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Asymptotic analysis of a viscous fluid–thin plate interaction: Periodic flow

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ABSTRACT

The interaction "viscous fluid–thin plate" is considered when the thickness of the plate, ε, tends to zero, while the density and the Young's modulus of the plate are of order ε^{-1} and ε^{-3}, respectively. The thickness of the fluid layer is of the order of one. An asymptotic expansion is constructed and the error estimates are proved. The leading term of the asymptotic expansion is the solution of the interaction problem "fluid-Kirchoff plate". The method of asymptotic partial domain decomposition is discussed: the main part of the plate is described by a $1D$ model while a small part is simulated by the $2D$ elasticity equations, with appropriate junction conditions.

1. Introduction

The interaction of a fluid with a deformable structure has important applications in biomathematics, medicine, hydroelasticity, etc. The paper [1] considered one of such fluid–plate interaction models but the plate's thickness was supposed to be equal to zero, i.e. the dimension reduction arguments were applied. More precisely, [1] deals with the viscous fluid-elastic plate interaction problem, where the plate is described by Sophie Germain's hyperbolic equation of the fourth order in the space variable. This equation is a limit model for the elasticity equation set in a thin domain with a given force at the lateral boundary (see [2, Ch. 3] and the bibliography there). However, the coupled system "viscous fluid flow–thin elastic layer" has not been provided earlier. In the present paper we consider this dimensional reduction problem when the thickness of the layer tends to zero. More precisely, we consider the small parameter ε that is the ratio of the thickness of the plate and its length, while the density and the Young's modulus of the plate material are of order ε^{-1} and ε^{-3}, respectively. The plate lies on the fluid which occupies a thick domain. The complete asymptotic expansion is constructed when ε tends to zero and it is proved that the leading term of the expansion satisfies the equations of [1]. So, the first goal of the present paper is an asymptotic derivation and justification of the model considered in [1]. The second goal is the partial asymptotic decomposition formulation of the original problem when the main part of the plate is described by a $1D$ model while a small part is simulated by the $2D$ elasticity equations. The appropriate junction conditions based on the previous asymptotic analysis are proposed at the interface point between the $1D$ and $2D$ equations. The error of the method is evaluated.

2. Statement of the problem

Consider the domains $D^- = (0,1) \times (-1,0)$, $D_\varepsilon^+ = (0,1) \times (0,\varepsilon)$ occupied by the viscous fluid and by the elastic plate, which is much thinner than the fluid layer, respectively. The problem is 1-periodic with respect to x_1, so that it is set in the layer $\mathbb{R} \times (-1,\varepsilon)$. The small parameter of the problem, ε, is defined as the ratio between the thickness of the elastic domain

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and the length of the period of the motion. Consider as well the periodic extensions of D^- and D_ε^+: $\Delta^- = \mathbb{R} \times (-1, 0)$ and $\Delta_\varepsilon^+ = \mathbb{R} \times (0, \varepsilon)$, respectively. Denote $\Gamma^- = \{(x_1, -1) \mid x_1 \in (0, 1)\}$, $\Gamma^0 = \{(x_1, 0) \mid x_1 \in (0, 1)\}$, $\Gamma_\varepsilon^+ = \{(x_1, \varepsilon) \mid x_1 \in (0, 1)\}$.

The characteristics of the elastic medium are described by the variable density, $\tilde{\rho}_+ = \varepsilon^{-1}\rho_+(\xi_2)$, by the 2×2-matrix-valued coefficients $\tilde{A}_{ij} = \tilde{A}_{ij}(\xi_2)$, $i, j \in \{1, 2\}$, by the Young's modulus $\tilde{E} = \tilde{E}(\xi_2)$ and by the Poisson's ratio $\hat{\nu} = \hat{\nu}(\xi_2)$, with $\xi_2 = \frac{x_2}{\varepsilon}$. In general, the Young's modulus has very big values. We take $\tilde{E}(\xi_2) = \varepsilon^{-3}E(\xi_2)$, where E is of order of one. The matrices $\tilde{A}_{ij} = (\tilde{a}_{ij}^{kl})_{1 \leqslant k,l \leqslant 2}$ are defined by

$$\tilde{a}_{ij}^{kl} = \varepsilon^{-3}a_{ij}^{kl}, \qquad a_{ij}^{kl} = \frac{E}{2(1+\hat{\nu})}\left(\frac{2\hat{\nu}}{1-2\hat{\nu}}\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right)$$

such that:

