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Refined theory of bi-layer beams for a transversely isotropic body

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A B S T R A C T

Based on elastic theory, the refined theory of bi-layer beams for a transversely isotropic body is studied. Using the Elliott–Lodge (E–L) solution and Luré method, the refined theory of beams is derived from continuity conditions without ad hoc assumptions. It is shown that the displacements and stresses of the beam can be represented by displacements and stresses of the interface of two layers of different materials. The governing equations about the transverse displacement of the interface can be obtained directly from the refined theory under transverse surface loading. Approximate solutions are derived for beams by dropping terms of high order. In addition, one example is examined to illustrate the application of the theory proposed in this paper.

1. Introduction

With the development of engineering, monolayer structures cannot meet demand for high-strength, high-modulus and low-density industrial materials, and the laminated composite structures have been used in more and more fields. The bi-layer beams as a special laminated composite structure attracts broad attention. In general, two different models have been used to study bi-layer beams: single-layer theory and layerwise theory [1]. Using single-layer theory, the whole plate is considered as one monolayer structure, and a magnified error will appear. Using layerwise theory, each layer is analyzed, and the precision of layerwise theory will be affected by monolayer structure theory. Moreover, Carrera [2,3] makes a summing up of the theories for multilayered structures and gives one-dimensional formulations for the analysis of multilayered structures.

Research ideas of refined analysis were proposed by Cheng [4] in 1979. A parallel development of Cheng's theory has been obtained by Barrett and Ellis [5] for isotropic plates under transverse surface loadings. Wang and Shi [6] further study Cheng's refined theory by using the Papkovich–Neuber solution and discussed isotropic plate in the case of surface transverse load. They finally got the plate deflection control equations and shear control equations. Luo and Wang [7] obtained the refined theory of generalized plane-stress problems in elasticity.

Wang [8] applied Cheng's refined theory approach to the study of transversely isotropic plate, and a refined equation for transversely isotropic plates with homogeneous boundary conditions was obtained. Wang substituted the sum of the general integrals of the three differential equations of the plate problem for the solution to the refined equation, but he did not prove the rationality of the substitution. Zhao et al. [9] studied the transcendental equation of the transversely isotropic plate, and proved the rationality of the substitution.

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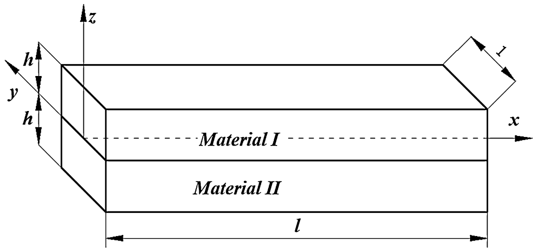

Fig. 1. Schematic diagram of the bi-layer beam structure.

Cheng's refined theory was extended to transversely isotropic plates under transverse surface loadings by Yin and Wang [10] through an Elliott–Lodge (E–L) solution. Gao and Wang [11,12] extended Cheng's theory to monolayer isotropic beams, and gave the refined theory for rectangular elastic beam problems. Gao et al. [13] got the refined theory of beams for a transversely isotropic body. Lu et al. [14] got the refined theory of transversely isotropic thermoporoelastic beam. Based on a refined theory of a single layer, Zhao and others [15,16] got the refined analysis of bi-layer beams and a refined theory of sandwich beams without an ad hoc assumption.

It is the purpose of this paper to extend our previous work to the bi-layer beams for a transversely isotropic body and the refined theory is given.

2. Structure and general solutions

In this paper, we study the bi-layer beams. Two beams are made of different transversely isotropic materials. The x–y plane of each layer beam is an isotropic plane and the interface of two materials is also an isotropic plane, and $z=0$. As shown in Fig. 1, the isotropic planes (x–y plane, $z=0$) of the two beams are stacked. The beam length in the x-direction is l, the beam width in the y-direction is 1, the beam height in the z-direction is $2h$, and $l \gg 2h \gg 1$. Therefore, it is plausible to set the components of stress $\sigma_y = \tau_{xy} = \tau_{yz} = 0$. This is a plane stress assumption, and the width in the y-direction is stress free. The mark of superscript (1) and (2) denote the top beam and the bottom beam of the bi-layer beam, respectively. And we set the nonhomogeneous boundary conditions to be:

$$\sigma_{zx}^{(1)} = 0, \qquad \sigma_{zz}^{(1)} = \frac{q}{2} \quad (\text{on } z=h); \qquad \sigma_{zx}^{(2)} = 0, \qquad \sigma_{zz}^{(2)} = -\frac{q}{2} \quad (\text{on} \quad z=-h).$$

The constitutive equations for the transversely isotropic body in two-dimensional linear elasticity are described to be:

$$\begin{cases} \sigma_{xx}^{(i)} = c_{11}^{(i)} \dfrac{\partial u_x^{(i)}}{\partial x} + c_{13}^{(i)} \dfrac{\partial u_z^{(i)}}{\partial z} \\ \sigma_{zz}^{(i)} = c_{13}^{(i)} \dfrac{\partial u_x^{(i)}}{\partial x} + c_{33}^{(i)} \dfrac{\partial u_z^{(i)}}{\partial z} \qquad (i=1,2) \\ \sigma_{xz}^{(i)} = c_{44}^{(i)} (\dfrac{\partial u_z^{(i)}}{\partial x} + \dfrac{\partial u_x^{(i)}}{\partial z}) \end{cases} \tag{1}$$

The general solution to the transversely isotropic elastic body is:

$$u_x^{(i)} = \left(\psi_1^{(i)} + \psi_2^{(i)}\right)_{,x}, \qquad u_z^{(i)} = \left(k_1^{(i)} \psi_1^{(i)} + k_2^{(i)} \psi_2^{(i)}\right)_{,z} \tag{2}$$

where the constants $k_1^{(i)}$ and $k_2^{(i)}$ satisfy:

$$\begin{aligned} \frac{c_{11}^{(i)}}{c_{44}^{(i)} + (c_{13}^{(i)} + c_{44}^{(i)}) k_1^{(i)}} &= \frac{c_{13}^{(i)} + c_{44}^{(i)} (1 + k_1^{(i)})}{c_{33}^{(i)} k_1^{(i)}} = \left(s_1^{(i)}\right)^2 \\ \frac{c_{11}^{(i)}}{c_{44}^{(i)} + (c_{13}^{(i)} + c_{44}^{(i)}) k_2^{(i)}} &= \frac{c_{13}^{(i)} + c_{44}^{(i)} (1 + k_2^{(i)})}{c_{33}^{(i)} k_2^{(i)}} = \left(s_2^{(i)}\right)^2 \quad (i=1,2) \end{aligned} \tag{3}$$

and $(s_1^{(i)})^2$ and $(s_2^{(i)})^2$ are two characteristic roots of the following quadratic algebra equation of $(s^{(i)})^2$,

$$c_{33}^{(i)} c_{44}^{(i)} \left(s^{(i)}\right)^4 + \left[\left(c_{13}^{(i)}\right)^2 + 2 c_{13}^{(i)} c_{44}^{(i)} - c_{11}^{(i)} c_{33}^{(i)}\right] \left(s^{(i)}\right)^2 + c_{11}^{(i)} c_{44}^{(i)} = 0 \tag{4}$$

We obtain the two roots $(s_1^{(i)})^2$ and $(s_2^{(i)})^2$ of the algebra equation (4) and assume that they are distinct. So the potential functions $\psi_i^{(1)}$ and $\psi_i^{(2)}$ satisfy the following equation:

$$\nabla_{\text{i}}^2 \psi_i^{(1)} = 0, \qquad \nabla_{\text{i}}^2 \psi_i^{(2)} = 0 \quad (i=1,2) \tag{5}$$

with $\nabla_{\text{i}}^2 = \partial_x^2 + \partial_z^2 / (s_i^{(1)})^2$, $\nabla_{\text{i}}^2 = \partial_x^2 + \partial_z^2 / (s_i^{(2)})^2$, $\partial_x = \partial/\partial x$, and $\partial_z = \partial/\partial z$.

