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Buckling load and critical length of nanowires on an elastic substrate

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ABSTRACT

This paper considers the stability of nanowires on an elastic substrate. The problem is converted to a generalized Euler problem containing rotational spring restraint. When distributed loading and tip forces are simultaneously applied, the buckling problem of a heavy nanocolumn with rotational spring junction is reduced to an integral equation. An approximate buckling load equation is derived explicitly. The critical length of nanocantilevers is given in closed form. Results indicate that spring stiffness increases the critical length of nanowires. The effect of self-weight on the critical length is pronounced for small tip forces, and becomes weaker for larger tip forces.

1. Introduction

One-dimensional nanostructures such as nanowires, nanobelts, and nanorods have attracted considerable interest in recent years. These materials are promising candidates used as nanoscale interconnects, active components of optical electronic devices, and nanoelectromechanical systems. A typical example is nanoimprint lithography formulated by Chou et al. [1] in 1995, which has been used broadly in recent years due to its low cost, easy manipulation, and high throughput. An urgent problem to be solved in this technology is how to fabricate a nanoimprint stamp. A stamp with the desired nanoscale feature is fabricated by various techniques and is deposited onto a polymer substrate. Polymer substrates used in nanoimprint deform easily under an applied pressure and at high temperature. When a long and slender nanocolumn is loaded axially by its self-weight and by a compressive force at the tip, buckling may occur if the axial load exceeds its critical load. Therefore, a mechanical instability is one of the major reasons that cause mechanical failure of nanostructures during the nanoimprinting process [2]. To avoid occurrence of buckling, many researchers have made considerable work in this field.

Stability analysis of such structures is of fundamental issue for understanding their load-carrying capacity. Experimental and numerical investigations of the stability of nanocolumns under axial compressive force have been made by nanoindentation [3,4]. Wang et al. [5] gave a comparison of the critical stress of GaN nanotubes under uniaxial compression using molecular dynamics simulations and elastic column model. Using the classical Euler theory, Wang et al. [6] investigated the stability of a vertical single-walled carbon nanotube only under its own weight without tip force. Due to extraordinary mechanical properties of nanocolumns, Riaz et al. [7] studied the buckling characterization of vertically aligned single-crystal

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ZnO nanorods grown on different elastic substrates. For an elastic substrate, Lin et al. [8] gave an analysis of collective buckling of an array of nanocolumns on an elastic substrate.

In addition, another resource of a column being partially embedded in an elastic base is encountered in civil engineering. Building codes and common design practices generally assume that lateral support provided by soil is sufficient to prevent moving. This means that the column structure is clamped at the junction between the column and soil. Nevertheless, soil is sometimes so soft that standing columns supported by soil may translate and rotate at the junction position, which gives rise to a reduction of the load-carrying capacity of the column. Therefore, buckling analysis is of much concern for some designs related to an elastic column with its own weight and standing on a soft elastic foundation. Frisch-Fay [9] studied buckling of masonry pier under its own weight with the aid of Bessel functions and determined the buckling mode explicitly by a seven-order polynomial fitting. Using the Rayleigh quotient method, Wang and Ang [10] dealt with buckling of an elastic column subjected to a tip force and axially distributed load under various end restraints, including fixed, pinned and free ends. Elishakoff [11] formulated a new semi-inverse method for determining the closed-form buckling load for a column under its own weight. In recent years, some other new approaches have been presented to solve buckling problems for a column subjected to compressive force and distributed axial load, even for tapered columns with non-uniform cross-section [12–17]. Furthermore, Kerr [18] analyzed the effect of soft soil on the buckling load of a water tower standing on soft soil through the Bessel functions, in which the junction between the water tower bottom and soil is simplified as a rotational spring. Gluck and Gellert [19] also dealt with the stability of elastically supported cantilevers with rotational spring restraint. Wang [20] studied the dependence of critical load on self-weight for a column standing on an elastic base. Maretic and Atanackovic [21] considered buckling of a heavy elastic column loaded by a concentrated force at the top, in which the built-in end was assumed fixed to a rigid plate adhesively contacting with an elastic half-space. For some specific non-uniform columns with varying axial load, Li [22] made use of special functions to analytically tackle buckling problem for some cases of interest. In most of the above-mentioned researches, although exact solution can be derived through special functions such as the Bessel functions, it is still inconvenient in engineering applications, since buckling loads are expressed in terms of the solution to some transcendent equations. Therefore, in practice, it is highly desirable to obtain a simple yet accurate expression for the buckling load.