(i) $a_{ij}^{kl}(\xi_2) = a_{kj}^{il}(\xi_2) = a_{ji}^{lk}(\xi_2)$, $\forall i, j, k, l \in \{1, 2\}$, $\forall \xi_2 \in [0, 1]$,

(ii) $\exists \kappa > 0$ independent of ε such that

$$\sum_{i,j,k,l=1}^{2} a_{ij}^{kl}(\xi_2)\eta_j^l\eta_i^k \geqslant \kappa \sum_{j,l=1}^{2}\left(\eta_j^l\right)^2, \quad \forall \xi_2 \in [0, 1],\ \forall \eta = \left(\eta_j^l\right)_{1 \leqslant j,l \leqslant 2}$$

with $\eta_j^l = \eta_l^j$.

The variable density of the elastic medium has the property

$$\exists \alpha, \beta > 0 \text{ independent of } \varepsilon \quad \text{such that} \quad \alpha \leqslant \rho_+(\xi_2) \leqslant \beta, \quad \forall \xi_2 \in [0, 1] \tag{1}$$

The characteristics of the viscous fluid, independent of ε, are the positive constants ρ_- and ν representing its density and its viscosity, respectively. In addition to the data ρ_+, A_{ij}, E, $\hat{\nu}$ (for the elastic medium) and ρ_-, ν (for the viscous fluid), we also know the forces $\mathbf{g}$ and $\mathbf{f}$ which act on the elastic medium and on the fluid, respectively.

The unknowns of the fluid-elastic layer interaction problem are: $\mathbf{u}_\varepsilon$, representing the displacement of the elastic medium and $\mathbf{v}_\varepsilon$, p_ε, representing the velocity and the pressure of the viscous fluid, respectively. Denote the velocity strain tensor $D(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$.

The coupled system which describes the fluid-elastic layer interaction is the following:

$$\begin{cases} \varepsilon^{-1}\rho_+\dfrac{\partial^2\mathbf{u}_\varepsilon}{\partial t^2} - \varepsilon^{-3}\displaystyle\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial\mathbf{u}_\varepsilon}{\partial x_j}\right) = \varepsilon^{-1}\mathbf{g} \quad \text{in } \Delta_\varepsilon^+ \times (0, T) \\ \rho_-\dfrac{\partial\mathbf{v}_\varepsilon}{\partial t} - 2\nu\,\mathrm{div}\big(D(\mathbf{v}_\varepsilon)\big) + \nabla p_\varepsilon = \mathbf{f} \quad \text{in } \Delta^- \times (0, T) \\ \mathrm{div}\,\mathbf{v}_\varepsilon = 0 \quad \text{in } \Delta^- \times (0, T) \\ \displaystyle\sum_{j=1}^{2} A_{2j}(1)\frac{\partial\mathbf{u}_\varepsilon}{\partial x_j}(x_1, \varepsilon, t) = \mathbf{0} \quad \text{for } (x_1, t) \in \mathbb{R} \times (0, T) \\ \mathbf{v}_\varepsilon(x_1, -1, t) = \mathbf{0} \quad \text{for } (x_1, t) \in \mathbb{R} \times (0, T) \\ \mathbf{v}_\varepsilon(x_1, 0, t) = \dfrac{\partial\mathbf{u}_\varepsilon}{\partial t}(x_1, 0, t) \quad \text{for } (x_1, t) \in \mathbb{R} \times (0, T) \\ -p_\varepsilon(x_1, 0, t)\mathbf{e}_2 + 2\nu D\big(\mathbf{v}_\varepsilon(x_1, 0, t)\big)\mathbf{e}_2 = \varepsilon^{-3}\displaystyle\sum_{j=1}^{2} A_{2j}(0)\frac{\partial\mathbf{u}_\varepsilon}{\partial x_j}(x_1, 0, t) \quad \text{for } (x_1, t) \in \mathbb{R} \times (0, T) \\ \mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon \quad 1\text{-periodic in } x_1 \\ \mathbf{u}_\varepsilon(x_1, x_2, 0) = \dfrac{\partial\mathbf{u}_\varepsilon}{\partial t}(x_1, x_2, 0) = \mathbf{0} \quad \text{in } \Delta_\varepsilon^+, \qquad \mathbf{v}_\varepsilon(x_1, x_2, 0) = \mathbf{0} \quad \text{in } \Delta^- \end{cases} \tag{2}$$

