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Volumic method for the variational sum of a 2D discrete model

Méthode volumique pour la sommation variationnelle d'un modèle 2D discret

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A B S T R A C T

The geometric complexity of some heterogeneous materials (for example, fibers distributed randomly or deterministically with high conductivity [5,2]) can make it difficult to model their macroscopic behavior. In some cases, it is convenient to simplify the geometry by cutting it into "simple" elements, so that the first study is performed only on these items. The difficulties arise from the reconstruction of the material. In such study, we describe a method for reconstructing a material cut into thin plates having undergone a size reduction (see [6] and [5], for example). The method used is of variational summation limit.

R É S U M É

La complexité géométrique de certains matériaux hétèrogènes (inclusions distribuées aléatoirement dans différentes directions (voir, par exemple, [5,2])) rend difficile la modélisation du comportement à l'échelle macroscopique. Dans certains cas, il est commode de simplifier la géométrie par des éléments plus « simples », afin de ne travailler uniquement que sur ces éléments. La difficulté est alors déplacée vers la reconstruction du modèle 3D. Dans cette étude, nous décrivons une méthode de reconstruction 3D d'un matériau coupé en fines tranches ayant subi une réduction de dimension (voir [6] et [5], par exemple). Cette méthode constitue un passage à la limite par sommation variationnelle.

1. Introduction

Geometric complexities of some heterogeneous materials, especially in the case of random inclusions, can make it difficult to model its macroscopic behavior. In some cases, it is convenient to simplify the geometry by cutting it into n simpler elements, and when n goes to infinity our estimate should be close to the physical reality. These elements will be considered "almost" independent. Although the issue of the connection between the elements becomes important, it is not discussed

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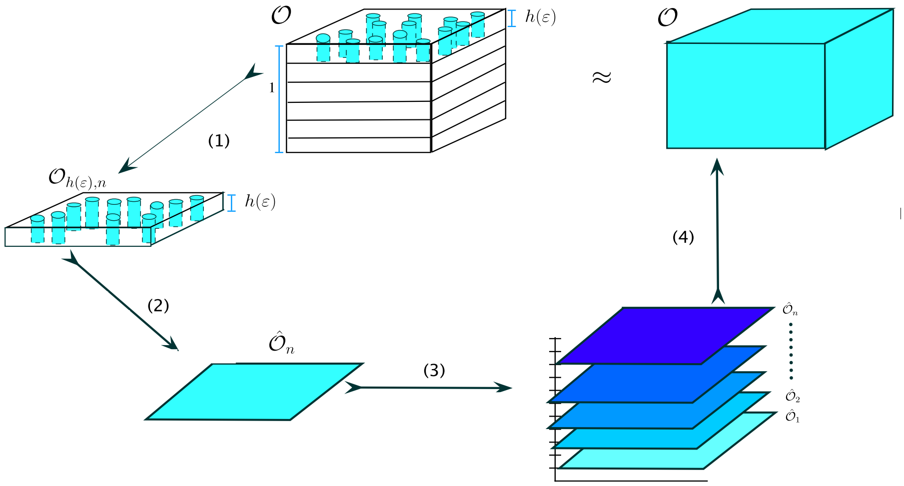

Fig. 1. (Color online.) Illustration of our strategy: step (1), we consider only one slice of material; step (2) we obtain by Γ-convergence a 2D equivalent model; step (3) we superpose all 2D models; step (4) presents the passage to the limit of the number plate by Γ-convergence (variational limit summation).

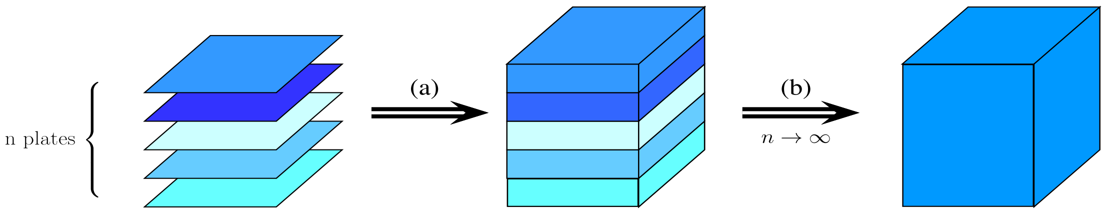

Fig. 2. (Color online.) Illustration of our strategy: step (a) gives a continuous training by part, step (b) is the passage to the limit of the number of plates.

in this study. Our work only addresses the question of reconstruction of the overall problem after studying each element in a simplified setting (Fig. 1). The difficulty arises when reconstructing the material, i.e. we expect that the final result is close to the original model. In this work, we describe a method for reconstructing a material cut into thin plates having undergone a reduction of dimension. The method used is a variational limit summation.