The purpose of this paper is to present the integral equation approach to handle the buckling problem of an elastic nanowire subjected to both distributed loading and a tip force. Emphasis is placed on deriving a simplified approximate formula for determining buckling loads of a column with rotational spring without use of special functions. Moreover, the dependence of the critical length of a heavy nanowire with a rotational spring is discussed for given forces at the tip.

2. Governing equation

Consider buckling of a nanowire grown on an elastic substrate under its own weight and subjected to a tip force. If denoting g as acceleration due to gravity, ρ as the density of the nanowire, and A as the cross-sectional area, axially distributed loading $p(x)$ takes a linearly-varying form:

$$p(x) = \rho A g(L - x) \tag{1}$$

where L is the nanocolumn length or height for a vertically standing nanowire, and x is the axial coordinate measured from the bottom (see Fig. 1). Within the framework of the classical theory of Euler–Bernoulli columns, the deflection w is related to the rotation angle θ by $\theta = \mathrm{d}w/\mathrm{d}x$, and for a standing heavy column, the rotation angle obeys the governing differential equation:

$$EI\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \left[P + q(L - x)\right]\theta = 0 \tag{2}$$

where EI is bending stiffness, P is an axial compressive force at the tip, and $q = \rho A g$. For the present treatment, it is more convenient to use the governing equation (2) than a governing equation for the deflection w.

For a standing elastic column embedded in an elastic foundation, the junction condition is usually chosen as clamped constraint. The displacement and rotation are prohibited. However, due to the fact that nanowires often have extraordinarily high Young's modulus, the elastic substrate on which nanowires grow are soft or compliant. Thus the nanowires are neither clamped nor freely rotatable, but slightly rotate as if a rotational spring is linked to the bottom. This case can be described by the following spring restraint:

$$M(0) = J\theta(0), \qquad M(L) = 0 \tag{3}$$

where J is rotational spring stiffness at the bottom tip, and M is bending moment. In principle, if the constant J is set to be sufficiently large, the restraint in this case corresponds to the condition $\theta = 0$, meaning that rotation at the end is forbidden, whereas if J is set to be zero, the restraint corresponds to the condition $M = 0$, meaning that the end can freely rotate.

In fact, for an elastic column with a rotational spring junction, a large number of articles treating the case of the presence of an axial compressive force have been published. The corresponding problem is related to a governing equation with constant coefficients and it is easily solved under various boundary conditions. If distributed loading is considered, the

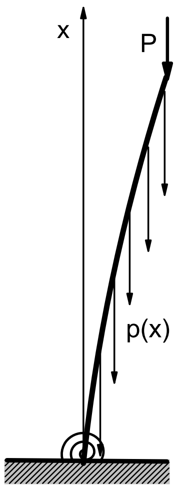

Fig. 1. Buckling of a column standing on an elastic substrate with a rotational spring junction subjected to a compressive force and its own weight.

governing equation is a differential equation with variable coefficients. Analytic approaches can be appealed to obtain the desired exact solutions only for some specific cases with the aid of special functions such as Bessel functions. Here we focus on the presence of axially distributed loading and present a simple and efficient method for determining buckling loads.

For later convenience, we introduce the following quantities:

$$\mu = \frac{PL^2}{EI}, \qquad \lambda = \frac{qL^3}{EI} \tag{4}$$

$$\xi = \frac{x}{L}, \qquad \theta = \frac{dw}{d\xi} \tag{5}$$

Then we rewrite the governing equation (2) as:

$$\theta'' + \big[\mu + \lambda(1-\xi)\big]\theta = 0 \tag{6}$$

where the prime denotes differentiation with respect to ξ.

3. Buckling load

In order to obtain buckling load, we perform the integration of Eq. (6) twice from 0 to ξ and have:

$$\theta + \int_0^{\xi} (\xi - s)\big[\mu + \lambda(1-s)\big]\theta(s)\,ds = a_1\xi + a_2 \tag{7}$$

where a_1 and a_2 are two constants to be determined.

