline{\mathcal{L}}_{(p_1,1)}(v,\psi) := \begin{cases} \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{mN}}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + \int_{\Gamma_{\mathrm{eN}}} d_{\mathrm{E}}\,\psi\,\mathrm{d}\mathcal{H}_2 + (\bar{\mu}_1)^{1/2}\int_{\gamma_N}\left(\int_{-1}^1 d_{\mathrm{I}}(\cdot,x_3)\,\mathrm{d}x_3\right)\psi\,\mathrm{d}\mathcal{H}_2 & p_1 = 1 \\ \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{mN}}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + \int_{\Gamma_{\mathrm{eN}}} d_{\mathrm{E}}\,\psi\,\mathrm{d}\mathcal{H}_2 & p_1 \geq 2 \end{cases}$$

$$V_1 := \left\{v\in H^1_{\Gamma_{\mathrm{mD}}}(\Omega;\mathbb{R}^3);\ \hat{v}\in H^1(S;\mathbb{R}^2)\right\},\quad V_2 := H^1_{\Gamma_{\mathrm{mD}}}(\Omega;\mathbb{R}^3),\quad V_3 = V_4 := H^1_{\Gamma_{\mathrm{mD}}}(\Omega\setminus S;\mathbb{R}^3)$$

$$\Psi_1 := \left\{\psi\in H^1_{\Gamma_{\mathrm{eD}}}(\Omega);\ \psi\in H^1_{\gamma_{\mathrm{D}}}(S)\right\},\quad \Psi_2 := H^1_{\Gamma_{\mathrm{eD}}}(\Omega),\quad \Psi_3 = \Psi_4 := H^1_{\Gamma_{\mathrm{eD}}}(\Omega\setminus S)$$

*If $p_2 > 1$, when h goes to $\bar{h}$, $(T_1^\varepsilon u_p^h,\Phi_p^h)$ converges strongly in $H^1(\Omega\setminus S;\mathbb{R}^3)\times H^1(S)$ toward $(\bar{u}_p,\overline{\Phi}_p)$ the unique solution to*

$$\left(\overline{\mathcal{P}}_p\right): \quad \begin{cases} \text{Find } (u,\phi) \text{ in } \big(0,q\,\gamma_o(\varphi_o)\big) + V_{p_1}\times\Psi_p \text{ such that} \\ \overline{\mathcal{M}}_p\big((u,\phi),(v,\psi)\big) = \overline{\mathcal{L}}_p(v,\psi) \quad \forall (v,\psi)\in V_{p_1}\times\Psi_p \end{cases}$$

*where*

$$\overline{\mathcal{M}}_{(1,p_2)}\big((u,\phi),(v,\psi)\big) := \begin{cases} \int_\Omega a_E\, e(u)\cdot e(v)\,\mathrm{d}x + \bar{\mu}_1\int_S \widetilde{M_{\mathrm{I}p}}\big(e(\hat{u}),\nabla\phi\big)\cdot\big(e(\hat{v}),\nabla\psi\big)\,\mathrm{d}\hat{x} & p_2 = 2 \\ \int_\Omega a_E\, e(u)\cdot e(v)\,\mathrm{d}x + \bar{\mu}_1\int_S \widetilde{M_{\mathrm{I}p}}\big(e(\hat{u}),(\bar{g}_p)_3\big)\cdot\big(e(\hat{v}),0\big)\,\mathrm{d}\hat{x} & p_2 \geq 3 \\ \quad \text{with } \widetilde{M_{\mathrm{I}p}}\big(e(\hat{u}),(\bar{g}_p)_3\big)\cdot\big(0,e_3\big) = 0 \text{ if } p_2 = 3 \\ \qquad (\bar{g}_p)_3 = \triangle\varphi_{o\mathrm{I}} \text{ if } p_2 = 4 \\ \int_\Omega a_E\, e(u)\cdot e(v)\,\mathrm{d}x + \bar{\mu}_1\int_S \widetilde{M_{\mathrm{I},p}}(e(\hat{u}),\triangle\varphi_{o\mathrm{I}})\cdot(e(\hat{v}),0)\,\mathrm{d}\hat{x} & p_2 = 4 \end{cases}$$

$$\overline{\mathcal{L}}_{(1,p_2)}(v,\psi) := \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{mN}}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + (\bar{\mu}_1)^{1/2}\int_{\gamma_{\mathrm{N}}}\left(\int_{-1}^1 d_{\mathrm{I}}(\cdot,x_3)\,\mathrm{d}x_3\right)\psi\,\mathrm{d}\hat{x} \quad \forall p_2 \geq 2$$

