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Out of Equilibrium Dynamics

Bifurcations at the dawn of Modern Science

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ABSTRACT

In this article we review two classical bifurcation problems: the instability of an axisymmetric floating body studied by Archimedes, 2300 years ago and the multiplicity of images observed in curved mirrors, a problem which has been solved by Alhazen in the 11th century. We will first introduce these problems in trying to keep some of the flavor of the original analysis and then, we will show how they can be reduced to a question of extremal distances studied by Apollonius.

1. Introduction

Symmetry breaking bifurcations have received a great deal of attention in the past decencies. Linked to the lost of stability of a symmetric equilibrium, examples of bifurcations are found in many areas of physics and of other sciences. For a long time, the study by L. Euler [1] of the buckling of a vertical elastic beam, was believed to be the first rigorous bifurcation analysis. Surprisingly enough, recently, the nonlinear community [2] has rediscovered the second book of Archimedes on floating bodies [3]. In the first book on floating bodies, Archimedes introduces his famous law of buoyancy. In the second book, he is studying the stability of a paraboloid of revolution, floating in a fluid. For Joseph-Louis Lagranges: "This book is one of the most striking monument of Archimedes genius. It contains the theory of stability of floating bodies to which little was added by contemporary scientists".

Bifurcations are closely related to the mathematics of singularities, as it was strongly emphasized by the works of R. Thom on "Catastrophs". Christian Huygens in his "Optics" was the first [4] to describe the singularities of light rays and introduced the important concept of envelopes of families of curves. Envelop of the normals to a curve, its evolute, was discovered, although not recognized as such, by the great geometer Apollonius in his fifth book on "conic" [5]. In this book, he studied the minimum and maximum distances from arbitrary points to the curve. Extrema turn out to be normals to the conic passing through the point. Apollonius finds that the number of extrema changes as a function of the position of the point. For the French mathematician Michel Chasles, book V of Apollonius is "the monument of the genius of Apollonius … one can recognize there the germ of the theory of the envelopes of normals to a conic".

For centuries geometrical optics has been enough to describe correctly almost all optical phenomena. Optics prior to the 11th century was mainly concerned with the problem of finding mirrors and lenses which will efficiently light a fire at distance and the understanding of multiple images observed in a curved mirror. Geometrical optics constitutes a predilection ground for the study of singularities of curves [6], particularly with the question of caustic [7] which turns out to be the evolute of wavefronts. The first example of bifurcation in this domain is due to Ibn al-Haytham (Alhazen in Latin) [8,9]. Let M be a circular mirror in two dimension, A a point which represents the source and a point B which represents the observer. The problem apparently set by Ptolemee and solved by Alhazen is the following: how many light rays go from A to B with a single reflexion on M?

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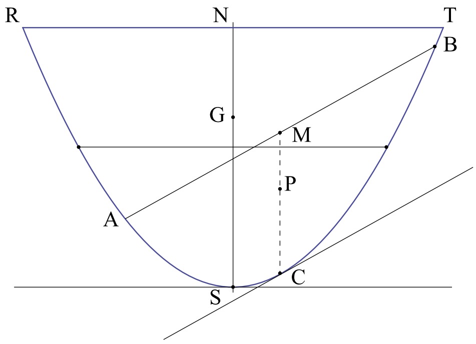

Fig. 1. Geometry of the floating parabolic body.

In this article we will discuss the three problems of bifurcation: the stability, the vertical position of an axisymmetric floating body, the Apollonius problem of the normals to a conic, and the "Alhazen billiard" problem. We will then show that the two problems of physics, in hydrostatics and in optics, can be understood qualitatively and quantitatively using the evolute of a particular curve, reducing them to a generalization of the problem of Apollonius.

2. The stability of floating bodies

Book II on floating bodies is concerned by the vertical stability of a segment of a paraboloid of revolution lighter than the fluid in which it is immersed. This book is divided into 10 propositions. In Proposition 8 Archimedes gives the condition of instability of the vertical position of the paraboloid and computes the angular value of the stable equilibrium position as a function of two dimensionless parameters, the ratio of densities of the fluid and the solid and the ratio of the two characteristic lengths of the paraboloid. In this section we reproduce the result of Archimedes, in the simpler case of a two dimensional parabolic body.

Let RST be a right segment of a parabola (see Fig. 1). In the Archimedes terminology, S is its vertex and ST its base. SN, where N is the middle of RT, is called the axe of the segment. Let us also consider the oblique segment of the parabola ACB. Its vertex C is defined as the point of the parallel of its base AB which "touches" the parabola, i.e. which is tangent to the parabola. The axe of the oblique segment of parabola is CM where M is the middle of AB. Following Archimedes, we will use two of his results.

(1) In the "Quadrature of the Parabola", Archimedes shows that the surface of a parabolic segment ABC is equal to $\frac{4}{3}$ of the surface of the inscribed triangle ABC.
(2) In the book II "On Equilibrium of Planes", Archimedes shows that the center of mass of the parabolic segment ABC, P, is such that $CP = \frac{3}{5}CM$.

In the following the right segment RST represents the parabolic floating body whose density is ρ_p. The oblique segment ACB represents the immersed part of the body, when its base AB does not cut the base RT (Archimedean condition).