Using the boundary conditions in (3), one easily gets:

$$a_1 = \int_0^1 \big[\mu + \lambda(1-s)\big]\theta(s)\,ds \tag{8}$$

$$a_2 = \frac{1}{\eta}\int_0^1 \big[\mu + \lambda(1-s)\big]\theta(s)\,ds \tag{9}$$

where

$$\eta = \frac{JL}{EI} \tag{10}$$

Consequently, we insert a_1 and a_2 from (8), (9) into (7), yielding:

$$\theta + \int_0^{\xi} (\xi - s)\big[\mu + \lambda(1-s)\big]\theta(s)\,ds - \frac{1+\eta\xi}{\eta}\int_0^1 \big[\mu + \lambda(1-s)\big]\theta(s)\,ds = 0 \tag{11}$$

This is a Fredholm integral equation. The remaining task is to determine the buckling load parameter μ or λ. This is equivalent to the determination of generalized eigenvalues of the resulting Fredholm integral equation. In this field, there are many approaches to obtain the eigenvalue of integral equations. Here we adopt a simple yet accurate method to solve the above-mentioned problem. To this end, we expand $\theta(\xi)$ as a series of ξ:

$$\theta(\xi) = \sum_{n=0}^{\infty} C_n \xi^n, \quad 0 \leqslant \xi \leqslant 1 \tag{12}$$

When substituting (12) into Eq. (11) and after some manipulations, we obtain a system of linear algebraic equations for C_n $(n = 0, 1, \ldots)$ as follows:

$$\sum_{n=0}^{\infty} (k_{jn} + \mu h_{jn} + \lambda m_{jn}) C_n = 0, \quad j = 0, 1, 2, \ldots \tag{13}$$

where

$$k_{jn} = \frac{1}{j+n+1} \tag{14}$$

$$h_{jn} = \frac{1}{(n+1)(n+2)} \left[\frac{1}{j+n+3} - \frac{2+n}{\eta} \left(\frac{1}{j+1} + \frac{\eta}{j+2} \right) \right] \tag{15}$$

$$m_{jn} = \frac{1}{(n+1)(n+2)(n+3)} \left[\frac{9+3n+2j}{(j+n+3)(j+n+4)} - \frac{3+n}{\eta} \left(\frac{1}{j+1} + \frac{\eta}{j+2} \right) \right] \tag{16}$$

To obtain its approximate solution, we truncate the above infinite series to become a sum of finite terms. Thus the above equations reduce to:

$$\sum_{n=0}^{N} (k_{jn} + \mu h_{jn} + \lambda m_{jn}) C_n = 0, \quad j = 0, 1, 2, \ldots N \tag{17}$$

In order for the above system in C_n $(n = 1, 2, \ldots, N)$ to have a non-trivial solution, the determinant of the coefficient matrix of the system has to vanish. That is:

$$\det(k_{jn} + \mu h_{jn} + \lambda m_{jn}) = 0 \tag{18}$$

This is an algebraic equation in μ and λ. By solving this equation, if λ is given, the lowest positive value of μ gives a critical load parameter denoted by μ_{cr}. On the contrary, if μ is prescribed, the lowest positive value of λ gives a critical distributed load parameter denoted by λ_{cr}. It is interesting to mention that for combined cases, negative critical loads are possible (see, e.g., [10,23]). In this study, we focus our attention on the case where μ and λ are both positive. Note that the above equation (18) is actually a polynomial equation in μ or λ. It is a simple matter to give its solution. In particular, for lower N values, its solution can be given in closed form. Owing to the assumption of (12), the boundary conditions in (3) at two ends might not be identically satisfied when the first $N + 1$ terms in (12) remain. Nevertheless, such a treatment does not bring a great influence on the buckling load. An alternative approach to remedy this drawback is to replace the last two equations (corresponding to $j = N, N - 1$) in (17) with the boundary conditions in (3), namely:

$$C_1 = \eta C_0 \tag{19}$$

$$\sum_{n=0}^{N} n C_n = 0 \tag{20}$$

In the following, we employ (19) and (20) to replace the last two equations in (17). In particular, taking the first three terms of (12) (i.e. $N = 2$), we get a linear equation in the variable λ and μ as follows:

$$5\left(\eta^2 + 6\eta + 12\right)\lambda + 8\left(2\eta^2 + 10\eta + 15\right)\mu = 40\eta(\eta + 3) \tag{21}$$