According to Wang and Wang [17], it can be further proved that $k_1^{(i)}k_2^{(i)}=1$. Lekhnitskii [18] proved that the numbers $(s_1^{(i)})^2$ and $(s_2^{(i)})^2$ for any transversely isotropic body can be real or complex (with a real part different from zero), but cannot be purely imaginary.

3. The displacement field and the stress states of monolayer beams

In this section, the displacement field and the stress field of a monolayer beam will be studied. Based on Lur'e's method [19] and with these requirements satisfied, treating Eq. (5) as an ordinary differential equation in z with constant coefficients, one obtains the following symbolic solution to Eq. (5):

$$\psi_1=\frac{\sin(s_1z\partial_x)}{s_1\partial_x}g_1+\cos(s_1z\partial_x)g_2,\qquad \psi_2=\frac{\sin(s_2z\partial_x)}{s_2\partial_x}g_3+\cos(s_2z\partial_x)g_4 \tag{6}$$

where

$$\begin{aligned}&\frac{\sin(s_iz\partial_x)}{s_i\partial_x}=z\left(1-\frac{1}{3!}s_i^2z^2\partial_x^2+\frac{1}{5!}s_i^4z^4\partial_x^4-\cdots\right),\\&\cos(s_iz\partial_x)=1-\frac{1}{2!}s_i^2z^2\partial_x^2+\frac{1}{4!}s_i^4z^4\partial_x^4-\cdots\end{aligned} \tag{7}$$

Substituting Eq. (6) into Eq. (2), one obtains:

$$\begin{aligned}&u_x=\frac{\sin(s_1z\partial x)}{s_1}g_1+\cos(s_1z\partial_x)g_2'+\frac{\sin(s_2z\partial_x)}{s_2}g_3+\cos(s_2z\partial_x)g_4'\\&u_z=k_1\cos(s_1z\partial x)g_1-k_1\sin(s_1z\partial x)s_1g_2'+k_2\cos(s_2z\partial_x)g_3-k_2\sin(s_2z\partial_x)s_2g_4'\end{aligned} \tag{8}$$

From Eq. (1) and Eq. (8), the components of stress can be indicated as:

$$\begin{aligned}\frac{\sigma_{xx}}{c_{44}}&=(1+k_1)s_1^2\frac{\sin(s_1z\partial_x)}{s_1}g_1'+(1+k_1)s_1^2\cos(s_1z\partial_x)g_2''\\&\quad+(1+k_2)s_2^2\frac{\sin(s_2z\partial_x)}{s_2}g_3'+(1+k_2)s_2^2\cos(s_2z\partial_x)g_4''\\\frac{\sigma_{zz}}{c_{44}}&=-(1+k_1)\frac{\sin(s_1z\partial_x)}{s_1}g_1'-(1+k_1)\cos(s_1z\partial_x)g_2''\\&\quad-(1+k_2)\frac{\sin(s_2z\partial x)}{s_2}g_3'-(1+k_2)\cos(s_2z\partial_x)g_4''\\\frac{\sigma_{zx}}{c_{44}}&=(1+k_1)\cos(s_1z\partial_x)g_1'-(1+k_1)\sin(s_1z\partial_x)s_1g_2''\\&\quad+(1+k_2)\cos(s_2z\partial_x)g_3'-(1+k_2)\sin(s_2z\partial_x)s_2g_4''\end{aligned} \tag{9}$$

According to Eq. (8) and (9), the displacement and the stress of the interface of the top beam and of the bottom beam can be obtained as follows:

$$\begin{aligned}&U=u_x|_{z=0}=g_2'+g_4',\qquad W=u_z|_{z=0}=k_1g_1+k_2g_3\\&X=\sigma_{zx}|_{z=0}=c_{44}(1+k_1)g_1'+c_{44}(1+k_2)g_3',\\&Z=\sigma_{zz}|_{z=0}=-c_{44}(1+k_1)g_2''-c_{44}(1+k_2)g_4''\end{aligned} \tag{10}$$

Form Eq. (10), the following form of a matrix can be derived:

$$\begin{bmatrix}g_1\\g_2\\g_3\\g_4\end{bmatrix}=\begin{bmatrix}0&\frac{1+k_2}{k_1-k_2}&-\frac{k_2}{c_{44}(k_1-k_2)\partial_x}&0\\-\frac{1+k_2}{(k_1-k_2)\partial_x}&0&0&-\frac{1}{c_{44}(k_1-k_2)\partial_x^2}\\0&-\frac{1+k_1}{k_1-k_2}&\frac{k_1}{c_{44}(k_1-k_2)\partial_x}&0\\\frac{1+k_1}{(k_1-k_2)\partial_x}&0&0&\frac{1}{c_{44}(k_1-k_2)\partial_x^2}\end{bmatrix}\begin{bmatrix}U\\W\\X\\Z\end{bmatrix} \tag{11}$$

Substituting Eq. (11) into Eq. (9), one obtains:

$$\begin{aligned}\frac{\sigma_{xx}}{c_{44}}&=\left[-(1+k_1)s_1^2\cos(s_1z\partial_x)\frac{1+k_2}{k_1-k_2}+(1+k_2)s_2^2\cos(s_2z\partial_x)\frac{1+k_1}{k_1-k_2}\right]\partial_xU\\&\quad+\left[(1+k_1)s_1^2\frac{\sin(s_1z\partial_x)}{s_1}\frac{1+k_2}{k_1-k_2}-(1+k_2)s_2^2\frac{\sin(s_2z\partial_x)}{s_2}\frac{1+k_1}{k_1-k_2}\right]\partial_xW\end{aligned}$$

$$
\begin{aligned}
&+\left[-(1+k_1)s_1^2\frac{\sin(s_1z\partial_x)}{s_1}\frac{k_2}{c_{44}(k_1-k_2)}+(1+k_2)s_2^2\frac{\sin(s_2z\partial_x)}{s_2}\frac{k_1}{c_{44}(k_1-k_2)}\right]X\\
&+\left[-(1+k_1)s_1^2\cos(s_1z\partial_x)\frac{1}{c_{44}(k_1-k_2)}+(1+k_2)s_2^2\cos(s_2z\partial_x)\frac{1}{c_{44}(k_1-k_2)}\right]Z\\
\frac{\sigma_{zz}}{c_{44}}=&\left[(1+k_1)\cos(s_1z\partial_x)\frac{1+k_2}{k_1-k_2}-(1+k_2)\cos(s_2z\partial_x)\frac{1+k_1}{k_1-k_2}\right]\partial_xU\\
&+\left[-(1+k_1)\frac{\sin(s_1z\partial_x)}{s_1}\frac{1+k_2}{k_1-k_2}+(1+k_2)\frac{\sin(s_2z\partial_x)}{s_2}\frac{1+k_1}{k_1-k_2}\right]\partial_xW\\
&+\left[(1+k_1)\frac{\sin(s_1z\partial_x)}{s_1}\frac{k_2}{c_{44}(k_1-k_2)}-(1+k_2)\frac{\sin(s_2z\partial_x)}{s_2}\frac{k_1}{c_{44}(k_1-k_2)}\right]X\\
&+\left[(1+k_1)\cos(s_1z\partial_x)\frac{1}{c_{44}(k_1-k_2)}-(1+k_2)\cos(s_2z\partial_x)\frac{1}{c_{44}(k_1-k_2)}\right]Z\\
\frac{\sigma_{zx}}{c_{44}}=&\left[(1+k_1)\sin(s_1z\partial_x)s_1\frac{1+k_2}{k_1-k_2}-(1+k_2)\sin(s_2z\partial_x)s_2\frac{1+k_1}{k_1-k_2}\right]\partial_xU\\
&+\left[(1+k_1)\cos(s_1z\partial_x)\frac{1+k_2}{k_1-k_2}-(1+k_2)\cos(s_2z\partial_x)\frac{1+k_1}{k_1-k_2}\right]\partial_xW\\
&+\left[-(1+k_1)\cos(s_1z\partial_x)\frac{k_2}{c_{44}(k_1-k_2)}+(1+k_2)\cos(s_2z\partial_x)\frac{k_1}{c_{44}(k_1-k_2)}\right]X\\
&+\left[(1+k_1)\sin(s_1z\partial_x)s_1\frac{1}{c_{44}(k_1-k_2)}-(1+k_2)\sin(s_2z\partial_x)s_2\frac{1}{c_{44}(k_1-k_2)}\right]Z
\end{aligned}
\tag{12}
$$

4. The refined equations of bi-layer transversely isotropic beam

Making use of the refined theory of transversely isotropic beams obtained in the previous section, we will investigate the rectangular straight beam with inhomogeneous boundary conditions on the neutral layer in this section. Namely, the following boundary conditions are prescribed:

$$
\sigma_{zx}^{(1)}=0,\qquad \sigma_{zz}^{(1)}=\frac{q}{2}\quad(\text{on}\quad z=h);\qquad \sigma_{zx}^{(2)}=0,\qquad \sigma_{zz}^{(2)}=-\frac{q}{2}\quad(\text{on}\quad z=-h)
\tag{13}
$$

The mark of superscript (1) and (2) denote the top beam and the bottom beam of the bi-layer beam, respectively. The physical quantities U of the top beam and of the bottom beam are the same. The other physical quantities W', X, and Z have the same properties as U.