For example, Texsol strategy modeling consists in cutting along x_3 our thin structure dependent on a small parameter ε. For sufficiently small ε, we agree to consider the vertical fibers in each plate in such a sense that we take a leadership in guiding the wire network. Our initial problem is decomposed into n-type plate models to which we previously wanted to give a two-dimensional formulation. A modification made to this approach allows us to model the macroscopic behavior of porous media.

For the moment, this model deals with various steady-state situations, such as, for example, heat diffusion and electrostatic states. This work aims to present an original strategy for modeling complex materials. It must be complemented by future work. For illustration purposes, one can consider the material displayed in Fig. 1.

For more details on step (2), the reader is referred to [2,5]. Our work is only concerned with the question of the reconstruction of the overall problem, after studying each simplified element. In the initial material, the fibers must be moderately gilded, and the plate direction must be privileged. But the challenge here is to reconstruct the material with adequate modeling error.

We present a variational method for the reconstruction of a volumetric model from a surface model obtained, for example, via size reduction. Note that the original material geometry is defined as the cube $\mathcal{O} := \hat{\mathcal{O}} \times (0,1)$. We consider five plates from a material whose dimension has been reduced (Fig. 2). This cutting can be justified for some cases of material with complex geometry (see [1] for example).

To reconstruct a volumic model by "gluing" n two-dimensional problems obtained by size reduction, we consider the discrete energy:

$$E_n(u) = \sum_{k=0}^{n-1} \frac{1}{n}\left(\int_{\hat{\mathcal{O}}} \psi_0\left(\frac{k}{n}, \nabla u\left(\hat{x}, \frac{k}{n}\right)\right) \mathrm{d}\hat{x} - \int_{\hat{\mathcal{O}}} \bar{L}\left(\hat{x}, \frac{k}{n}\right).u\left(\hat{x}, \frac{k}{n}\right) \mathrm{d}\hat{x}\right) \quad \text{if } u \in Step_{3,n}(\mathcal{O})$$

where $\psi_0(\frac{k}{n}, .)$ is an energy density of plate $\frac{k}{n}$ depending on ∇u; this energy depends on x_3 because it is obtained by reduction of dimension. And the set $Step_{3,n}(\mathcal{O})$ represents the following set of functions defined by several parts:

$$u \in Step_{3,n}(\mathcal{O}) \quad \Leftrightarrow \quad u(x) = \sum_{k=0}^{n-1} u\left(\hat{x}, \frac{k}{n}\right) 1_{[\frac{k}{n}, \frac{k+1}{n}[}(x_3) \quad \text{with } (\hat{x}, x_3) \in \hat{\mathcal{O}} \times (0,1)$$

Remark 1. In order to keep a general framework in our study, we consider a volume loading between each pair of plates, assuming that in most cases the load is constant.

Remark 2. The connection between the plates can be taken into account for example by boundary condition (see [6]), which in some cases does not add any difficulty to the method.

Here, we just constructed a discrete macroscopic energy but, to obtain a variational limit model, we must first make a regularization of our energy. In Fig. 2, step (a), we obtain a multiphase material, and this material will be homogenized using a variational approach. To obtain a continuous formulation, it is convenient to construct a "continuous by part" expression of the energy E_n like a Riemann sum, illustrated by (a) in Fig. 1. Indeed,

$$\widetilde{\psi}_0^n(x_3, s) := \psi_0\left(\frac{k}{n}, s\right) \quad \text{and} \quad \widetilde{L}^n(x) := L\left(\hat{x}, \frac{k}{n}\right) \quad \text{if } x_3 \in \left[\frac{k}{n}, \frac{k+1}{n}\right[$$

so the energy E_n becomes:

$$E_n(u) = \int_{\mathcal{O}} \widetilde{\psi}_0^n(x_3, \nabla u)\,\mathrm{d}x - \int_{\mathcal{O}} \widetilde{L}^n(x).u(x)\,\mathrm{d}x \quad \text{if } u \in Step_{3,n}(\mathcal{O})$$