with T a positive given constant.

3. The variational analysis of the problem

This section provides us some qualitative properties of the unknowns of the coupled system (2), $\mathbf{u}_\varepsilon$, $\mathbf{v}_\varepsilon$, p_ε, such as existence, regularity and uniqueness. For performing the variational analysis of the problem (2), we choose the following regularity for the data:

$$\rho_+, a_{ij}^{kl} \in L^\infty(0, 1), \qquad \mathbf{g} \in H^1\big(0, T; \big(L^2\big(D_\varepsilon^+\big)\big)_{per}^2\big), \qquad \mathbf{f} \in H^1\big(0, T; \big(L^2\big(D^-\big)\big)_{per}^2\big) \tag{3}$$

Here and in what follows the subscript *per* denotes the 1-periodicity of functions of the corresponding space in x_1, i.e. the space is a closure of the set of infinitely smooth 1-periodic in x_1 functions with respect to the norm of the corresponding space. Taking into account the properties of the displacement and of the velocity, given by (2), let us introduce the spaces:

$$U=\left\{\mathbf{z}\in\left(H^1\left(D_\varepsilon^+\right)\right)_{per}^2 \,\Big|\, \int_0^1 z_2(x_1,0)\,\mathrm{d}x_1=0\right\}$$
$$V=\left\{\mathbf{w}\in\left(H^1\left(D^-\right)\right)_{per}^2 \,\big|\, \operatorname{div}\mathbf{w}=0,\ \mathbf{w}=\mathbf{0}\text{ on }\Gamma^-\right\}$$
$$S=\left\{(\mathbf{z},\mathbf{w})\in U\times V \,\big|\, \mathbf{z}=\mathbf{w}\text{ on }\Gamma^0\right\}$$
$$H_u=\left\{\mathbf{z}\in H^1(0,T;U) \,\Big|\, \frac{\partial^2\mathbf{z}}{\partial t^2}\in L^2\left(0,T;U'\right)\right\}$$
$$H_v=\left\{\mathbf{w}\in L^2(0,T;V) \,\Big|\, \frac{\partial\mathbf{w}}{\partial t}\in L^2\left(0,T;V'\right)\right\}$$

The variational formulation for the physical problem (2) is given by

$$\begin{cases}\text{Find }(\mathbf{u}_\varepsilon,\mathbf{v}_\varepsilon)\in H_u\times H_v\text{ such that}\\ \varepsilon^{-1}\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{D_\varepsilon^+}\rho_+\left(\frac{x_2}{\varepsilon}\right)\frac{\partial\mathbf{u}_\varepsilon}{\partial t}\cdot\mathbf{z}+\varepsilon^{-3}\int_{D_\varepsilon^+}\sum_{i,j=1}^2 A_{ij}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial\mathbf{u}_\varepsilon}{\partial x_j}\cdot\frac{\partial\mathbf{z}}{\partial x_i}+\rho_-\frac{\mathrm{d}}{\mathrm{d}t}\int_{D^-}\mathbf{v}_\varepsilon\cdot\mathbf{w}+2\nu\int_{D^-}D(\mathbf{v}_\varepsilon):D(\mathbf{w})\\ \quad=\varepsilon^{-1}\displaystyle\int_{D_\varepsilon^+}\mathbf{g}\cdot\mathbf{z}+\int_{D^-}\mathbf{f}\cdot\mathbf{w},\quad\forall(\mathbf{z},\mathbf{w})\in S\\ \mathbf{v}_\varepsilon=\dfrac{\partial\mathbf{u}_\varepsilon}{\partial t}\quad\text{a.e. on }\Gamma^0\times(0,T),\qquad\mathbf{u}_\varepsilon(0)=\dfrac{\partial\mathbf{u}_\varepsilon}{\partial t}(0)=\mathbf{0}\quad\text{a.e. in }D_\varepsilon^+,\qquad\mathbf{v}_\varepsilon(0)=\mathbf{0}\quad\text{a.e in }D^-\end{cases}\tag{4}$$