Substituting the stress expressions in Eq. (12) into the boundary conditions (13) of beams, we get the following equations:

$$
\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14}\\ L_{21} & L_{22} & L_{23} & L_{24}\\ L_{31} & L_{32} & L_{33} & L_{34}\\ L_{41} & L_{42} & L_{43} & L_{44}\end{bmatrix}
\begin{bmatrix} U'\\ W'\\ X\\ Z\end{bmatrix}=
\begin{bmatrix} 0\\ \frac{q}{2}(k_1^{(1)}-k_2^{(1)})\\ 0\\ \frac{q}{2}(k_1^{(2)}-k_2^{(2)})\end{bmatrix}
\tag{14}
$$

where L_{ij} is given in Appendix A as Eq. (25).

Taking the adjoint matrix L^* on both sides of Eq. (14), one obtains:

$$
\begin{aligned}
&|L|U'=L_{21}^*\frac{q}{2}\big(k_1^{(1)}-k_2^{(1)}\big)+L_{41}^*\frac{q}{2}\big(k_1^{(2)}-k_2^{(2)}\big),\\
&|L|W'=L_{22}^*\frac{q}{2}\big(k_1^{(1)}-k_2^{(1)}\big)+L_{42}^*\frac{q}{2}\big(k_1^{(2)}-k_2^{(2)}\big),\\
&|L|X=L_{23}^*\frac{q}{2}\big(k_1^{(1)}-k_2^{(1)}\big)+L_{43}^*\frac{q}{2}\big(k_1^{(2)}-k_2^{(2)}\big),\\
&|L|Z=L_{24}^*\frac{q}{2}\big(k_1^{(1)}-k_2^{(1)}\big)+L_{44}^*\frac{q}{2}\big(k_1^{(2)}-k_2^{(2)}\big).
\end{aligned}
\tag{15}
$$

where $|L|$ equals:

$$
\begin{aligned}
&\big(c_{44}^{(1)}p_1^{(1)}\big)^2\big(-2+a^{(1)}SN_1^{(1)}SN_2^{(1)}+2CS_1^{(1)}CS_2^{(1)}\big)\big(d^{(2)}-p_1^{(2)}f^{(2)}SN_1^{(2)}SN_2^{(2)}-p_1^{(2)}e^{(2)}CS_1^{(2)}CS_2^{(2)}\big)\\
&\quad+\big(c_{44}^{(2)}p_1^{(2)}\big)^2\big(-d^{(1)}+p_1^{(1)}f^{(1)}SN_1^{(1)}SN_2^{(1)}+p_1^{(1)}e^{(1)}CS_1^{(1)}CS_2^{(1)}\big)\big(2-a^{(2)}SN_1^{(2)}SN_2^{(2)}-2CS_1^{(2)}CS_2^{(2)}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+c_{44}^{(1)}p_1^{(1)}c_{44}^{(2)}p_1^{(2)}\big(t^{(1)}-n^{(1)}SN_1^{(1)}SN_2^{(1)}-t^{(1)}CS_1^{(1)}CS_2^{(1)}\big)\big(p_2^{(2)}-r^{(2)}SN_1^{(2)}SN_2^{(2)}-p_2^{(2)}CS_1^{(2)}CS_2^{(2)}\big)\\
&+c_{44}^{(1)}p_1^{(1)}c_{44}^{(2)}p_1^{(2)}\big(-p_2^{(1)}+r^{(1)}SN_1^{(1)}SN_2^{(1)}+p_2^{(1)}CS_1^{(1)}CS_2^{(1)}\big)\big(-t^{(2)}+n^{(2)}SN_1^{(2)}SN_2^{(2)}+t^{(2)}CS_1^{(2)}CS_2^{(2)}\big)\\
&+c_{44}^{(1)}p_1^{(1)}c_{44}^{(2)}p_1^{(2)}l^{(1)}l^{(2)}\big(s_1^{(1)}SN_1^{(1)}CS_2^{(1)}-s_2^{(1)}SN_2^{(1)}CS_1^{(1)}\big)\left(-\frac{1}{s_2^{(2)}}SN_2^{(2)}CS_1^{(2)}+\frac{1}{s_1^{(2)}}SN_1^{(2)}CS_2^{(2)}\right)\\
&+c_{44}^{(1)}p_1^{(1)}c_{44}^{(2)}p_1^{(2)}l^{(1)}l^{(2)}\left(-\frac{1}{s_2^{(1)}}SN_2^{(1)}CS_1^{(1)}+\frac{1}{s_1^{(1)}}SN_1^{(1)}CS_2^{(1)}\right)\big(s_1^{(2)}SN_1^{(2)}CS_2^{(2)}-s_2^{(2)}SN_2^{(2)}CS_1^{(2)}\big).
\end{aligned}
\tag{16}
$$

L_{2j}^* and L_{4j}^* $(j=1,2,3,4)$ are given in Appendix A as Eq. (26), and all the coefficient of determinant $|L|$ can be found in Eq. (27) and (28).

Since Eq. (16) is of high-infinite power, it does not lead to the engineering application. The following equations can be obtained by omitting the high-order terms:

$$
|L|=\alpha_1h^4\partial_x^4+\alpha_2h^6\partial_x^6=\alpha_1h^4\partial_x^4\big(1+\xi h^2\partial_x^2\big)
\tag{17}
$$

where

$$
\begin{aligned}
\alpha_1&=\frac{1}{12}\big\{c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]l^{(2)}\big\}^2+\frac{1}{12}\big\{c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]l^{(1)}\big\}^2\\
&\quad+\frac{7}{6}c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]l^{(1)}l^{(2)}\\
\alpha_2&=\big\{c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]\big\}^2\left\{\frac{1}{24}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]-\frac{1}{180}l^{(2)}\big[\big(s_1^{(1)}\big)^2+\big(s_2^{(1)}\big)^2\big]\right\}l^{(2)}\\
&\quad+\big\{c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]\big\}^2\left\{\frac{1}{24}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]-\frac{1}{180}l^{(1)}\big[\big(s_1^{(2)}\big)^2+\big(s_2^{(2)}\big)^2\big]\right\}l^{(1)}\\
&\quad-\frac{1}{24}c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big\{\big[\big(1+3k_1^{(2)}-k_2^{(2)}+k_1^{(2)}k_2^{(2)}\big)\big(s_1^{(2)}\big)^4+\big(1+3k_2^{(2)}-k_1^{(2)}+k_1^{(2)}k_2^{(2)}\big)\big(s_2^{(2)}\big)^4\\
&\quad-2p_1^{(2)}\big(s_1^{(2)}\big)^2\big(s_2^{(2)}\big)^2\big]\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]l^{(1)}+\big[\big(1+3k_1^{(1)}-k_2^{(1)}+k_1^{(1)}k_2^{(1)}\big)\big(s_1^{(1)}\big)^4\\
&\quad+\big(1+3k_2^{(1)}-k_1^{(1)}+k_1^{(1)}k_2^{(1)}\big)\big(s_2^{(1)}\big)^4-2p_1^{(1)}\big(s_1^{(1)}\big)^2\big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]l^{(2)}\big\}\\
&\quad-\frac{8}{45}c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}l^{(1)}l^{(2)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]\big(\big(s_1^{(1)}\big)^2+\big(s_2^{(1)}\big)^2+\big(s_1^{(2)}\big)^2+\big(s_2^{(2)}\big)^2\big),\\
\xi&=\alpha_2/\alpha_1.
\end{aligned}
\tag{18}
$$