We assume that $s \mapsto \psi_0(.,s)$ is a convex function and satisfies the Lipschitz property

$$\left|\psi_0(x_3, s) - \psi_0\big(x_3, s'\big)\right| \le \ell\left|s - s'\right|\left(1 + |s|^{p-1} + \left|s'\right|^{p-1}\right) \tag{1}$$

for all $(s, s') \in \mathbb{R}^3 \times \mathbb{R}^3$ where ℓ is a positive constant independent of x_3. It is easy to verify that $\widetilde{\psi}_0^n$ satisfies the standard growth condition of order $p > 1$ uniformly in x_3, i.e that there exist two positives constants α and β independent of n, such that for all s of $\mathbb{R}^3$

$$\alpha|s|^p \le \widetilde{\psi}_0^n(x_3, s) \le \beta\big(1 + |s|^p\big) \tag{2}$$

for all x_3 fixed in $(0,1)$.

2. Convergence of the discrete problem

To find a limit formulation of this problem, we need to make a second regularization for the potential energy in each plate. For any bounded function $h : \mathbb{R} \to \mathbb{R}$, we introduce its lower and upper λ-Lipschitz approximations.

Lemma 2.1. *Assume that $h : \mathbb{R} \to \mathbb{R}$ is bounded, upper semi-continuous and $\lambda > 0$. The regularization h^λ is*

$$h^\lambda(x) = \sup_{t \in \mathbb{R}} \big\{h(t) - \lambda|x - t|\big\}$$

and verifies:

i) $|h^\lambda(x) - h^\lambda(x')| \le \lambda|x - x'|$ *for all x and all x' in $\mathbb{R}$;*
ii) $h \le h^\lambda$ *and $(h^\lambda)_{\lambda>0}$ decreasing;*
iii) $\lim_{\lambda \to +\infty} h^\lambda = h$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded function and lower semi-continuous, then Baire's regularization h_λ defined by

$$h_\lambda(x) = \inf_{t \in \mathbb{R}} \big\{h(t) + \lambda|x - t|\big\}$$

verifies the following properties:

i′) $|h_\lambda(x) - h_\lambda(x')| \le \lambda|x - x'|$ *for all x and all x' in $\mathbb{R}$;*
ii′) $h \ge h_\lambda$ *and $(h_\lambda)_{\lambda>0}$ increasing;*
iii′) $\lim_{\lambda \to +\infty} h_\lambda = h$.

The proofs of i), ii), iii) are done in [1]: Theorem 9.2.1 (as well as i′), ii′), iii′) with $\tilde{h} = -h$).

Thereafter, we apply this lemma with function $h = \psi_0(.,s)$. The following result allows us to avoid any trouble with the continuity of the strain tensor according to x_3. Indeed we have the following Lusin's results, compensating for the lack of continuity of $x_3 \mapsto \psi_0(.,x_3)$. For all $s \in \mathbb{R}^3$ fixed, the function $\psi_0(.,s) : (0,1) \to \mathbb{R}$ is assumed to be upper semi-continuous.

Lemma 2.2. *For any $\gamma > 0$, there exists a compact subset $K_\gamma \subset [0,1]$ verifying $|[0,1]\backslash K_\gamma| < \gamma$, such that for all $s \in \mathbb{R}^3$, restriction $\psi_0(.,s)$ on K_γ is continuous.*

This result gives us the possibility to have an estimate of the energy limit in cases where the energies are continuous in almost all the plates. We will first check the compactness of the sequences of finite energy, then study the variational limit of this energy.

2.1. Compactness lemma

Lemma 2.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of set $Step_{3,n}(\mathcal{O})$ such that $\sup_{n\in\mathbb{N}} E_n < +\infty$. Then, there exists a subsequence u_n, and $u \in L^p(\mathcal{O})$ such that: $u_n \rightharpoonup u$ in $L^p(\mathcal{O})$.*

The classical proof tends to show $\|\nabla u\|_{L^p(\mathcal{O})} < \infty$. For more details, see the similar case in [4], and the complete proof using Korn inequality.

2.2. The volumic limit problem

In order to obtain a volumic limit of our total energy, we want to use the Γ-convergence theory, see [3] and [4], for example.