This gives an approximate dependence of all the three parameters. To get a more accurate relation, there is a need to take a larger N value and the detail is similar to that of the buckling analysis of a standing and hanging heavy column under a compressive force where the constrained end is completely clamped to the foundation [23,17]. Moreover, the selection of orthogonal polynomials in place of ξ^n in (12) can guarantee the stability of the solution and avoid a large condition number for larger N values in solving the characteristic equation. Here we do not go ahead along this line. Instead, we employ the accurate results for two limit cases in the case of $\eta \to \infty$, namely

$$\lambda_{cr} = 7.8373 \quad \text{if } \mu = 0 \tag{22}$$

$$\mu_{cr} = \frac{\pi^2}{4} \quad \text{if } \lambda = 0 \tag{23}$$

We further rewrite the above result (21) in the following form:

$$\frac{\eta^2 + 6\eta + 12}{7.8373\eta(\eta + 3)}\lambda + \frac{4(\eta^2 + 5\eta + 7.5)}{\pi^2\eta(\eta + 3)}\mu = 1 \tag{24}$$

Therefore, if μ is given, one has the critical buckling load λ_{cr}:

$$\lambda_{cr} = \frac{7.8373\eta(\eta + 3)}{\eta^2 + 6\eta + 12} - \frac{3.1763(\eta^2 + 5\eta + 7.5)}{\eta^2 + 6\eta + 12}\mu \tag{25}$$

On the other hand, if λ is given, one has the critical buckling load μ_{cr}:

$$\mu_{cr} = \frac{\pi^2\eta(\eta + 3)}{4(\eta^2 + 5\eta + 7.5)} - \frac{0.3148(\eta^2 + 6\eta + 12)}{\eta^2 + 5\eta + 7.5}\lambda \tag{26}$$

4. Critical length

When an axial compressive force and axially distributed loading along the column length are prescribed, the critical length is determined here. Obviously, in the presence of both an axial force and distributed load, the critical length is computed by the following equation:

$$\frac{\eta^2 + 6\eta + 12}{7.8373}qL_{cr}^3 + \frac{4(\eta^2 + 5\eta + 7.5)}{\pi^2}PL_{cr}^2 = EI\eta(\eta + 3) \tag{27}$$

where $\eta = JL_{cr}/EI$ is still related to the critical length. Therefore, the critical length can be obtained by numerically solving the above algebraic equation. It is unlikely to give its solution in closed form, except for several special cases. A simplest case is that the bottom may rotate freely, meaning that $\eta = 0$. In this case, we find $L_{cr} = 0$. It turns out that when the bottom may rotate freely, the column cannot bear any load, as expected.

Consider a case where the bottom is clamped, corresponding to $\eta \to \infty$; Eq. (27) in this case reduces to a cubic equation in L_{cr}^{-1} as an unknown:

$$L_{cr}^{-3} - \frac{4}{\pi^2 EI}PL_{cr}^{-1} - \frac{1}{7.8373EI}q = 0 \tag{28}$$

By solving this cubic equation, one readily finds the desirable positive root L_{cr} to be:

$$L_{cr} = \frac{\pi}{2\chi}\sqrt{\frac{3EI}{P}} \tag{29}$$

with

$$\chi = \begin{cases} 2\cos\left[\frac{1}{3}\cos^{-1}(\phi)\right] & \text{if } \phi \leqslant 1 \\ \tan^{1/3}\left[\frac{1}{2}\sin^{-1}\left(\frac{1}{\phi}\right)\right] + \frac{1}{\tan^{1/3}\left[\frac{1}{2}\sin^{-1}\left(\frac{1}{\phi}\right)\right]} & \text{if } \phi > 1 \end{cases} \tag{30}$$

where

$$\phi = \frac{\pi^3 q(EI)^{1/2}}{24.1326P^{3/2}} \tag{31}$$

As a check, if setting axially distributed loading $q = 0$, we find $\phi = 0$ and then using the above formula $\chi = \sqrt{3}$. As a result, the critical length collapses to $L_{cr} = (\pi/2)\sqrt{EI/P}$, in exact agreement with the well-known result. On the other hand, when setting the tip force $P \to 0$, we find $\phi \to \infty$ and $1/\phi \to 0$. Thus we have $\chi = (2\phi)^{1/3}$, substitution of which into (29) leads to the critical length $L_{cr} = \sqrt[3]{7.83EI/q}$, coinciding with the well-known result, as expected.