And Eq. (15) can be rewritten in the following form:

$$
\begin{aligned}
&\alpha_1h^4\partial_x^4\big(1+\xi h^2\partial_x^2\big)U'=\beta_1h^2\partial_x^2\big(1+\eta_1h^2\partial_x^2\big)q\\
&\alpha_1h^4\partial_x^4\big(1+\xi h^2\partial_x^2\big)W'=\beta_2h\partial_x\big(1+\eta_2h^2\partial_x^2\big)q\\
&\alpha_1h^4\partial_x^4\big(1+\xi h^2\partial_x^2\big)X=\beta_3h^3\partial_x^3\big(1+\eta_3h^2\partial_x^2\big)q\\
&\alpha_1h^4\partial_x^4\big(1+\xi h^2\partial_x^2\big)Z=\beta_4h^4\partial_x^4\big(1+\eta_4h^2\partial_x^2\big)q
\end{aligned}
\tag{19}
$$

where the coefficients of the Eq. (19) are given in Appendix B as Eq. (29).

Taking the operator $(1-\xi h^2\partial_x^2)$ on both sides of Eq. (19) and then omitting all the terms associated with h^4 or the higher-order terms, we obtain:

$$
\begin{aligned}
&\alpha_1h^4\partial_x^4U'=\beta_1h^2\partial_x^2\big[1+(\eta_1-\xi)h^2\partial_x^2\big]q\\
&\alpha_1h^4\partial_x^4W'=\beta_2h\partial_x\big[1+(\eta_2-\xi)h^2\partial_x^2\big]q\\
&\alpha_1h^4\partial_x^4X=\beta_3h^3\partial_x^3\big[1+(\eta_3-\xi)h^2\partial_x^2\big]q\\
&\alpha_1h^4\partial_x^4Z=\beta_4h^4\partial_x^4\big[1+(\eta_4-\xi)h^2\partial_x^2\big]q
\end{aligned}
\tag{20}
$$

Eq. (20) are the governing equations of the bi-layer transversely isotropic beams.

According to Eq. (20), if the coefficients of every layer are the same, the refined theory of bi-layer beams for a transversely isotropic body can be degenerated into the refined theory of the monolayer beam for a transversely isotropic body. We can obtain $U'=0$, $Z=0$ and

$$
\frac{2c_{44}(1+k_1)(1+k_2)(s_2^2-s_1^2)h^3}{3(k_1-k_2)}W''''=\left[1-\frac{5[k_1(1+k_2)s_2^2-k_2(1+k_1)s_1^2]+(k_2-k_1)(s_1^2+s_2^2)}{10(k_1-k_2)}h^2\partial_x^2\right]q
\tag{21}
$$

Eq. (21) provides the same conclusion as in Gao et al. [13].

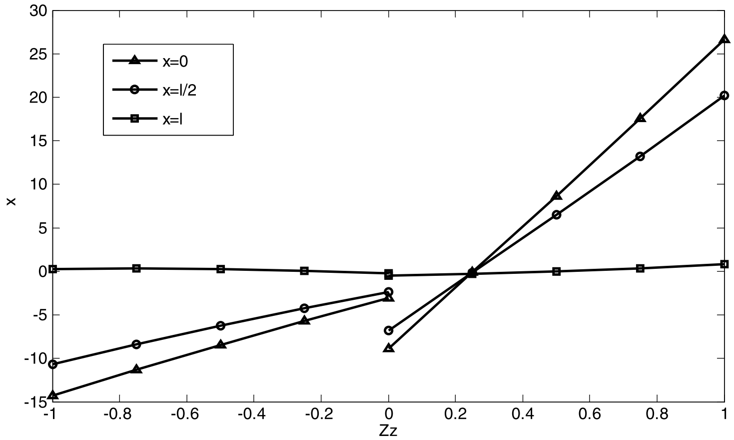

Fig. 2. Distribution curve of the stress σ_x along the axial direction z_z.

5. Numerical results

In this section, the simply supported beam is studied, and the deformations are symmetrical about the z-axis. The material chosen for the top beam is Ti [20] and the material chosen for the bottom beam is Mg [21]. The physical data for the two materials are given below:

$$c_{11}^{(1)} = 18.78 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}, \qquad c_{12}^{(1)} = 8.76 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}, \qquad c_{13}^{(1)} = 8 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2},$$
$$c_{33}^{(1)} = 18.2 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}, \qquad c_{44}^{(1)} = 5.06 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}.$$
$$c_{11}^{(2)} = 5.97 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}, \qquad c_{12}^{(2)} = 2.62 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}, \qquad c_{13}^{(2)} = 2.17 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2},$$
$$c_{33}^{(2)} = 6.17 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}, \qquad c_{44}^{(2)} = 1.51 \times 10^{10}\ \text{N}{\cdot}\text{m}^{-2}.$$

Let $2l = 10$ m, $2h = 2$ m and the load of the top beam and of the bottom beam be $\pm\frac{q}{2}$, which is a constant.
The boundary condition has following form:

$$x = -l: \quad \int_{-h}^{h} \sigma_{zx}\text{d}z = ql, \qquad \int_{-h}^{h} \sigma_{xx}z\text{d}z = 0, \qquad W = 0$$
$$x = l: \quad \int_{-h}^{h} \sigma_{zx}\text{d}z = -ql, \qquad \int_{-h}^{h} \sigma_{xx}z\text{d}z = 0, \qquad W = 0. \tag{22}$$

5.1. The distribution of the normal stresses along the axial direction

In order to make drawing more convenient, the following dimensionless quantities are introduced:

$$\sigma_x = \frac{\sigma_{xx}}{q}, \qquad \sigma_z = \frac{\sigma_{zz}}{q}, \qquad z_z = \frac{z}{h}. \tag{23}$$

Fig. 2 depicts the stress distribution σ_x against the thickness z_z for $x = 0$, $x = 0.5l$ and $x = l$. Fig. 3 shows the variation of the stress distribution σ_z against the thickness z_z.

5.2. The deflection curve of the beams

According to the boundary condition (22) and to Eq. (20), the displacement W has the following form:

$$\frac{W}{q} = 0.17895 \times 10^{-10}\left(\frac{1}{24}x^4 - 6.7804x^2 + 143.4784\right)\ (\text{m}^3/\text{N}) \tag{24}$$

And the deflection curve of the beams can be drawn as in Fig. 4.

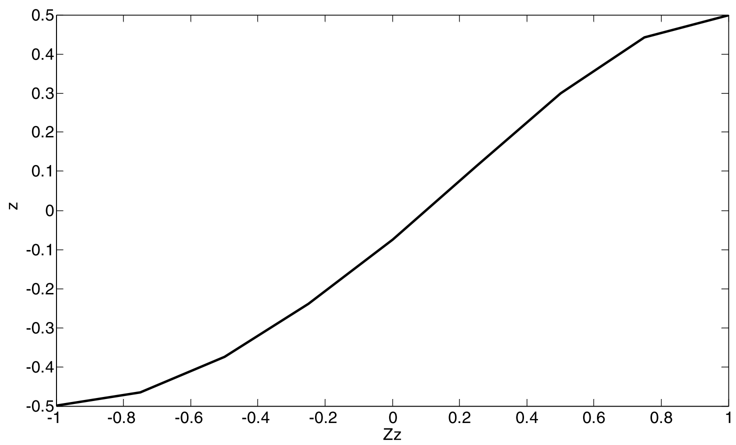

Fig. 3. Distribution curve of the stress σ_z along the axial direction z_z.

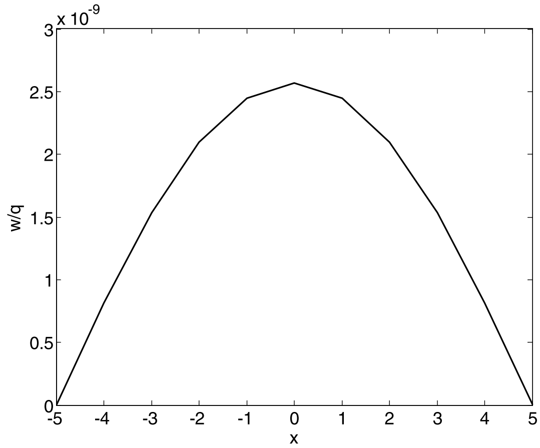

Fig. 4. Deflection curve of the beams.

