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Steady Navier–Stokes system with nonhomogeneous boundary conditions in the axially symmetric case

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ABSTRACT

The nonhomogeneous boundary value problem for the steady Navier–Stokes equations is studied in a three-dimensional axially symmetric bounded domain with multiply connected Lipschitz boundary. We assume that the boundary value is axially symmetric. Our results imply, in particular, the existence of the solution with arbitrary large fluxes over the connected components of the boundary, provided that all these components intersect the axis of the symmetry. The proof uses the Bernoulli law for a weak solution to the Euler equations and the one-side maximum principle for the total head pressure corresponding to this solution.

RÉSUMÉ

Des conditions aux limites non-homogènes des équations de Navier–Stokes sont étudiées dans une région bornée tridimensionnelle ayant symétrie axiale et la frontière multiplement connexe. En particulier, dans le cas où toutes les composantes connexes de la frontière intersectent l'axe de symétrie, les résultats obtenus impliquent l'existence d'une solution pour flux arbitrairement grands. La démonstration est basée sur la loi de Bernoulli pour la solution faible des équations d'Euler et sur le principe de maximum pour la fonction de Bernoulli correspondante à cette solution.

1. Introduction

Let $\Omega = \Omega_0 \setminus \bigcup_{j=1}^{N} \Omega_j$ be a bounded domain in $\mathbb{R}^3$ with multiply connected Lipschitz boundary $\partial\Omega$ consisting of $N+1$ disjoint components $\partial\Omega_j = \Gamma_j$: $\partial\Omega = \Gamma_0 \cup \cdots \cup \Gamma_N$, $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$. Consider in Ω the stationary Navier–Stokes system with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega \end{cases} \tag{1}$$

The continuity equation (1_2) implies the necessary compatibility condition for the solvability of problem (1):

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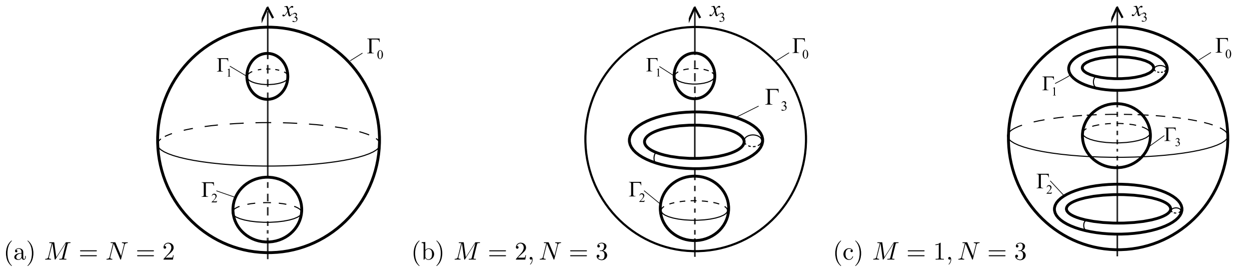

Fig. 1. Domain Ω.

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \sum_{j=0}^{N} \int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \sum_{j=0}^{N} \mathcal{F}_i = 0 \tag{2}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to Ω) normal to $\partial\Omega$ and $\mathcal{F}_j = \int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S$.

Starting from the famous paper of J. Leray [1] published in 1933, problem (1) was a subject of investigation in many papers (for a detailed survey of these results one can see the recent papers [2] or [3,4]). However, for a long time the existence of a weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (1) was proved only under the condition of zero fluxes:

$$\mathcal{F}_j = \int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = 0, \quad j = 0, 1, \ldots, N \tag{3}$$

or assuming the fluxes $\mathcal{F}_j$ to be sufficiently small. Note that condition (3) requires the net flux $\mathcal{F}_j$ of the boundary value $\mathbf{a}$ to be zero separately across each component Γ_j of the boundary $\partial\Omega$, while the compatibility condition (2) means only that the total flux is zero. Thus, (3) is stronger than (2) (condition (3) does not allow the presence of sinks and sources).