Furthermore, we examine the effect of rotational spring stiffness on the critical length. Without axially distributed loading $q = 0$ and only under the action of the tip force P, the critical length for a column is determined by:

$$L_{cr}^3 + \frac{5EI}{J}L_{cr}^2 + \left[\frac{15}{2}\left(\frac{EI}{J}\right)^2 - \frac{\pi^2 EI}{4P}\right]L_{cr} - \frac{3(\pi EI)^2}{4PJ} = 0 \tag{32}$$

This is still a cubic equation on L_{cr}. Although its analytic solution is also obtained using the above analogous technique, the solution is omitted here. For a general case, we prefer to choose a numerical method to get the desired solution.

Table 1
Critical load parameter λ_{cr} with η.

η	Eq. (25)	Exact [20]
0	0 (0.0%)	0
0.1	0.193 (1.53%)	0.196
0.2	0.379 (1.30%)	0.384
0.5	0.899 (0.88%)	0.907
1	1.650 (0.30%)	1.655
2	2.799 (0.14%)	2.795
5	4.679 (0.17%)	4.671
10	5.924 (0.07%)	5.920
20	6.776 (0.0%)	6.776
50	7.386 (0.0%)	7.386
∞	7.837 (0.0%)	7.837

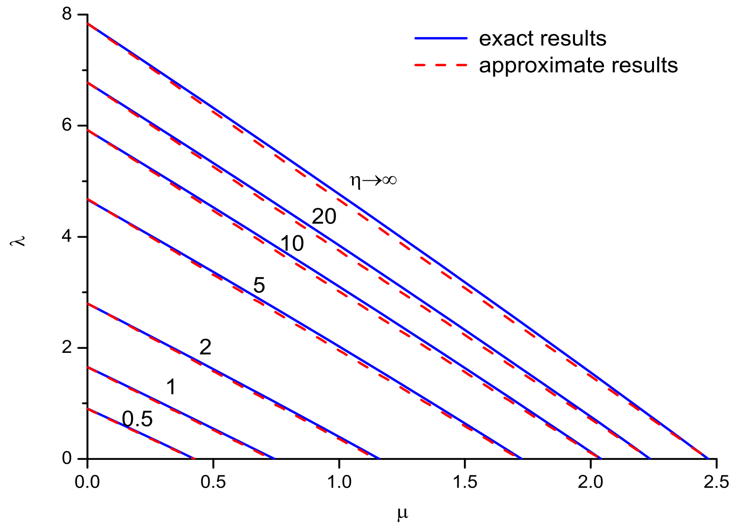

Fig. 2. Buckling curves of a standing column for various values of the rotational spring stiffness.

5. Results and discussion

In this section, we examine the validity of the above resulting formulae (25) and (26). As an example, we only calculate numerical results without tip forces; the evaluated results are listed in Table 1. For comparison, we also tabulate the analytical results in Table 1. The corresponding analytic results are obtainable by solving a transcendent equation [18–20]:

$$G_2'(z_1)\left[\eta G_1(z_0)+\lambda^{1/3}G_1'(z_0)\right]-G_1'(z_1)\left[\eta G_2(z_0)+\lambda^{1/3}G_2'(z_0)\right]=0 \tag{33}$$

with

$$G_1(z)=z^{1/2}J_{-1/3}\left(\frac{2}{3}z^{3/2}\right),\qquad G_2(z)=z^{1/2}J_{1/3}\left(\frac{2}{3}z^{3/2}\right) \tag{34}$$

$$z_0=\lambda^{-2/3}(\mu+\lambda),\qquad z_1=\lambda^{-2/3}\mu \tag{35}$$

where J_ν is the Bessel function of order ν of the first kind. From Table 1, we find that our approximate results are quite accurate and the relative error does not exceed 2%. This indicates that the obtained formulae are simple and efficient. For engineering application, there is no need to solve the above transcendent equation and instead it is only sufficient to determine solutions to simple algebraic equations (25) and (26).