The main result of this section is the theorem which gives the existence, the regularity and the uniqueness of the solution of (4).

Theorem 3.1. *The problem* (4) *has a unique solution,* $(\mathbf{u}_\varepsilon,\mathbf{v}_\varepsilon)$. *Moreover, for* $\mathbf{g}$, $\mathbf{f}$ *with the regularity given by* (3) *we get*

$$\frac{\partial^2\mathbf{u}_\varepsilon}{\partial t^2}\in L^\infty\left(0,T;\left(L^2\left(D_\varepsilon^+\right)\right)^2\right),\qquad\frac{\partial\mathbf{v}_\varepsilon}{\partial t}\in L^\infty\left(0,T;\left(L^2\left(D^-\right)\right)^2\right)\tag{5}$$

4. The asymptotic analysis

Assume that ρ_+, a_{ij}^{kl} are piecewise smooth functions i.e. $\rho_+, a_{ij}^{kl}\in C^1(\zeta_i,\zeta_{i+1})$, where $0=\zeta_0<\zeta_1<\cdots<\zeta_p=1$ and

$$\begin{cases}\mathbf{g}=\mathbf{g}(x_1,t)\in C^\infty\left([0,T],\left(C^\infty(\mathbb{R})\right)^2\right),\quad\text{1-periodic in }x_1\\ \mathbf{f}\in C^\infty\left([0,T],\left(C^\infty\left(\mathbb{R}\times[-1,0]\right)\right)^2\right),\quad\text{1-periodic in }x_1\\ \exists\tau_0<T\quad\text{such that}\quad\mathbf{f}(x_1,x_2,t)=\mathbf{0},\qquad\mathbf{g}(x_1,t)=\mathbf{0},\quad\forall t\in[0,\tau_0]\end{cases}\tag{6}$$

We look for the asymptotic solution of order J for (2) in the form

$$\mathbf{u}_\varepsilon^{(J)}(x_1,x_2,t)=\sum_{q+l=0}^J\varepsilon^{q+l}N_{q,l}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial^{q+l}\mathbf{w}_\varepsilon^{(J)}(x_1,t)}{\partial t^q\partial x_1^l}+\sum_{q+l=0}^J\varepsilon^{q+l+2}M_{q,l}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial^{q+l}\mathbf{z}_\varepsilon^{(J)}(x_1,t)}{\partial t^q\partial x_1^l}$$

$$\begin{cases}\mathbf{v}_\varepsilon^{(J)}(x_1,x_2,t)=\displaystyle\sum_{k=0}^J\varepsilon^k\mathbf{v}_k(x_1,x_2,t)\\ p_\varepsilon^{(J)}(x_1,x_2,t)=\displaystyle\sum_{k=0}^J\varepsilon^k p_k(x_1,x_2,t)\end{cases}$$

$$\mathbf{z}_\varepsilon^{(J)}(x_1,t)=2\nu D\left(\mathbf{v}_\varepsilon^{(J)}(x_1,0,t)\right)\mathbf{e}_2-p_\varepsilon^{(J)}(x_1,0,t)\mathbf{e}_2,\qquad\mathbf{w}_\varepsilon^{(J)}(x_1,t)=\sum_{k=0}^J\varepsilon^k\mathbf{w}_k(x_1,t)\tag{7}$$

To determine the asymptotic solution means to determine the matrices $N_{q,l} = N_{q,l}(\xi_2)$, $M_{q,l} = M_{q,l}(\xi_2)$, $N_{q,l}, M_{q,l} \in \mathbb{R}^{2\times 2}$ and the functions $\mathbf{v}_k = \mathbf{v}_k(x_1, x_2, t)$, $p_k = p_k(x_1, x_2, t)$, $\mathbf{w}_k = \mathbf{w}_k(x_1, t)$.