We shall study the problem in the axial symmetric case. Let $O_{x_1}, O_{x_2}, O_{x_3}$ be coordinate axis in $\mathbb{R}^3$, (θ, r, z) be cylindrical coordinates and v_θ, v_r, v_z be the projections of the vector $\mathbf{v}$ on the axes θ, r, z. A vector-valued function $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric* if h_θ, h_r and h_z do not depend on θ, and $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric without rotation* if $h_\theta = 0$ while h_r and h_z do not depend on θ. We will use the following symmetry assumptions:

(SO) $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary and O_{x_3} is the axis of symmetry of the domain Ω.

(AS) The assumptions (SO) are fulfilled and the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ is axially symmetric.

(ASwR) The assumptions (SO) are fulfilled and the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ is axially symmetric without rotation.

Assume that

$$\Gamma_j \cap O_{x_3} \neq \emptyset, \quad j = 0, \ldots, M, \qquad \Gamma_j \cap O_{x_3} = \emptyset, \quad j = M+1, \ldots, N$$

We will prove the existence theorem for the solution of problem (1), if one of the following two additional conditions is fulfilled:

$$M = N - 1, \qquad \mathcal{F}_N \geqslant 0 \tag{4}$$

or

$$|\mathcal{F}_j| < \delta, \quad j = M+1, \ldots, N \tag{5}$$

where $\delta = \delta(\nu, \Omega)$ is sufficiently small. In particular, (5) includes the case $N = M$, i.e., when each component of the boundary intersects the axis of symmetry. In both cases (4) and (5) the fluxes $\mathcal{F}_j$, $j = 0, 1, \ldots, M$, are arbitrary.

On Fig. 1 we show several possible domains Ω. In the case (a) all fluxes $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}_2$ are arbitrary; in the case (b) fluxes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ are arbitrary, while the flux $\mathcal{F}_3$ has to be nonnegative, but there are no restriction on its size; in the case (c) fluxes $\mathcal{F}_0, \mathcal{F}_1$ are arbitrary, while $\mathcal{F}_2$ and $\mathcal{F}_1$ has to be "sufficiently small".

The main result reads as follows:

Theorem 1.1. *Let the conditions* (AS), (2) *be fulfilled. Suppose also that one of the conditions* (4) *or* (5) *holds. Then the problem* (1) *admits at least one weak axially symmetric solution.*

If, in addition, the conditions (ASwR) *are fulfilled, then the problem* (1) *admits at least one weak axially symmetric solution without rotation.*

The analogous results for the two-dimensional case were obtained in [2].

The preprint version of the results of this paper (with detailed proofs) see in [5].

The proof of Theorem 1.1 uses the Bernoulli law for a weak solution of the Euler equations and the one-side maximum principle for the total head pressure corresponding to this solution (see the next section).

2. Euler equation

We shall study the Euler equations

$$\begin{cases} \lambda_0(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla p=0 & \text{in } \Omega \\ \operatorname{div}\mathbf{v}=0 & \text{in } \Omega \end{cases} \tag{6}$$

under the following assumption:

(E) *Let the conditions* (SO) *be fulfilled. Suppose that axially symmetric functions* $\mathbf{v}\in W^{1,2}(\Omega)$ *and* $p\in W^{1,3/2}(\Omega)$ *satisfy the Euler system* (6) *for almost all* $x\in\Omega$. *Moreover, suppose that*

$$\int\limits_{\Gamma_j} \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S=0 \quad \forall j=0,1,\dots,N \tag{7}$$

Denote $P_+=\{(0,x_2,x_3)\colon\ x_2>0,\ x_3\in\mathbb{R}\}$. On P_+ the coordinates x_2,x_3 coincides with coordinates r,z. From the conditions (SO) it follows that

(S_1) $\mathcal{D}:=\Omega\cap P_+$ is a bounded plane domain with Lipschitz boundary. Moreover, $C_j:=P_+\cap\Gamma_j$ is a connected set for each $j=0,\dots,N$.

From the last equality in (6) and from (7) it follows that there exists a stream function $\psi\in W^{2,2}_{\mathrm{loc}}(\mathcal{D})$:

$$\frac{\partial\psi}{\partial r}=-rv_z,\qquad \frac{\partial\psi}{\partial z}=rv_r \tag{8}$$

Denote by $\Phi=p+\lambda_0\frac{|\mathbf{v}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{v},p)$.