6. Conclusion

By using Luré's method and the refined theory of a transversely isotropic beam, the refined theory of bi-layer beams for a transversely isotropic body has been deduced without ad hoc assumption. For the beams under transverse loadings, approximate equations are reached. When the coefficients of the top beam and the bottom beam are the same, the refined equations and governing equations of bi-layer beams for a transversely isotropic body can be degenerated into these of monolayer beams for a transversely isotropic body.

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Appendix A

This appendix gives the operational parameters of the process.

$$L_{11} = c_{44}^{(1)}\left(1+k_1^{(1)}\right)\left(1+k_2^{(1)}\right)\left[\sin\left(s_1^{(1)}h\partial_x\right)s_1^{(1)} - \sin\left(s_2^{(1)}h\partial_x\right)s_2^{(1)}\right],$$
$$L_{12} = c_{44}^{(1)}\left(1+k_1^{(1)}\right)\left(1+k_2^{(1)}\right)\left[\cos\left(s_1^{(1)}h\partial_x\right) - \cos\left(s_2^{(1)}h\partial_x\right)\right],$$
$$L_{13} = -k_2^{(1)}\left(1+k_1^{(1)}\right)\cos\left(s_1^{(1)}h\partial_x\right) + k_1^{(1)}\left(1+k_2^{(1)}\right)\cos\left(s_2^{(1)}h\partial_x\right),$$
$$L_{14} = \left(1+k_1^{(1)}\right)\sin\left(s_1^{(1)}h\partial_x\right)s_1^{(1)} - \left(1+k_2^{(1)}\right)\sin\left(s_2^{(1)}h\partial_x\right)s_2^{(1)},$$
$$L_{21} = c_{44}^{(1)}\left(1+k_1^{(1)}\right)\left(1+k_2^{(1)}\right)\left[\cos\left(s_1^{(1)}h\partial_x\right) - \cos\left(s_2^{(1)}h\partial_x\right)\right],$$

$$
\begin{aligned}
L_{22} &= -c_{44}^{(1)}\left(1+k_1^{(1)}\right)\left(1+k_2^{(1)}\right)\left[\sin\left(s_1^{(1)}h\partial_x\right)/s_1^{(1)} - \sin\left(s_2^{(1)}h\partial_x\right)/s_2^{(1)}\right],\\
L_{23} &= \left(1+k_1^{(1)}\right)\sin\left(s_1^{(1)}h\partial_x\right)k_2^{(1)}/s_1^{(1)} - \left(1+k_2^{(1)}\right)\sin\left(s_2^{(1)}h\partial_x\right)k_1^{(1)}/s_2^{(1)},\\
L_{24} &= \left(1+k_1^{(1)}\right)\cos\left(s_1^{(1)}h\partial_x\right) - \left(1+k_2^{(1)}\right)\cos\left(s_2^{(1)}h\partial_x\right),\\
L_{31} &= -c_{44}^{(2)}\left(1+k_1^{(2)}\right)\left(1+k_2^{(2)}\right)\left[\sin\left(s_1^{(2)}h\partial_x\right)s_1^{(2)} - \sin\left(s_2^{(2)}h\partial_x\right)s_2^{(2)}\right],\\
L_{32} &= c_{44}^{(2)}\left(1+k_1^{(2)}\right)\left(1+k_2^{(2)}\right)\left[\cos\left(s_1^{(2)}h\partial_x\right) - \cos\left(s_2^{(2)}h\partial_x\right)\right],\\
L_{33} &= -k_2^{(2)}\left(1+k_1^{(2)}\right)\cos\left(s_1^{(2)}h\partial_x\right) + k_1^{(2)}\left(1+k_2^{(2)}\right)\cos\left(s_2^{(2)}h\partial_x\right),\\
L_{34} &= -\left(1+k_1^{(2)}\right)\sin\left(s_1^{(2)}h\partial_x\right)s_1^{(2)} + \left(1+k_2^{(2)}\right)\sin\left(s_2^{(2)}h\partial_x\right)s_2^{(2)},\\
L_{41} &= c_{44}^{(2)}\left(1+k_1^{(2)}\right)\left(1+k_2^{(2)}\right)\left[\cos\left(s_1^{(2)}h\partial_x\right) - \cos\left(s_2^{(2)}h\partial_x\right)\right],\\
L_{42} &= c_{44}^{(2)}\left(1+k_1^{(2)}\right)\left(1+k_2^{(2)}\right)\left[\sin\left(s_1^{(2)}h\partial_x\right)/s_1^{(2)} - \sin\left(s_2^{(2)}h\partial_x\right)/s_2^{(2)}\right],\\
L_{43} &= -\left(1+k_1^{(2)}\right)\sin\left(s_1^{(2)}h\partial_x\right)k_2^{(2)}/s_1^{(2)} + \left(1+k_2^{(2)}\right)\sin\left(s_2^{(2)}h\partial_x\right)k_1^{(2)}/s_2^{(2)},\\
L_{44} &= \left(1+k_1^{(2)}\right)\cos\left(s_1^{(2)}h\partial_x\right) - \left(1+k_2^{(2)}\right)\cos\left(s_2^{(2)}h\partial_x\right). \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
L_{21}^* &= -c_{44}^{(1)}p_1^{(1)}\left(CS_1^{(1)} - CS_2^{(1)}\right)\left(d^{(2)} - p_1^{(2)}e^{(2)}CS_1^{(2)}CS_2^{(2)} - p_1^{(2)}f^{(2)}SN_1^{(2)}SN_2^{(2)}\right)\\
&\quad + c_{44}^{(2)}p_1^{(2)}\left(-m_1^{(1)}k_2^{(1)}CS_1^{(1)} + m_2^{(1)}k_1^{(1)}CS_2^{(1)}\right)\left(-p_2^{(2)} + p_2^{(2)}CS_1^{(2)}CS_2^{(2)} + r^{(2)}SN_1^{(2)}SN_2^{(2)}\right)\\
&\quad - c_{44}^{(2)}p_1^{(2)}l^{(2)}\left(m_1^{(1)}SN_1^{(1)}s_1^{(1)} - m_2^{(1)}SN_2^{(1)}s_2^{(1)}\right)\left(-\frac{SN_2^{(2)}CS_1^{(2)}}{s_2^{(2)}} + \frac{SN_1^{(2)}CS_2^{(2)}}{s_1^{(2)}}\right),\\
L_{41}^* &= -c_{44}^{(2)}p_1^{(2)}\left(CS_1^{(2)} - CS_2^{(2)}\right)\left(-d^{(1)} + p_1^{(1)}e^{(1)}CS_1^{(1)}CS_2^{(1)} + p_1^{(1)}f^{(1)}SN_1^{(1)}SN_2^{(1)}\right)\\
&\quad + c_{44}^{(1)}p_1^{(1)}\left(-m_1^{(2)}k_2^{(2)}CS_1^{(2)} + m_2^{(2)}k_1^{(2)}CS_2^{(2)}\right)\left(p_2^{(1)} - p_2^{(1)}CS_1^{(1)}CS_2^{(1)} - r^{(1)}SN_1^{(1)}SN_2^{(1)}\right)\\
&\quad - c_{44}^{(1)}p_1^{(1)}l^{(1)}\left(-m_1^{(2)}SN_1^{(2)}s_1^{(2)} + m_2^{(2)}SN_2^{(2)}s_2^{(2)}\right)\left(-\frac{SN_2^{(1)}CS_1^{(1)}}{s_2^{(1)}} + \frac{SN_1^{(1)}CS_2^{(1)}}{s_1^{(1)}}\right),\\
L_{22}^* &= c_{44}^{(1)}p_1^{(1)}\left(SN_1^{(1)}s_1^{(1)} - SN_2^{(1)}s_2^{(1)}\right)\left(d^{(2)} - p_1^{(2)}e^{(2)}CS_1^{(2)}CS_2^{(2)} - p_1^{(2)}f^{(2)}SN_1^{(2)}SN_2^{(2)}\right)\\
&\quad - c_{44}^{(2)}p_1^{(2)}l^{(2)}\left(-m_1^{(1)}k_2^{(1)}CS_1^{(1)} + m_2^{(1)}k_1^{(1)}CS_2^{(1)}\right)\left(-s_2^{(2)}SN_2^{(2)}CS_1^{(2)} + s_1^{(2)}SN_1^{(2)}CS_2^{(2)}\right)\\
&\quad + c_{44}^{(2)}p_1^{(2)}\left(m_1^{(1)}SN_1^{(1)}s_1^{(1)} - m_2^{(1)}SN_2^{(1)}s_2^{(1)}\right)\left(-t^{(2)} + t^{(2)}CS_1^{(2)}CS_2^{(2)} + n^{(2)}SN_1^{(2)}SN_2^{(2)}\right),\\
L_{42}^* &= c_{44}^{(2)}p_1^{(2)}\left(-SN_1^{(2)}s_1^{(2)} + SN_2^{(2)}s_2^{(2)}\right)\left(-d^{(1)} + p_1^{(1)}e^{(1)}CS_1^{(1)}CS_2^{(1)} + p_1^{(1)}f^{(1)}SN_1^{(1)}SN_2^{(1)}\right)\\
&\quad - c_{44}^{(1)}p_1^{(1)}l^{(1)}\left(-m_1^{(2)}k_2^{(2)}CS_1^{(2)} + m_2^{(2)}k_1^{(2)}CS_2^{(2)}\right)\left(-s_2^{(1)}SN_2^{(1)}CS_1^{(1)} + s_1^{(1)}SN_1^{(1)}CS_2^{(1)}\right)\\
&\quad + c_{44}^{(1)}p_1^{(1)}\left(-m_1^{(2)}SN_1^{(2)}s_1^{(2)} + m_2^{(2)}SN_2^{(2)}s_2^{(2)}\right)\left(t^{(1)} - t^{(1)}CS_1^{(1)}CS_2^{(1)} - h^{(1)}SN_1^{(1)}SN_2^{(1)}\right),\\
L_{23}^* &= -c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\left(SN_1^{(1)}s_1^{(1)} - SN_2^{(1)}s_2^{(1)}\right)\left(-p_2^{(2)} + p_2^{(2)}CS_1^{(2)}CS_2^{(2)} + r^{(2)}SN_1^{(2)}SN_2^{(2)}\right)\\
&\quad + c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}l^{(2)}\left(CS_1^{(1)} - CS_2^{(1)}\right)\left(-s_2^{(2)}SN_2^{(2)}CS_1^{(2)} + s_1^{(2)}SN_1^{(2)}CS_2^{(2)}\right)\\
&\quad - c_{44}^{(2)}p_1^{(2)}\left(m_1^{(1)}SN_1^{(1)}s_1^{(1)} - m_2^{(1)}SN_2^{(1)}s_2^{(1)}\right)\left(2 - 2CS_1^{(2)}CS_2^{(2)} - a^{(2)}SN_1^{(2)}SN_2^{(2)}\right),\\
L_{43}^* &= -c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\left(-SN_1^{(2)}s_1^{(2)} + SN_2^{(2)}s_2^{(2)}\right)\left(p_2^{(1)} - p_2^{(1)}CS_1^{(1)}CS_2^{(1)} - r^{(1)}SN_1^{(1)}SN_2^{(1)}\right)\\
&\quad + c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}l^{(1)}\left(CS_1^{(2)} - CS_2^{(2)}\right)\left(-s_2^{(1)}SN_2^{(1)}CS_1^{(1)} + s_1^{(1)}SN_1^{(1)}CS_2^{(1)}\right)\\
&\quad - c_{44}^{(1)}p_1^{(1)}\left(-m_1^{(2)}SN_1^{(2)}s_1^{(2)} + m_2^{(2)}SN_2^{(2)}s_2^{(2)}\right)\left(-2 + 2CS_1^{(1)}CS_2^{(1)} + a^{(1)}SN_1^{(1)}SN_2^{(1)}\right),\\
L_{24}^* &= c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}l^{(2)}\left(SN_1^{(1)}s_1^{(1)} - SN_2^{(1)}s_2^{(1)}\right)\left(-\frac{SN_2^{(2)}CS_1^{(2)}}{s_2^{(2)}} + \frac{SN_1^{(2)}CS_2^{(2)}}{s_1^{(2)}}\right)\\
&\quad - c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\left(CS_1^{(1)} - CS_2^{(1)}\right)\left(-t^{(2)} + t^{(2)}CS_1^{(2)}CS_2^{(2)} + n^{(2)}SN_1^{(2)}SN_2^{(2)}\right)\\
&\quad + c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\left(-m_1^{(1)}k_2^{(1)}CS_1^{(1)} + m_2^{(1)}k_1^{(1)}CS_2^{(1)}\right)\left(2 - 2CS_1^{(2)}CS_2^{(2)} - a^{(2)}SN_1^{(2)}SN_2^{(2)}\right),\\
L_{44}^* &= c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}l^{(1)}\left(-SN_1^{(2)}s_1^{(2)} + SN_2^{(2)}s_2^{(2)}\right)\left(-\frac{SN_2^{(1)}CS_1^{(1)}}{s_2^{(1)}} + \frac{SN_1^{(1)}CS_2^{(1)}}{s_1^{(1)}}\right)\\
&\quad - c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\left(CS_1^{(2)} - CS_2^{(2)}\right)\left(t^{(1)} - t^{(1)}CS_1^{(1)}CS_2^{(1)} - n^{(1)}SN_1^{(1)}SN_2^{(1)}\right)
\end{aligned}
$$