Replacing into the left hand side of (2) $\mathbf{u}_\varepsilon$ with its asymptotic expansion $(7)_1$ and denoting $\langle F\rangle = \int_0^1 F(s)\,\mathrm{d}s$, we are led to the following second order differential problem for $N_{q,l}$, $\forall q + l > 0$

$$\begin{cases} \left(A_{22}N'_{q,l} + A_{21}N_{q,l-1}\right)' = -A_{12}N'_{q,l-1} - A_{11}N_{q,l-2} + \varepsilon^2\rho_+ N_{q-2,l} + h_{q,l} \\ h_{q,l} = \left\langle A_{12}N'_{q,l-1} + A_{11}N_{q,l-2} - \varepsilon^2\rho_+ N_{q-2,l}\right\rangle \\ A_{22}(0)N'_{q,l}(0) = -A_{21}(0)N_{q,l-1}(0) \\ \langle N_{q,l}\rangle = O_2 \end{cases} \tag{8}$$

and $N_{0,0} = I_2$. The problem for $M_{q,l}$, $q + l \geqslant 0$ is the following:

$$\begin{cases} \left(A_{22}M'_{q,l} + A_{21}M_{q,l-1}\right)' = -A_{12}M'_{q,l-1} - A_{11}M_{q,l-2} + \varepsilon^2\rho_+ M_{q-2,l} + h^M_{q,l} \\ h^M_{q,l} = \left\langle A_{12}M'_{q,l-1} + A_{11}M_{q,l-2} - \varepsilon^2\rho_+ M_{q-2,l}\right\rangle - \varepsilon^2 I_2\delta_{q0}\delta_{l0} \\ A_{22}(0)M'_{q,l}(0) = \varepsilon^2 I_2\delta_{q0}\delta_{l0} \\ M_{q,l}(0) = O_2 \end{cases} \tag{9}$$

All the other unknown functions appearing in the definition of the asymptotic solution are obtained from a coupled problem for $((w_k)_2, \mathbf{v}_k, p_k)$ and a problem for $(w_k)_1$. In order to simplify the writing of these problems we introduce the notations

$$\begin{cases} \hat{E} = \left\langle \dfrac{E}{1-\hat{\nu}^2}\right\rangle, \qquad \hat{\hat{E}} = \left\langle \dfrac{E}{1-\hat{\nu}^2}\left(\dfrac{1}{2} - \xi_2\right)\right\rangle, \qquad \hat{\hat{\hat{E}}} = \dfrac{1}{2}\left\langle \dfrac{E}{1-\hat{\nu}^2}\right\rangle - \left\langle \displaystyle\int_0^{\xi_2} \dfrac{E(s)}{1-\hat{\nu}^2(s)}\,\mathrm{d}s\right\rangle \\ \hat{J} = \left\langle \left(\overline{\dfrac{E}{1-\hat{\nu}^2}\left(\dfrac{1}{2} - \xi_2\right)}\right)\right\rangle, \qquad \hat{\hat{J}} = \hat{E}^{-1}(\hat{\hat{E}}\cdot\hat{\hat{\hat{E}}} - \hat{E}\cdot\hat{J}) \end{cases} \tag{10}$$

where $\bar{F}(x) = x\langle F\rangle - \int_0^x F(s)\,\mathrm{d}s$. The triplet $((w_k)_2, \mathbf{v}_k, p_k)$ is obtained as the unique solution of