In the case of the presence of an axial force, Fig. 2 shows the dependence of the critical buckling load parameters λ and μ for various values of the rotational spring stiffness η. From (25) one finds that λ_{cr} is linearly dependent on μ. However, the exact buckling curve is not straight, but is slightly convex, as seen in Fig. 2, where solid lines correspond to the exact results and dashed lines to approximate results obtained here. From Fig. 2, it is seen that our approximate results are very close to the exact ones. Furthermore, Fig. 3 displays the variation of the critical load λ_{cr} as a function of η for given values of μ. For another case where axially distributed load λ is known, the critical load μ_{cr} as a function of η is demonstrated in Fig. 4.

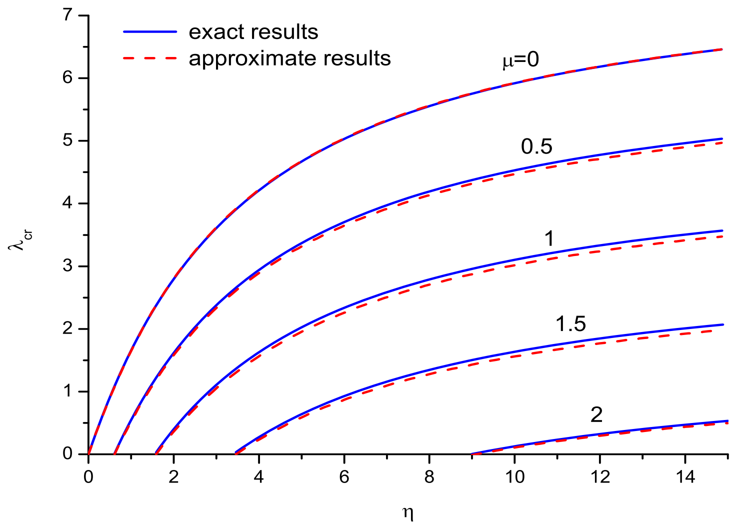

Fig. 3. Buckling load parameter λ_{cr} against rotational spring stiffness parameter η for a standing column subjected to various compressive forces.

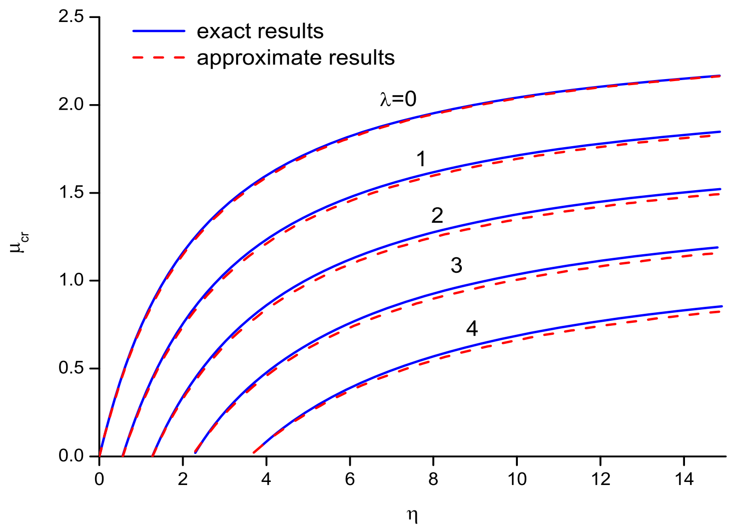

Fig. 4. Buckling load parameter μ_{cr} against rotational spring stiffness parameter η for a standing column subjected to axially distributed loading.

To promote the load-carrying capacity of nanowires, some compressing tests for vertically well-aligned ZnO nanowires grown on glass substrates have been performed by nanoindentation [3]. If the length, L, of ZnO nanowires is given, we can apply the above formula to evaluate the buckling load, P_{cr}, when considering the effects of their own weight, i.e. axially distributed loading, q, and soft substrates. If these two factors are both neglected, the classical Euler formula together with its extension of a parabola fit to the Euler curve is used [24,25]. On the contrary, if a compressive force, P, and its own weight described by q, are prescribed, there is a need to modify the Euler length of nanowires. In this case, Fig. 5 shows the variation of the ratio of the corresponding critical lengths, $L_{cr}/L_{cr}^{q=0}$, where L_{cr} and $L_{cr}^{q=0}$ denote the critical lengths with and without its own weight q. In the calculation, we have taken $PA/EI = 0.001$. From Fig. 5, it is found that when considering its own weight, the critical length clearly becomes small. Moreover, the larger the gravity load, the shorter the nanocolumn subjected to tip compression. Similarly, Fig. 6 gives the corresponding dependence of $L_{cr}/L_{cr}^{P=0}$, $L_{cr}^{P=0}$ being the critical length only under its own weight q, where $qA^{3/2}/EI = 10^{-4}$ is assumed. The trend observed in Fig. 6 is found to be similar to that in Fig. 5.