$$+c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big(-m_1^{(2)}k_2^{(2)}CS_1^{(2)}+m_2^{(2)}k_1^{(2)}CS_2^{(2)}\big)\big(-2+2CS_1^{(1)}CS_2^{(1)}+a^{(1)}SN_1^{(1)}SN_2^{(1)}\big). \tag{26}$$

where

$$SN_1^{(i)}=\sin\big(s_1^{(i)}h\partial_x\big),\qquad SN_2^{(i)}=\sin\big(s_2^{(i)}h\partial_x\big),\qquad CS_1^{(i)}=\cos\big(s_1^{(i)}h\partial_x\big),\qquad CS_2^{(i)}=\cos\big(s_2^{(i)}h\partial_x\big) \tag{27}$$

$$\begin{aligned}
&a^{(i)}=\frac{(s_1^{(i)})^2+(s_2^{(i)})^2}{s_1^{(i)}s_2^{(i)}},\qquad d^{(i)}=\big(1+k_1^{(i)}\big)^2k_2^{(i)}+\big(1+k_2^{(i)}\big)^2k_1^{(i)},\qquad e^{(i)}=k_1^{(i)}+k_2^{(i)},\\
&f^{(i)}=\frac{k_1^{(i)}s_1^{2\,(i)}+k_2^{(i)}s_2^{2\,(i)}}{s_1^{(i)}s_2^{(i)}},\qquad l^{(i)}=k_1^{(i)}-k_2^{(i)},\qquad m_1^{(i)}=1+k_1^{(i)},\qquad m_2^{(i)}=1+k_2^{(i)},\\
&n^{(i)}=\frac{(1+k_2^{(i)})k_1^{(i)}(s_1^{(i)})^2+(1+k_1^{(i)})k_2^{(i)}(s_2^{(i)})^2}{s_1^{(i)}s_2^{(i)}},\qquad p_1^{(i)}=\big(1+k_1^{(i)}\big)\big(1+k_2^{(i)}\big),\\
&r^{(i)}=\frac{(1+k_2^{(i)})(s_1^{(i)})^2+(1+k_1^{(i)})(s_2^{(i)})^2}{s_1^{(i)}s_2^{(i)}},\qquad t^{(i)}=k_1^{(i)}\big(1+k_2^{(i)}\big)+k_2^{(i)}\big(1+k_1^{(i)}\big),
\end{aligned} \tag{28}$$

where $I=1,2$.