Theorem 2.4. *This sequence of energy functionals $(E_n)_{n\in\mathbb{N}}$, Γ-converges in the weak sense to E_0 in $L^p(\mathcal{O})$ defined by*

$$E_0(u) := \int_{\mathcal{O}} \psi_0(x_3, \nabla u)\,\mathrm{d}x - \int_{\mathcal{O}} L(x)\cdot u(x)\,\mathrm{d}x$$

2.2.1. Sketch of the proof

By definition of Γ-convergence, we must establish two assertions for any displacement field $u \in L^p(\mathcal{O})$:

i) for any sequence $(u_n)_{n\in\mathbb{N}}$ in $Step_{3,n}(\mathcal{O})$ verifying $u_n \rightharpoonup u$, we have $\liminf_{n\to\infty} E_n(u_n) \geq E_0(u)$;
ii) there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $Step_{3,n}(\mathcal{O})$ such that $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} E_n(u_n) \leq E_0(u)$.

Proof of i). For any $\lambda > 0$ and $s \in \mathbb{R}^3$ fixed, we introduce the lower Baire's regularization

$$f_{0,\lambda}(x_3, s) := \inf_{t\in\mathbb{R}}\big[\psi_0(t,s) + \lambda|x_3 - t|\big]$$

Let the compact set K_γ be included in $[0,1]$ and obtained by Lemma 2.2. So, for all s fixed, the function $x_3 \mapsto \psi_0(x_3, s)$ is continuous in K_γ, and verifies all the hypotheses of Lemma 2.1.

Recall that $\psi_{0,\lambda}$ is λ-Lipchitz i.e., for all x_3 and x_3' of K_γ,

$$\big|f_{0,\lambda}(x_3,s) - f_{0,\lambda}\big(x_3', s\big)\big| \leq \lambda\big|x_3 - x_3'\big|$$

and that for all $x_3 \in K_\gamma$,the function $\psi_0(.,s)$ is lower semi-continuous; this is why we have

$$\lim_{\lambda\to+\infty} \psi_{0,\lambda}(x_3,s) = f_0(x_3,s) \tag{3}$$

More precisely, $(\psi_{0,\lambda})_\lambda$ is a strictly increasing sequence converging to ψ when $\lambda \to +\infty$ for all $x_3 \in K_\gamma$, where ψ_0 is lower semi-continuous.

Furthermore, it is easy to see:

$$\liminf_{n\to+\infty} E_n(u_n) \geq \liminf_{n\to+\infty} E_{\lambda,n}(u_n) \tag{4}$$

and $\lim_{n\to+\infty} \int_{\mathcal{O}} u_n(x)\cdot\widetilde{L}^n(x)\,\mathrm{d}x = \int_{\mathcal{O}} u(x)\cdot\widetilde{L}(x)\,\mathrm{d}x$.

Now, we can estimate the limit:

$$\begin{aligned}
\liminf_{n\to+\infty} E_{\lambda,n}(u_n) &= \liminf_{n\to+\infty} \int_{\mathcal{O}} \widetilde{\psi}^n_{0,\lambda}\big(x_3, \nabla u_n(x)\big)\,\mathrm{d}x - \int_{\mathcal{O}} u(x)\cdot\widetilde{L}(x)\,\mathrm{d}x \\
&\geq \liminf_{n\to+\infty} \int_{\mathcal{O}} \big[\widetilde{\psi}^n_{0,\lambda}(x_3, \nabla u_n) - \psi_{0,\lambda}(x_3, \nabla u_n)\big]\,\mathrm{d}x + \int_{\mathcal{O}} \psi_{0,\lambda}(x_3, \nabla u_n)\,\mathrm{d}x \\
&\quad - \int_{\mathcal{O}} u(x)\cdot\widetilde{L}(x)\,\mathrm{d}x
\end{aligned} \tag{5}$$

By the Lipchitz property of $\psi^n_{0,\lambda}$, we have:

$$\int_{\mathcal{O}} \left|\widetilde{\psi}^n_{0,\lambda}(x_3, u_n) - \psi_{0,\lambda}(x_3, \nabla u_n)\right| \mathrm{d}x = \sum_{k=0}^{n-1} \frac{1}{n} \int_{\hat{\mathcal{O}}} \left|\psi_{0,\lambda}\big(k, \nabla u_n(\hat{x}, k)\big) - \psi_{0,\lambda}\big(x_3, \nabla u_n(x)\big)\right|$$
$$\leq \sum_{k=0}^{n-1} \frac{\lambda}{n} \int_{\hat{\mathcal{O}}} |k - x_3| \mathrm{d}\hat{x} \leq |\hat{\mathcal{O}}| \frac{\lambda}{n} \tag{6}$$