Fig. 7 displays the critical length ratio L_{cr}/L_{cr}^{c} versus a dimensionless rotational spring stiffness $J\sqrt{A}/EI$ for a column for various values of its own weight, where L_{cr}^{c} specifies the critical length of a column with a clamped bottom end. From Fig. 7, it is seen that with $J\sqrt{A}/EI$ increasing, the critical length gradually approaches that for a column with a clamped end. This reveals that if the substrate becomes stiffer, nanowires grown on the substrate become stronger and have a larger length. In other words, the load-carrying capacity of nanowires grown on the substrate can be enhanced.

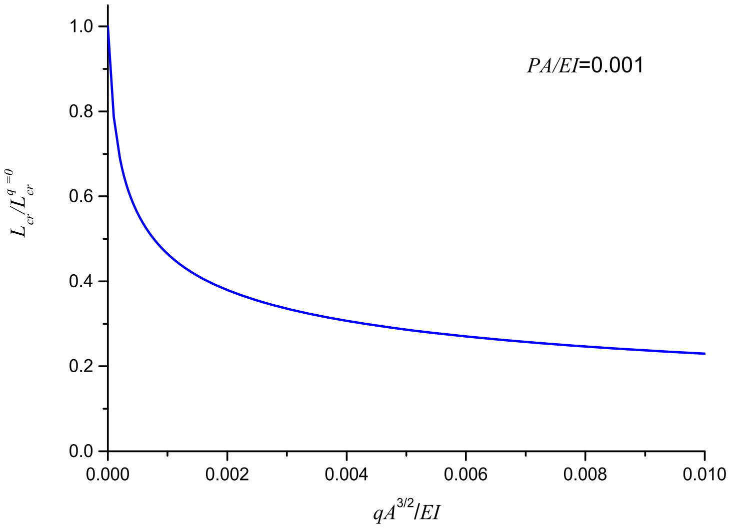

Fig. 5. Ratio of the critical length $L_{\mathrm{cr}}/L_{\mathrm{cr}}^{q=0}$ with its own weight to that without its own weight as a function of $qA^{3/2}/EI$ for a cantilevered column.

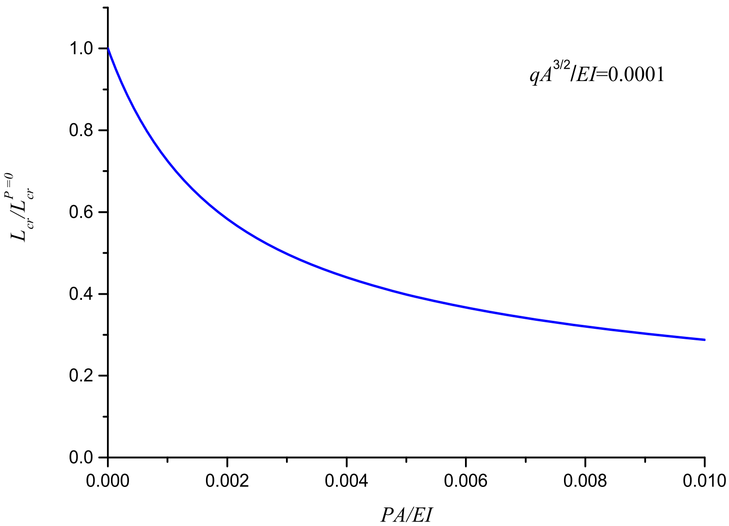

Fig. 6. Ratio of the critical length $L_{\mathrm{cr}}/L_{\mathrm{cr}}^{P=0}$ under a tip force to that without a tip force as a function of PA/EI for a cantilevered column.