Appendix B

$$\begin{aligned}
&\eta_1=\frac{\gamma_1}{\beta_1},\qquad \eta_2=\frac{\gamma_2}{\beta_2},\qquad \eta_3=\frac{\gamma_3}{\beta_3},\qquad \eta_4=\frac{\gamma_4}{\beta_4},\\
&\beta_1=\frac{1}{2}\big[\big(A_1'+A_2'\big)l^{(1)}+\big(C_1'+C_2'\big)l^{(2)}\big],\qquad \beta_2=\frac{1}{2}\big[\big(E_1'+E_2'\big)l^{(1)}+\big(G_1'+G_2'\big)l^{(2)}\big],\\
&\beta_3=\frac{1}{2}\big[\big(I'\big)l^{(1)}+\big(L'\big)l^{(2)}\big],\qquad \beta_4=\frac{1}{2}\big[\big(N_1'+N_2'\big)l^{(1)}+\big(R_1'+R_2'\big)l^{(2)}\big],\\
&\gamma_1=\frac{1}{2}\big[\big(B_1'+B_2'+B_3'\big)l^{(1)}+\big(D_1'+D_2'+D_3'\big)l^{(2)}\big],\\
&\gamma_2=\frac{1}{2}\big[\big(F_1'+F_2'+F_3'\big)l^{(1)}+\big(H_1'+H_2'+H_3'\big)l^{(2)}\big],\\
&\gamma_3=\frac{1}{2}\big[\big(J_1'+J_2'+J_3'\big)l^{(1)}+\big(M_1'+M_2'+M_3'\big)l^{(2)}\big]\\
&\gamma_4=\frac{1}{2}\big[\big(Q_1'+Q_2'+Q_3'\big)l^{(1)}+\big(T_1'+T_2'+T_3'\big)l^{(2)}\big]
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
&A_1'=-\frac{1}{2}c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]\big(l^{(2)}\big)^2,\qquad A_2'=\frac{1}{2}c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]l^{(1)}l^{(2)},\\
&B_1'=-c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]\bigg\{\frac{1}{4}\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]p_1^{(2)}l^{(2)}-\frac{1}{24}\big[\big(s_1^{(1)}\big)^2+\big(s_2^{(1)}\big)^2\big]\big(l^{(2)}\big)^2\bigg\},\\
&B_2'=c_{44}^{(2)}p_1^{(2)}\bigg\{\frac{1}{24}l^{(1)}\big[\big(-3k_1^{(2)}+k_2^{(2)}-2\big)\big(s_1^{(2)}\big)^4+\big(-3k_2^{(2)}+k_1^{(2)}-2\big)\big(s_2^{(2)}\big)^4+\big(m_1^{(2)}+m_2^{(2)}\big)\big(s_1^{(2)}s_2^{(2)}\big)^2\big]\\
&\qquad+\frac{1}{4}\big(m_1^{(1)}k_2^{(1)}s_1^{2\,(1)}-m_2^{(1)}k_1^{(1)}s_2^{2\,(1)}\big)\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]l^{(2)}\bigg\},\\
&B_3'=-\frac{1}{3}c_{44}^{(2)}p_1^{(2)}l^{(2)}\big[m_1^{(1)}\big(s_1^{(1)}\big)^2-m_2^{(1)}\big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big],\\
&C_1'=\frac{1}{2}c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]\big(l^{(1)}\big)^2,\qquad C_2'=-\frac{1}{2}c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]l^{(1)}l^{(2)},\\
&D_1'=-c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2-\big(s_2^{(2)}\big)^2\big]\bigg[-\frac{1}{4}\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]p_1^{(1)}l^{(1)}+\frac{1}{24}\big[\big(s_1^{(2)}\big)^2+\big(s_2^{(2)}\big)^2\big]l^{2\,(1)}\bigg],\\
&D_2'=\bigg\{-\frac{1}{24}l^{(2)}\big[\big(-3k_1^{(1)}+k_2^{(1)}-2\big)\big(s_1^{(1)}\big)^4+\big(-3k_2^{(1)}+k_1^{(1)}-2\big)\big(s_2^{(1)}\big)^4+\big(m_1^{(1)}+m_2^{(1)}\big)\big(s_1^{(1)}s_2^{(1)}\big)^2\big]\\
&\qquad-\frac{1}{4}\big(m_1^{(2)}k_2^{(2)}\big(s_1^{(2)}\big)^2-m_2^{(2)}k_1^{(2)}\big(s_2^{(2)}\big)^2\big)\big[\big(s_1^{(1)}\big)^2-\big(s_2^{(1)}\big)^2\big]l^{(1)}\bigg\}c_{44}^{(1)}p_1^{(1)},
\end{aligned}$$

$$D'_3 = \frac{1}{3}c_{44}^{(1)}p_1^{(1)}l^{(1)}\big[m_1^{(2)}\big(s_1^{(2)}\big)^2 - m_2^{(2)}\big(s_2^{(2)}\big)^2\big]\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big],$$

$$E'_1 = -c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big(l^{(2)}\big)^2, \qquad E'_2 = -c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]l^{(1)}l^{(2)},$$

$$F'_1 = c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\left\{-\frac{1}{2}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]p_1^{(2)} + \frac{1}{6}\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big]l^{(2)}\right\}l^{(2)},$$

$$F'_2 = -c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\left\{\frac{1}{2}\big[m_1^{(1)}k_2^{(1)}\big(s_1^{(1)}\big)^2 - m_2^{(1)}k_1^{(1)}\big(s_2^{(1)}\big)^2\big] - \frac{1}{6}\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big]l^{(1)}\right\}l^{(2)},$$

$$F'_3 = -\frac{1}{2}c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\big[m_1^{(1)}\big(s_1^{(1)}\big)^2 - m_2^{(1)}\big(s_2^{(1)}\big)^2\big]l^{(2)},$$

$$G'_1 = -c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\big(l^{(1)}\big)^2, \qquad G'_2 = -c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]l^{(1)}l^{(2)},$$

$$H'_1 = c_{44}^{(2)}p_1^{(2)}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\left\{\frac{1}{6}\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big]l^{(1)} - \frac{1}{2}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]p_1^{(1)}\right\}l^{(1)},$$

$$H'_2 = -c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\left\{\frac{1}{2}\big[m_1^{(2)}k_2^{(2)}\big(s_1^{(2)}\big)^2 - m_2^{(2)}k_1^{(2)}\big(s_2^{(2)}\big)^2\big] - \frac{1}{6}\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big]l^{(2)}\right\}l^{(1)},$$

$$H'_3 = -\frac{1}{2}c_{44}^{(1)}p_1^{(1)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[m_1^{(2)}\big(s_1^{(2)}\big)^2 - m_2^{(2)}\big(s_2^{(2)}\big)^2\big]l^{(1)},$$

$$I' = -c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]l^{(2)},$$

$$\begin{aligned} J'_1 = -\bigg\{&\frac{1}{24}\big[\big(-3k_1^{(2)} + k_2^{(2)} - 2\big)\big(s_1^{(2)}\big)^4 + \big(-3k_2^{(2)} + k_1^{(2)} - 2\big)\big(s_2^{(2)}\big)^4 + \big(m_1^{(2)} + m_2^{(2)}\big)\big(s_1^{(2)}s_2^{(2)}\big)^2\big] \\ &- \frac{1}{12}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big]l^{(2)}\bigg\}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}, \end{aligned}$$

$$J'_2 = \frac{c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}}{24}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\big\{2\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big] + \big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big]\big\}l^{(2)},$$

$$J'_3 = -\frac{1}{12}\big[c_{44}^{(2)}p_1^{(2)}\big]^2\big[m_1^{(1)}\big(s_1^{(1)}\big)^2 - m_2^{(1)}\big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]^2,$$

$$L' = -c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]l^{(1)},$$

$$\begin{aligned} M'_1 = -\bigg\{&\frac{1}{24}\big[\big(-3k_1^{(1)} + k_2^{(1)} - 2\big)\big(s_1^{(1)}\big)^4 + \big(-3k_2^{(1)} + k_1^{(1)} - 2\big)\big(s_2^{(1)}\big)^4 + \big(m_1^{(1)} + m_2^{(1)}\big)\big(s_1^{(1)}s_2^{(1)}\big)^2\big] \\ &- \frac{1}{12}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big]l^{(1)}\bigg\}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}, \end{aligned}$$