$$\begin{cases} \langle\rho_+\rangle \dfrac{\partial^2(w_k)_2}{\partial t^2} + \hat{\hat{J}}\dfrac{\partial^4(w_k)_2}{\partial x_1^4} + \left(2\nu\dfrac{\partial(v_k)_2}{\partial x_2} - p_k\right)\Bigg|_{x_2=0} \\ \quad = g_2\delta_{k0} - \mathbf{R}_{k-1}\cdot\mathbf{e}_2 - \hat{E}^{-1}\cdot\hat{\hat{E}}\left(\dfrac{\partial g_1}{\partial x_1}\delta_{k1} - \dfrac{\partial\mathbf{R}_{k-1}}{\partial x_1}\cdot\mathbf{e}_1\right) \quad \text{in } \mathbb{R}\times(0,T) \\ \rho_-\dfrac{\partial\mathbf{v}_k}{\partial t} - 2\nu\,\mathrm{div}\big(D(\mathbf{v}_k)\big) + \nabla p_k = \mathbf{f}\delta_{k0} \\ \mathrm{div}\,\mathbf{v}_k = 0 \quad \text{in } \Delta^-\times(0,T) \\ \mathbf{v}_k(x_1,-1,t) = \mathbf{0} \quad \text{in } \mathbb{R}\times(0,T) \\ (v_k)_1(x_1,0,t) = \dfrac{\partial(w_k)_1}{\partial t}(x_1,t) + \mathbf{a}_{k-1}(x_1,t)\cdot\mathbf{e}_1 \quad \text{in } \mathbb{R}\times(0,T) \\ (v_k)_2(x_1,0,t) = \dfrac{\partial(w_k)_2}{\partial t}(x_1,t) + \mathbf{a}_{k-1}(x_1,t)\cdot\mathbf{e}_2 \quad \text{in } \mathbb{R}\times(0,T) \\ (w_k)_2, \mathbf{v}_k, p_k \quad \text{1-periodic in } x_1 \\ \mathbf{v}_k(x_1,x_2,0) = \mathbf{0} \quad \text{in } \Delta^-, \qquad (w_k)_2(x_1,0) = \dfrac{\partial(w_k)_2}{\partial t}(x_1,0) = 0 \quad \text{in } \mathbb{R} \end{cases} \tag{11}$$

with $\mathbf{R}_{k-1}$, $\mathbf{a}_{k-1}$ representing known functions, which depend on $(\mathbf{w}_j, \mathbf{v}_j, p_j)$ and on their derivatives, $j < k$.

The problem for $(w_k)_1$ is the following second order differential problem:

$$\begin{cases} \hat{E}\dfrac{\partial^2(w_k)_1}{\partial x_1^2} = -\hat{\hat{E}}\dfrac{\partial^3(w_{k-1})_2}{\partial x_1^3} - g_1\delta_{k-1,1} + \mathbf{R}_{k-2}\cdot\mathbf{e}_1, \quad (x_1,t)\in\mathbb{R}\times(0,T) \\ (w_k)_1 \quad \text{1-periodic in } x_1, \qquad \displaystyle\int_0^1 (w_k)_1(x_1,t)\,\mathrm{d}x_1 = 0 \end{cases} \tag{12}$$

Taking $k=0$ in (11) and in (12) we get $(w_0)_1=0$ and for $((w_0)_2, \mathbf{v}_0, p_0)$ the problem

$$\begin{cases} \langle\rho_+\rangle \dfrac{\partial^2 (w_0)_2}{\partial t^2} + \hat{\hat{J}} \dfrac{\partial^4 (w_0)_2}{\partial x_1^4} - p_0\Big|_{x_2=0} = g_2 \quad \text{in } \mathbb{R}\times(0,T) \\ \rho_- \dfrac{\partial \mathbf{v}_0}{\partial t} - \nu\Delta\mathbf{v}_0 + \nabla p_0 = \mathbf{f} \\ \operatorname{div}\mathbf{v}_0 = 0 \quad \text{in } \Delta^-\times(0,T) \\ \mathbf{v}_0(x_1,-1,t) = \mathbf{0} \quad \text{in } \mathbb{R}\times(0,T) \\ \mathbf{v}_0(x_1,0,t) = \dfrac{\partial (w_0)_2}{\partial t}(x_1,t)\mathbf{e}_2 \quad \text{in } \mathbb{R}\times(0,T) \\ (w_0)_2, \mathbf{v}_0, p_0 \quad \text{1-periodic in } x_1 \\ \mathbf{v}_0(x_1,x_2,0) = \mathbf{0} \quad \text{in } \Delta^-, \qquad (w_0)_2(x_1,0) = \dfrac{\partial (w_0)_2}{\partial t}(x_1,0) = 0 \quad \text{in } \mathbb{R} \end{cases} \tag{13}$$

which means that $(\mathbf{u}_0, \mathbf{v}_0, p_0)$ represents the exact solution of the interaction problem between a viscous fluid and an elastic membrane in the periodic case, studied in [1].