*and for* $2 \le p_1, p_2 \le 4$

$$\overline{\mathcal{M}}_{(p_1,p_2)}\big((u,\phi),(v,\psi)\big) := \begin{cases} \int_\Omega a_{\mathrm{E}}\, e(u)\cdot e(v)\,\mathrm{d}x + \bar{\mu}_2 \int_S M_I([\![u]\!]\otimes_S e_3, (\nabla\phi, (\bar{g}_p)_3))\cdot([\![v]\!]\otimes_S e_3, (\nabla\psi, 0))\,\mathrm{d}\hat{x} \\ \textit{with}\ \ M_I([\![u]\!]\otimes_S e_3, (\nabla\phi, (\bar{g}_p)_3))\cdot(0, e_3) = 0 \textit{ if } p_2 = 2, 3 \\ \qquad (\bar{g}_p)_3 = 0 \textit{ if } p_2 = 4 \textit{ or } p_1 = 2 \end{cases}$$

$$\overline{\mathcal{L}}_{(p_1,p_2)}(v,\psi) = \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{mN}}} F\cdot v\,\mathrm{d}\mathcal{H}_2$$

$$\Psi_{(p_1,2)} := H^1_{\gamma_{\mathrm{D}}}(S) \textit{ or } H^1_m(S) \textit{ according to the positivity of the length of } \gamma_D,\ \Psi_{(p_1,3)} = \Psi_{(p_1,4)} := \{0\}, p_1 \neq 2$$

$$\Psi_{(2,p_2)} := \{0\}, 2 \le p_2 \le 4$$

## 3. Concluding remarks

In the case of piezoelectric adhesive and adherents ($p_2 = 1$), our results extend those obtained in elasticity (see [4–7]). The asymptotic behavior of the adhesive strongly depends on the magnitude of the stiffness compared to that of the thickness. When the magnitude of the stiffness is of the order of the inverse of the thickness, the adhesive is replaced by a *material piezoelectric surface* perfectly bonded to the adherents. When it is lesser, the adhesive is replaced by an *electromechanical constraint* between the two adherents which can be perfect adhesion, electromechanical pull-back or free separation, according to the order of magnitude of the stiffness which is, respectively, larger, equal or lower than that of the thickness. There is a large discrepancy between our results and that of [8,9] obtained by formal or questionable arguments.

Similarly, in the case of a thin piezoelectric layer embedded between two elastic adherents, depending on the magnitude of the stiffness, the adhesive is replaced by a material elastic surface perfectly bonded to the adherents or by a mechanical constraint between the adherents. In the case of electrically impermeable interfaces, the material surface has a *non-local* elastic behavior (since the additional state variable of electric nature $\phi$ can be eliminated!), the constitutive equations being derived from the asymptotic behavior of a thin piezoelectric plate acting as a sensor (case $p = 1$ in [10]). When one interface is electrically impermeable while the other is electroded, the material surface is an elastic membrane. When the two interfaces are electroded, the material surface is an elastic membrane with residual stress. In these last two cases, the constitutive equations are derived from the asymptotic behavior of a thin piezoelectric plate acting as an actuator (case $p = 2$ in [10]). The mechanical constraint is perfect adhesion, elastic pull-back or free separation according to the order of magnitude of the stiffness. In the case of electrically impermeable interfaces, the elastic pull-back is of non-local nature (since the state variable of electric nature $\phi$, additional to the relative displacement, can be eliminated). In the two other cases, the elastic pull-back is local. When the two interfaces are electroded, it is similar to the purely elastic case, while, if only one interface is electroded, piezoelectric and dielectric coefficients enter the constitutive equations.

The realistic dual situation [11] in which an elastic layer is embedded between two piezoelectric bodies can be treated within the same framework.

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