Consequently (6), (4) and the lower semi-continuity of $u \mapsto \int_{\mathcal{O}} \psi_{0,\lambda}(x_3, \nabla u_n)\,\mathrm{d}x$ (note that $s \mapsto \psi_0(x_3, s)$ is convex) give:

$$\liminf_{n\to+\infty} E_n(u_n) \geq \liminf_{n\to+\infty} E_{\lambda,n}(u_n) \geq \liminf_{n\to+\infty} \int_{\mathcal{O}} \psi_{0,\lambda}(x_3, \nabla u_n)\,\mathrm{d}x\,\mathrm{d}x - \int_{\hat{\mathcal{O}}} u(x).\widetilde{L}(x)\,\mathrm{d}x$$
$$\geq \int_{\mathcal{O}\times K_\gamma} \psi_{0,\lambda}(x_3, \nabla u)\,\mathrm{d}x\,\mathrm{d}x - \int_{\mathcal{O}} u(x)\cdot\widetilde{L}(x)\,\mathrm{d}x$$

When λ goes to $+\infty$, by (3) and the monotone convergence theorem, we get:

$$\liminf_{n\to+\infty} E_n(u_n) \geq \int_{\hat{\mathcal{O}}\times K_\gamma} \psi_0\big(x_3, \nabla(u)\big)\,\mathrm{d}x\,\mathrm{d}x - \int_{\hat{\mathcal{O}}} u(x)\cdot\widetilde{L}(x)\,\mathrm{d}x$$
$$= \int_{\mathcal{O}} \psi_0(x_3, \nabla u)\,\mathrm{d}x\,\mathrm{d}x - \int_{\mathcal{O}\times[[0,1]\setminus K_\gamma]} \psi_0(x_3, \nabla u)\,\mathrm{d}x\,\mathrm{d}x - \int_{\hat{\mathcal{O}}} u(x)\cdot\widetilde{L}(x)\,\mathrm{d}x$$

The theorem is proven when $\gamma \to 0$. $\square$

Proof of ii). This part of the proof is the most difficult, as it builds a proper method to underestimate the energy limit.
For all $\lambda > 0$ and $s \in \mathbb{R}^3$ fixed, we define the upper regularization $\psi^\lambda_0(.,s)$

$$\psi^\lambda_0(x_3, s) := \sup_{t\in\mathbb{R}}\big[\psi_0(t, s) - \lambda|x_3 - t|\big].$$

We have the following inequality:

$$\limsup_{n\to+\infty} E^\lambda_n(u_n) \geq \limsup_{n\to+\infty} E_n(u_n)$$

For any displacement field $u \in L^p(\mathcal{O})$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of $Step_{3,n}(\mathcal{O})$ with $u_n \rightharpoonup u$ and

$$\limsup_{n\to\infty}\ E^\lambda_n(u_n) \leq E^\lambda_0(u) \overset{\lambda\to+\infty}{\longrightarrow} E_0(u)$$

by Lebesgue's dominated convergence theorem. The proof is divided in twosteps.

Step 1. Let a field $u \in \mathcal{C}_c(\mathcal{O})$, we construct $(u_n)_{n\in\mathbb{N}}$ that converges weakly to u in $L^p(\mathcal{O})$ and $\lim_{n\to\infty} E^\lambda_n(u_n) = E^\lambda_0(u)$.
Indeed, we define the step function

$$\widetilde{u}_n(\hat{x}, x_3) := 1_{[\frac{k}{n}, \frac{k+1}{n}[}\, u\left(\hat{x}, \frac{k}{n}\right)$$

clearly $\widetilde{u}_n \in Step_{3,n}(\mathcal{O})$ and $\widetilde{u}_n \to u$ in $L^p(\mathcal{O})$ when n goes to $+\infty$.
By continuity of ψ^λ_0 and of the strain tensor, (6), (1) and with same decomposition strategy as in (5)