Theorem 4.1. *Let $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon)$ be the exact solution of (2) and $(\mathbf{u}_\varepsilon^{(J)}, \mathbf{v}_\varepsilon^{(J)}, p_\varepsilon^{(J)})$ the asymptotic solution of order J, defined by (7). Then the error between these two solutions is given by*

$$\begin{cases} \left\| \dfrac{\partial^i}{\partial t^i}\big(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\big)\right\|_{L^\infty(0,T;(L^2(D_\varepsilon^+))^2)} = \mathcal{O}\big(\varepsilon^{J+3/2}\big), \quad i=1,2 \\ \big\|\mathcal{E}_x\big(\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\big)\big\|_{L^\infty(0,T;(L^2(D_\varepsilon^+))^4)} = \mathcal{O}\big(\varepsilon^{J+1/2}\big) \\ \left\| \dfrac{\partial^i}{\partial t^i}\big(\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon^{(J)}\big)\right\|_{L^\infty(0,T;(L^2(D^-))^2)} = \mathcal{O}\big(\varepsilon^{J+1}\big), \quad i=0,1 \\ \big\|D_x\big(\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon^{(J)}\big)\big\|_{L^2(0,T;(L^2(D^-))^4)} = \mathcal{O}\big(\varepsilon^{J+1}\big) \\ \big\|p_\varepsilon - p_\varepsilon^{(J)}\big\|_{L^2(0,T;H^1(D^-))} = \mathcal{O}\big(\varepsilon^{J+1}\big) \end{cases} \tag{14}$$

5. The method of partial asymptotic decomposition of the domain

Let us apply the method of partial asymptotic decomposition of the domain for the fluid-elastic layer interaction problem (2) (see [2, Chapter 6]). Namely, let us replace Eq. $(2)_1$ by some special 1D in the space equations in the part of the domain corresponding to the values $x_1 \in (1/3, 2/3)$ with some special interface conditions between the 2D and 1D parts at the lines $x_1 = 1/3$ and $x_1 = 2/3$. To this end, let us introduce, as follows, the partially decomposed spaces: the main, $H_{dec}^{(J)}$, for the solution, and the other space, $S_{dec}^{(J)}$, representing the space of the test functions, which is the space of the traces for a fixed t of functions from $H_{dec}^{(J)}$. We first define

$$\tilde{H}_{dec}^{(J)} = \Bigg\{(\varphi,\mathbf{v}) \in H_u\times H_v \;\Big|\; \varphi(x_1,x_2,t) = \sum_{q+l=0}^{J} \varepsilon^{q+l} N_{q,l}\Big(\frac{x_2}{\varepsilon}\Big)\frac{\partial^{q+l}\mathbf{w}(x_1,t)}{\partial t^q \partial x_1^l} + \sum_{q+l=0}^{J} \varepsilon^{q+l+2} M_{q,l}\Big(\frac{x_2}{\varepsilon}\Big)\frac{\partial^{q+l}\psi(x_1,t)}{\partial t^q \partial x_1^l},$$

$$x_1 \in (1/3, 2/3),\ \mathbf{w}, \psi \in H^{J+2}\big(0,T; H^{J+1}(1/3,2/3)\big),\ \frac{\partial\varphi(x_1,0,t)}{\partial t} = \mathbf{v}(x_1,0,t)\Bigg\}$$