Finally, we consider the critical height of typical elastomers of circular cross-section and with Young's modulus $E = 100$ MPa and mass density $\rho = 2000\ \mathrm{kg/m^3}$. In this case, the height may be calculated by:

$$L_{\mathrm{cr}}^3 = \frac{7.8373EI\eta(\eta+3)}{\rho g A(\eta^2+6\eta+12)} \tag{36}$$

For several different values of Jr/EI, we depict the critical height of the column against the radius of the column in Fig. 8. The critical height obviously increases with rotational spring stiffness rising. The largest value corresponds to that for a cantilevered column with a clamped bottom end. In particular, when Jr/EI is larger than 10^{-2}, the change in height of the column is not apparent. Thus, it is again seen that an increase in the stiffness of the substrate is capable of supporting longer nanowires. It also implies that the load-carrying capacity of nanocolumns can be promoted by increasing the stiffness of the substrate if the length is known.

6. Conclusions

This paper studied a generalized Euler problem and presented an integral equation approach to solve the stability of a column standing on an elastic substrate. A governing integral equation for buckling loads was obtained for an Euler column

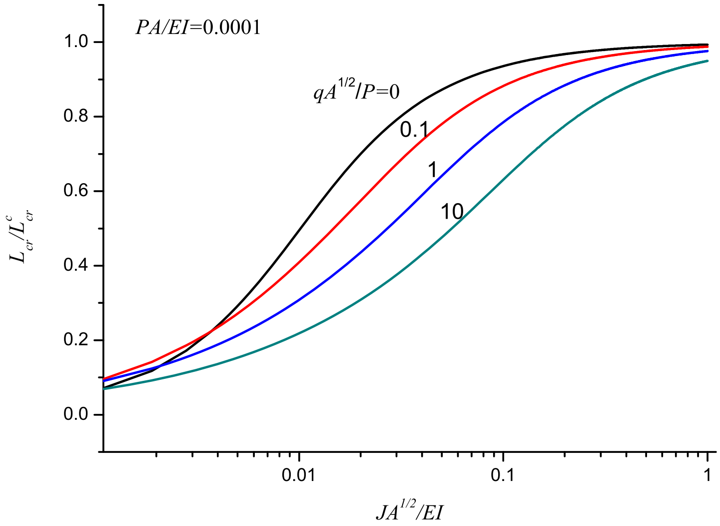

Fig. 7. Ratio of the critical length $L_{\mathrm{cr}}/L^{c}_{\mathrm{cr}}$ versus $JA^{1/2}/EI$ for a column with a rotational spring restraint.

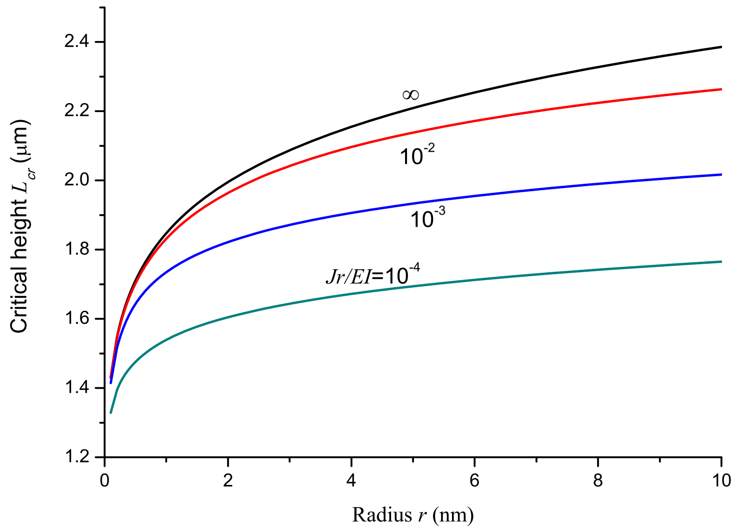

Fig. 8. Critical height of a nanocantilever as a function of its radius for different rotational spring stiffnesses.

with a rotational spring restraint at the junction and subjected to a compressive force and its own weight. Main results are drawn as follows:

- a simple yet accurate analytic relation linking the buckling loads and rotational spring stiffness was derived;
- the critical length of a cantilevered column was determined approximately in terms of a tip force and its own weight;
- soft substrates lower the load-carrying capacity of nanowires.

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