$$\lim_{n\to\infty}\ E^\lambda_n(\widetilde{u}_n) = \lim_{n\to\infty}\left[\int_{\mathcal{O}} \widetilde{\psi}^{n,\lambda}_0\big(x_3, \nabla\widetilde{u}_n(x)\big)\,\mathrm{d}x - \int_{\hat{\mathcal{O}}} \widetilde{u}_n(x).\widetilde{L}^n(x)\,\mathrm{d}x\right]$$
$$= \lim_{n\to\infty} \int_{\mathcal{O}} \psi^\lambda_0\big(x_3, \nabla\widetilde{u}_n(x)\big)\,\mathrm{d}x - \lim_{n\to\infty} \int_{\hat{\mathcal{O}}} \widetilde{u}_n(x).\widetilde{L}^n(x)\,\mathrm{d}x = E^\lambda_0(u)$$

Step 2. (End of proof.) Let $u \in L^p(\mathcal{O})$ be fixed. Then there exists $(u_\delta)_{\delta\in\mathbb{N}}$ of $\mathcal{C}_c(\mathcal{O})$ that converges strongly to u in $L^p(\mathcal{O})$ i.e.

$$\lim_{\delta\to\infty}\int_{\mathcal{O}} \widetilde{\psi}_0^{n,\lambda}\big(x_3,\nabla u_\delta(x)\big)\,\mathrm{d}x=\int_{\mathcal{O}} \psi_0^{\lambda}\big(x_3,\nabla u(x)\big)\,\mathrm{d}x$$

By *Step 1* and the previous equality, we construct a subsequence $u_{n,\delta}\in Step_{3,n}(\mathcal{O})$ that converges strongly to u_δ if $n\to+\infty$ and $\lim_{n\to+\infty}E_n^\lambda(u_{n,\delta})=E_0^\lambda(u_\delta)$. So $\lim_{\delta\to\infty}\lim_{n\to\infty}E_n^\lambda(u_{n,\delta})=E_0^\lambda(u)$.

A standard diagonalization argument gives the function $n\mapsto\delta(n)$ such that

$$u_n:=u_{n,\delta(n)}\to u\quad \text{in } L^p(\mathcal{O})\quad\text{and}\quad \lim_{n\to\infty}E_n^\lambda(u_n)=E_0^\lambda(u)$$

We conclude this step and proof with

$$\limsup_{n\to\infty}\,E_n(u_n)\le\limsup_{n\to\infty}\,E_n^\lambda(u_n)=E_0^\lambda(u)\overset{\lambda\to+\infty}{\longrightarrow}E_0(u)\qquad\square$$

3. Conclusion

Thanks to the variational properties of Γ-convergence [3], we deduce the following corollary, corresponding to our limit problem.

Corollary 3.1. *Let $\bar{u}_n$ verify $\bar{E}_n(u_n)=\min\{E_n(u):u\in Step_{3,n}(\mathcal{O})\}$ and $s\mapsto\psi_0(x_3,s)$ be a function strictly convex for almost every $x_3\in(0,1)$. Then, there exists a subsequence $(\bar{u}_n)_{n\in\mathbb{N}}$ converging weakly to $\bar{u}$ in $L^p(\mathcal{O})$, the minimizer of E_0, so $\bar{u}$ is a solution of*

$$(\mathcal{P})\quad \nabla\bar{u}(x)=\partial\psi_0^*\big(x_3,L(x)\big),\quad \textit{for almost every } x\in\mathcal{O}$$

where $\partial\psi_0^*$ is the sub-differential of the Fenchel transform.

Remark 3. If $s\mapsto\psi_0(x_3,s)$ is not convex, the solution is not unique; therefore the equality in problem $(\mathcal{P})$ becomes '$\in$'.

We have presented a technique for reconstructing a three-dimensional model obtained as a variational summation from a two-dimensional problem. From the mechanical point of view, this variational summation limit enables us to keep a property gradient along a privileged direction. But the main limitation is that the inclination of the fibers should be limited and that this work was done in the scalar case. However, the preferred orientation can be different from one plate to another. We developed this reconstruction in the hyperelastic case with some modification. Moreover, the loading is very limited in such sense that only the volumic loading can be used. Of course, this is because of the cut materials. This strategy can certainly be adapted to other types of reconstruction. A future study will be aimed at considering a field obtained by tomography and defining a three-dimensional behavior.

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