$$\tilde{S}_{dec}^{(J)} = \Bigg\{(\varphi,\omega) \in U\times V \;\Big|\; \varphi(x_1,x_2) = \sum_{q+l=0}^{J} \varepsilon^{q+l} N_{q,l}\Big(\frac{x_2}{\varepsilon}\Big)\frac{\partial^{l}\mathbf{w}_q(x_1)}{\partial x_1^l} + \sum_{q+l=0}^{J} \varepsilon^{q+l+2} M_{q,l}\Big(\frac{x_2}{\varepsilon}\Big)\frac{\partial^{l}\psi_q(x_1)}{\partial x_1^l},$$

$$x_1 \in (1/3, 2/3),\ \mathbf{w}_q, \psi_q \in H^{J+1}(1/3,2/3),\ q=0,\ldots,J,\ \varphi=\omega \text{ on } \Gamma^0\Bigg\}$$

then we put

$$H_{dec}^{(J)} = \overline{\tilde{H}_{dec}^{(J)}}^{\|\cdot\|_{H_u\times H_v}}, \qquad S_{dec}^{(J)} = \overline{\tilde{S}_{dec}^{(J)}}^{\|\cdot\|_{U\times V}}$$

Consider the following variational formulation for the partially decomposed problem:

$$\begin{cases} \text{Find } \big(\mathbf{u}_{\varepsilon,dec}^{(J)}, \mathbf{v}_{\varepsilon,dec}^{(J)}\big) \in H_{dec}^{(J)} \text{ such that} \\ \varepsilon^{-1} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{D_\varepsilon^+} \rho_+\left(\frac{x_2}{\varepsilon}\right) \frac{\partial \mathbf{u}_{\varepsilon,dec}^{(J)}}{\partial t} \cdot \varphi + \varepsilon^{-3} \int_{D_\varepsilon^+} \sum_{i,j=1}^{2} A_{ij}\left(\frac{x_2}{\varepsilon}\right) \frac{\partial \mathbf{u}_{\varepsilon,dec}^{(J)}}{\partial x_j} \cdot \frac{\partial \varphi}{\partial x_i} \\ \quad + \rho_- \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{D^-} \mathbf{v}_{\varepsilon,dec}^{(J)} \cdot \omega + 2\nu \int_{D^-} D\big(\mathbf{v}_{\varepsilon,dec}^{(J)}\big) : D(\omega) = \varepsilon^{-1} \int_{D_\varepsilon^+} \mathbf{g} \cdot \varphi + \int_{D^-} \mathbf{f} \cdot \omega, \quad \forall (\varphi, \omega) \in S_{dec}^{(J)} \\ \mathbf{v}_{\varepsilon,dec}^{(J)} = \dfrac{\partial \mathbf{u}_{\varepsilon,dec}^{(J)}}{\partial t} \quad \text{on } \Gamma^0 \\ \mathbf{u}_{\varepsilon,dec}^{(J)}(0) = \dfrac{\partial \mathbf{u}_{\varepsilon,dec}^{(J)}}{\partial t}(0) = \mathbf{0} \quad \text{in } D_\varepsilon^+, \qquad \mathbf{v}_{\varepsilon,dec}^{(J)}(0) = \mathbf{0} \quad \text{in } D^- \end{cases} \tag{15}$$

We study the existence and the regularity of the solution of (15) by means of the Galerkin's method as for the solution to (4). Applying the same arguments as in the proof of Theorem 3.1, we get the existence and uniqueness of the solution $(\mathbf{u}_{\varepsilon,dec}^{(J)}, \mathbf{v}_{\varepsilon,dec}^{(J)})$ to the partially decomposed problem (15). Applying the estimates of Theorem 4.1 to the differences $\mathbf{u}_{\varepsilon,dec}^{(J)} - \mathbf{u}_\varepsilon^{(J)}$ instead of $\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}$ and $\mathbf{v}_{\varepsilon,dec}^{(J)} - \mathbf{v}_\varepsilon^{(J)}$ instead of $\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon^{(J)}$ we get for these differences the estimates (14). Then, applying the triangle inequality, we finally get the same estimates for $\mathbf{u}_{\varepsilon,dec}^{(J)} - \mathbf{u}_\varepsilon$ and $\mathbf{v}_{\varepsilon,dec}^{(J)} - \mathbf{v}_\varepsilon$, which justify the method of asymptotic partial domain decomposition.

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