If all functions are smooth, then from (8) the classical Bernoulli law follows immediately: "*The total head pressure* $\Phi(x)$ *is constant along any streamline of the flow*". In the general case the following assertion holds:

Theorem 2.1. *Let the conditions* (E) *be fulfilled. Then there exists a set* $A_{\mathbf{v}}$ *such that* $\mathfrak{H}^1(A_{\mathbf{v}})=0$ *and for any compact connected set* $K\subset P_+\cap\bar{\mathcal{D}}$ *with* $\psi|_K=\text{const}$, *the identities* $\Phi(x)=\Phi(y)$ *hold for all* $x,y\in K\setminus A_{\mathbf{v}}$.

Here and henceforth we denote by $\mathfrak{H}^k$ the k-dimensional Hausdorff measure, i.e., $\mathfrak{H}^k(F)=\lim_{t\to0+}\mathfrak{H}^k_t(F)$, where $\mathfrak{H}^k_t(F)=\inf\{\sum_{i=1}^\infty(\operatorname{diam}F_i)^k\colon\ \operatorname{diam}F_i\leqslant t,\ F\subset\bigcup_{i=1}^\infty F_i\}$.

In the two-dimensional case Theorem 2.1 was obtained in [6, see Theorem 1]. The detailed proof of it is given in [2]. The proof for the axial symmetric case is absolutely analogous.

Theorem 2.2. *Let the conditions* (E) *be fulfilled. Suppose, in addition, that* $\mathbf{v}|_{\partial\Omega}=0$. *Then for each* $j=0,\dots,N$ *there exists* $p_j\in\mathbb{R}$ *such that* $p(x)\equiv p_j$ *for all* $x\in C_j\setminus A_{\mathbf{v}}$. *In particular, by axial symmetry,*

$$p(x)\equiv p_j \quad \text{for } \mathfrak{H}^2\text{-almost all } x\in\Gamma_j,\ j=0,\dots,N \tag{9}$$

Moreover,

$$p_0=p_1=\dots=p_M \tag{10}$$

$$\max_{i,j=0,\dots N}|p_i-p_j|\leqslant\delta_1\lambda_0\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} \tag{11}$$

where the constant δ_1 *depends on* Ω *only.*

To prove the equalities (10), the Bernoulli law and the fact that the axis O_z is an "almost" stream line were used. More precisely, O_z is a singularity line for $\mathbf{v}$, ψ, p, but it can be approximated by usual stream lines (on which $\Phi=\text{const}$).

In order to prove the existence theorem (Theorem 1.1) under assumptions (5), the equalities (10) are sufficient. In the case of assumptions (4), we need also a weak one-side maximum principle for the total head pressure Φ.

Let $U\subset\mathbb{R}^2$ be a domain with Lipschitz boundary. We say that the function $f\in W^{1,s}(U)$ satisfies a *one-side maximum principle locally* in U, if

$$\operatorname*{ess\,sup}_{x\in U'}f(x)\leqslant\operatorname*{ess\,sup}_{x\in\partial U'}f(x) \tag{12}$$

holds for any strictly interior subdomain U' ($\bar{U}'\subset U$) with the boundary $\partial U'$ not containing singleton connected components. In (12) negligible sets are the sets of two-dimensional Lebesgue measure zero in the left *esssup*, and the sets of one-dimensional Hausdorff measure zero in the right *esssup*.

Theorem 2.3. *Let the conditions of Theorem* 2.2 *be fulfilled. Assume that there exists a sequence of functions* $\{\Phi_\mu\}$ *such that* $\Phi_\mu \in W^{1,s}_{\mathrm{loc}}(\mathcal{D})$ *and* $\Phi_\mu \rightharpoonup \Phi$ *weakly in the space* $W^{1,s}_{\mathrm{loc}}(\mathcal{D})$ *for all* $s \in [1,2)$. *If all functions* Φ_μ *satisfy the one-side maximum principle locally in* $\mathcal{D}$, *then*

$$\operatorname*{ess\,sup}_{x\in\mathcal{D}} \Phi(x) \leqslant \max_{j=0,\dots,N} p_j \tag{13}$$

In the two-dimensional case Theorem 2.3 was obtained in [6, see Theorem 2]; for the detailed proof of it see [2].