$$M'_2 = \frac{c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}}{24}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\big\{2\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big] + \big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big]\big\}l^{(1)},$$

$$M'_3 = -\frac{1}{12}\big[c_{44}^{(1)}p_1^{(1)}\big]^2\big[m_1^{(2)}\big(s_1^{(2)}\big)^2 - m_2^{(2)}\big(s_2^{(2)}\big)^2\big]\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]^2,$$

$$N'_1 = \frac{7}{12}c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]l^{(2)},$$

$$N'_2 = \frac{1}{12}\big[c_{44}^{(2)}p_1^{(2)}\big]^2\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]^2l^{(1)},$$

$$\begin{aligned} Q'_1 = &-c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big] \\ &\times\left\{\frac{1}{30}\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big] + \frac{1}{18}\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big]\right\}l^{(2)}, \end{aligned}$$

$$\begin{aligned} Q'_2 = &\frac{1}{48}c_{44}^{(1)}c_{44}^{(2)}p_1^{(1)}p_1^{(2)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big\{\big[\big(-3k_1^{(2)} - 2k_1^{(2)}k_2^{(2)} + k_2^{(2)}\big)\big(s_1^{(2)}\big)^4 \\ &+ \big(-3k_2^{(2)} - 2k_1^{(2)}k_2^{(2)} + k_1^{(2)}\big)\big(s_2^{(2)}\big)^4 + 2\big(k_1^{(2)} + k_2^{(2)} + 2k_1^{(2)}k_2^{(2)}\big)\big(s_1^{(2)}s_2^{(2)}\big)^2\big] \\ &- l^{(2)}\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big]\big\}, \end{aligned}$$

$$
\begin{aligned}
Q_3' &= \big[c_{44}^{(2)} p_1^{(2)}\big]^2 \big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]^2 \\
&\quad \times \left\{ -\frac{1}{180}\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big] l^{(1)} + \frac{1}{24}\big[m_1^{(1)} k_2^{(1)} \big(s_1^{(1)}\big)^2 - m_2^{(1)} k_1^{(1)} \big(s_2^{(1)}\big)^2\big] \right\}, \\
R_1' &= -\frac{7}{12} c_{44}^{(1)} c_{44}^{(2)} p_1^{(1)} p_1^{(2)} \big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big] l^{(1)}, \\
R_2' &= -\frac{1}{12}\big[c_{44}^{(1)} p_1^{(1)}\big]^2 \big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]^2 l^{(2)}, \\
T_1' &= c_{44}^{(1)} c_{44}^{(2)} p_1^{(1)} p_1^{(2)} \big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big] \\
&\quad \times \left\{ \frac{1}{30}\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big] + \frac{1}{18}\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big] \right\} l^{(1)}, \\
T_2' &= -\frac{1}{48} c_{44}^{(1)} c_{44}^{(2)} p_1^{(1)} p_1^{(2)} \big[\big(s_1^{(2)}\big)^2 - \big(s_2^{(2)}\big)^2\big]\big\{\big[\big(-3k_1^{(1)} - 2k_1^{(1)} k_2^{(1)} + k_2^{(1)}\big)\big(s_1^{(1)}\big)^4 \\
&\quad + \big(-3k_2^{(1)} - 2k_1^{(1)} k_2^{(1)} + k_1^{(1)}\big)\big(s_2^{(1)}\big)^4 + 2\big(k_1^{(1)} + k_2^{(1)} + 2k_1^{(1)} k_2^{(1)}\big)\big(s_1^{(1)} s_2^{(1)}\big)^2\big] \\
&\quad - l^{(1)}\big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]\big[\big(s_1^{(2)}\big)^2 + \big(s_2^{(2)}\big)^2\big]\big\}, \\
T_3' &= \big[c_{44}^{(1)} p_1^{(1)}\big]^2 \big[\big(s_1^{(1)}\big)^2 - \big(s_2^{(1)}\big)^2\big]^2 \\
&\quad \times \left\{ \frac{1}{180}\big[\big(s_1^{(1)}\big)^2 + \big(s_2^{(1)}\big)^2\big]\big(l^{(1)}\big)^2 - \frac{1}{24}\big[m_1^{(2)} k_2^{(2)} \big(s_1^{(2)}\big)^2 - m_2^{(2)} k_1^{(2)} \big(s_2^{(2)}\big)^2\big] \right\}.
\end{aligned}
\tag{30}
$$

References

[1] R. Khandan, S. Noroozi, P. Sewell, J. Vinney, The development of laminated composite plate theories: a review, J. Mater. Sci. 47 (2012) 5901–5910.
[2] E. Carrera, Historical review of zig–zag theories for multilayered plates and shells, Appl. Mech. Rev. 56 (2003) 287–308.
[3] E. Carrera, Refined one-dimensional formulations for laminated structure analysis, AIAA J. 50 (2012) 176–189.
[4] S. Cheng, Elasticity theory of plates and a refined theory, J. Appl. Mech. 46 (1979) 644–650.
[5] K.E. Barrett, S. Ellis, An exact theory of elastic plates, Int. J. Solids Struct. 24 (1988) 859–880.
[6] W. Wang, M.X. Shi, Thick plate theory based on general solutions of elasticity, Acta Mech. 123 (1997) 27–36.
[7] C.H. Luo, W. Wang, Refined theory of generalized plane-stress problems in elasticity, Acta Mech. Solida Sin. 24 (2003) 192–196.
[8] F.Y. Wang, Two-dimensional theories deduced from three-dimensional theory for a transversely isotropic body, I: plate problems, Int. J. Solids Struct. 26 (1990) 455–470.
[9] B.S. Zhao, D. Wu, M.Z. Wang, The refined theory and the decomposed theorem of a transversely isotropic elastic plate, Eur. J. Mech. A, Solids 39 (2013) 243–250.
[10] H.M. Yin, W. Wang, A refined theory of transversely isotropic plates, Acta Sci. Nat. Univ. Pekin 37 (2001) 23–33.
[11] Y. Gao, M.Z. Wang, Refined theory of deep rectangular beams based on the general solution of the elastic, Sci. China, Ser. G, Phys. Mech. Astron. 49 (2006) 291–303.
[12] Y. Gao, M.Z. Wang, A refined beam theory based on the refined plate theory, Acta Mech. 177 (2005) 191–197.
[13] Y. Gao, B.X. Xu, B.S. Zhao, The refined theory of beams for a transversely isotropic body, Acta Mech. 191 (2007) 109–122.
[14] G.X. Lu, B.S. Zhao, X.E. Wu, Refined theory and decomposed theorem of transversely isotropic thermoporoelastic beam, C. R. Mecanique 341 (2013) 701–708.
[15] B.S. Zhao, Y.T. Zhao, X.E. Wu, The refined analysis of bi-layer Beams without ad hoc assumption, in: J.H. Fan, J.Q. Zhang, H.B. Chen, Z.H. Jin (Eds.), Advances in Heterogeneous Material Mechanics 2011, DEStech, Lancaster, 2011.
[16] B.S. Zhao, Y. Gao, X. Yu, A refined theory of sandwich beams, in: J.H. Fan, H.B. Chen (Eds.), Advances in Heterogeneous Material Mechanics 2008, DEStech, Lancaster, 2008.
[17] M.Z. Wang, W. Wang, Completeness and nonuniqueness of general solutions of transversely isotropic elasticity, Int. J. Solids Struct. 32 (1995) 501–513.
[18] S.G. Lekhnitskii, Theory of Elasticity of an Anisotropic Body, Mir, Moscow, 1981.
[19] A.I. Lur'e, Three-dimensional Problems in the Theory of Elasticity, Wiley Interscience, New York, 1964.
[20] R. Kumar, T. Kansal, Rayleigh–Lamb waves in transversely isotropic thermoelastic diffusive layer, Int. J. Thermophys. 30 (2009) 710–733.
[21] M. Islam, S.H. Mallik, M. Kanorla, Dynamic response in two-dimensional transversely isotropic thick plate with spatially varying heat sources and body forces, Appl. Math. Mech. 32 (2011) 1315–1332.