3. Sketch of the proof of the existence theorem

The existence of the solution to problem (1) is proved by the Leray–Schauder fixed point theorem. To apply this theorem, we have to prove that the norms of all possible weak solutions $\mathbf{w}^{(\lambda)} \in \mathring{W}^{1,2}(\Omega)$ of the following problem:

$$\begin{cases} -\nu\Delta\mathbf{w}^{(\lambda)} + \lambda\big[\big(\mathbf{w}^{(\lambda)}\cdot\nabla\big)\mathbf{w}^{(\lambda)} + \big(\mathbf{w}^{(\lambda)}\cdot\nabla\big)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{w}^{(\lambda)}\big] + \nabla p^{(\lambda)} = \lambda\big[\nu\Delta\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{A}\big] & \text{in } \Omega \\ \operatorname{div}\mathbf{w}^{(\lambda)} = 0 & \text{in } \Omega \\ \mathbf{w}^{(\lambda)} = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

are uniformly bounded by a constant independent of $\lambda \in [0,1]$. In (14) $\mathbf{A} \in W^{1,2}(\Omega)$ is any axially symmetric solenoidal extension of the boundary value $\mathbf{a}$. The existence of such extension is proved, e.g., in [7, Corollary 2.3]. In the case of an axially symmetric domain and boundary values the extension $\mathbf{A}$ can be chosen axially symmetric too. Notice that in the proofs below we do not use any specific properties of $\mathbf{A}$.

We prove this a priori estimate using the contradiction argument proposed by J. Leray [1] (this argument was used also by many other authors). Assume the estimate is false. Then there exists a sequence $\{\lambda_k\} \subset [0,1]$ and weak solutions $\mathbf{w}^{(\lambda_k)} \in \mathring{W}^{1,2}(\Omega)$ to problem (14) such that

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in [0,1], \qquad \lim_{k\to\infty} J_k = \lim_{k\to\infty} \big\|\nabla\mathbf{w}^{(\lambda_k)}\big\|_{L^2(\Omega)} = \infty$$

Denote $\mathbf{v}^{(\lambda_k)} = J_k^{-1}\mathbf{w}^{(\lambda_k)}$. Since $\|\nabla\mathbf{v}^{(\lambda_k)}\|_{L^2(\Omega)} = 1$, we can assume without loss of generality (modulo subsequence), that $\{\mathbf{v}^{(\lambda_k)}\}$ converging weakly in $\mathring{W}^{1,2}(\Omega)$ to $\mathbf{v} \in \mathring{W}^{1,2}(\Omega)$, $\|\nabla\mathbf{v}\|_{L^2(\Omega)} \leqslant 1$. It can be proved that the limit function $\mathbf{v}$ together with the corresponding pressure function p satisfies the Euler equations (6) almost everywhere in Ω and $\mathbf{v}|_{\partial\Omega} = 0$. Since the domain Ω and the data of problem (14) are axially symmetric, we conclude that $\mathbf{v}$, p are axially symmetric too.

Multiplying (14) by $J_k^{-2}\mathbf{w}^{(\lambda_k)}$, integrating by parts in Ω and passing to a limit as $k \to \infty$ we obtain the following equality:

$$\nu = \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,\mathrm{d}x \tag{15}$$

On the other hand, from Euler equations (6) it follows that

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,\mathrm{d}x = -\int_{\partial\Omega} p\mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S \tag{16}$$

Using (2), (9), and (10) we can rewrite the last formula in the following equivalent form:

$$\int_{\partial\Omega} p\mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \sum_{j=0}^{N} p_j\mathcal{F}_j = p_0\sum_{j=0}^{M}\mathcal{F}_j + \sum_{j=M+1}^{N} p_j\mathcal{F}_j = \sum_{j=M+1}^{N}\mathcal{F}_j(p_j - p_0) \tag{17}$$

Hence,

$$\nu = -\sum_{j=M+1}^{N}\mathcal{F}_j(p_j - p_0) \tag{18}$$

Now, if the assumption (5) is fulfilled with $\delta = \frac{1}{\delta_1(N-M)}\nu$, where δ_1 is a constant from Theorem 2.2, then we have a contradiction of (11) with (18). The proof for this case is complete.

The proof in the case (4) is more complicated; it is based on Theorem 2.3 and follows the ideas from the paper [2].

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