The first condition for equilibrium is given by the Archimedes law (book I "On Floating Bodies") which reads simply

$$\rho_f S_{ACB} = \rho_p S_{RST}$$

where S_{ABC} stands for the surface of the parabolic segment ABC or the triangle ABC and ρ_f is the density of the fluid. Let H be the axe of the parabolic body and $2R$ its base. When the body is tilted by an angle θ, the abscissas of the extrema A and B of the base of the immersed segment are then given by

$$A_x = -R\rho^{\frac{1}{3}} + e$$
$$B_x = R\rho^{\frac{1}{3}} + e$$

where $e = \frac{R^2}{2H}\tan\theta$ is the abscissa of M. This result ceases to be correct outside the interval $[-\theta_a, \theta_a]$, where θ_a corresponds to the tilt such that B coincides with T, $\tan(\theta_a) = 2\frac{H}{R}(1-\rho^{1/3})$. The second condition for equilibrium is the cancellation of the torque. It occurs when the center of thrust P is aligned on the same vertical with the center of mass G of the floating body. The center of mass of the parabolic body is given by

$$G = \left(0, \frac{3}{5}H\right)$$

and the center of mass of the immersed part G by

$$P = \left(e, \frac{H}{R^2}e^2 + \frac{3}{5}H\rho^{\frac{2}{3}}\right)$$

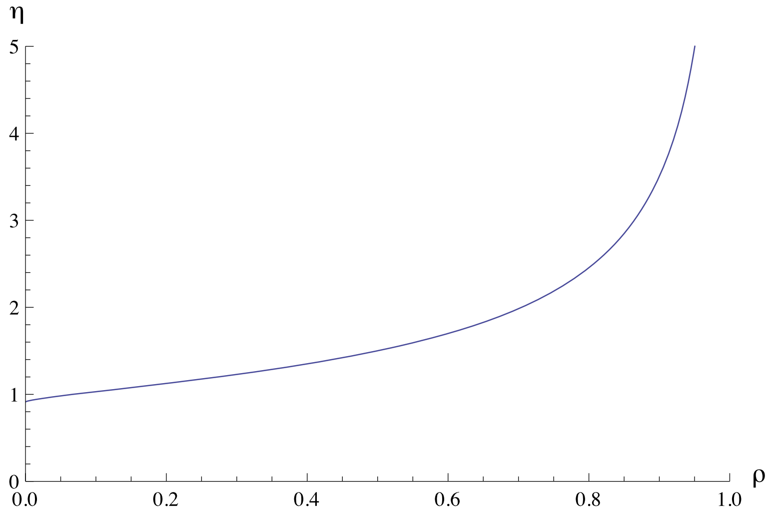

Fig. 2. Bifurcation curve. The vertical position of the body is unstable above this curve.

Since AB represents the horizontal plane in the titled geometry, the condition simply reads

$$\vec{PG}.\vec{AB}=0$$

Introducing the aspect ratio of the floating body $\eta=\frac{H}{R}$, the values of the equilibrium tilt read

$$p=0, \qquad p_{\pm}=\pm\sqrt{\frac{2}{5}}\left(6\eta^2\left(1-\rho^{\frac{2}{3}}\right)-5\right)^{\frac{1}{2}}$$

where $p=\tan\theta$ represents the slope of the base. The tilted equilibria exist if

$$\eta>\eta_c=\left(\frac{5}{6(1-\rho^{\frac{2}{3}})}\right)^{\frac{1}{2}}$$

$\eta_c(\rho)$ is an increasing monotonic function of ρ with a lower bound $\eta_c(0)=\sqrt{\frac{5}{6}}$ (see Fig. 2). When $H<\sqrt{\frac{5}{6}}R$, $\theta=0$ is the only possible equilibrium position of the body in the interval $[-\theta_a,\theta_a]$. The question of the stability reduces to the study of the sign the restoring torque $\tau=\vec{PG}.\vec{AB}$ as a function of the tilt around the equilibria.

$$\tau=\frac{p\rho^{2/3}R^2(2(6\eta^2(1-\rho^{2/3})-5)-5p^2)}{10\eta}$$

Tilted equilibria are stable and their basins of attraction given by $[-\theta_a,0]$ for θ_- and $[0,\theta_a]$ for θ_+ when they exist and $[-\theta_a,\theta_a]$ for $\theta=0$ when they do not exist. The case where the floating body is upside down can be analyzed in the same way.

In book II "On Floating Bodies", Archimedes solves the stability problem in the case of paraboloid in Proposition 8 for the body in normal position and in Proposition 9 when it is upside down, under the condition that the fluid does not touch the base of the body. This restriction is due to the fact that, in the three dimensional case, the application of the Archimedes law implies to find a solution which cannot be obtained by a compass and a ruler [2]. All the analysis relies on Euclid's Elements and his own results.

3. The Apollonius extremal distance problem

Apollonius was also a mathematician of the famous Alexandrian school whose works were probably done after the death of Archimedes. He is known as the "great geometer". His contribution to the theory of conic is considerable. He wrote 8 books on conic [10–13], the last has been lost. Books V–VII, the most important and original, have been found in Arabic language only. The first four books represent a compilation of what was known at that time on conic. In book V [5], Apollonius solves what it can be called the "problem of extremum distance". There the normals to the conic are studied per se. This work has been considered by many mathematicians as a precursor of the theory of curvature developed later by Huygens and Newton. It implicitly contains elements of the singularity theory. Let A be a point in the plane (see Fig. 3). What points M on the conic correspond to the local extrema of the distance AM? Apollonius analyzes this question for all types of conic and discovers a curve in the plane where, when the point A crosses it, the number of solutions to the extremal distance problem changes from n to $n-2$ ($n=4$ for the ellipse and $n=3$ for the parabola and the hyperbola).

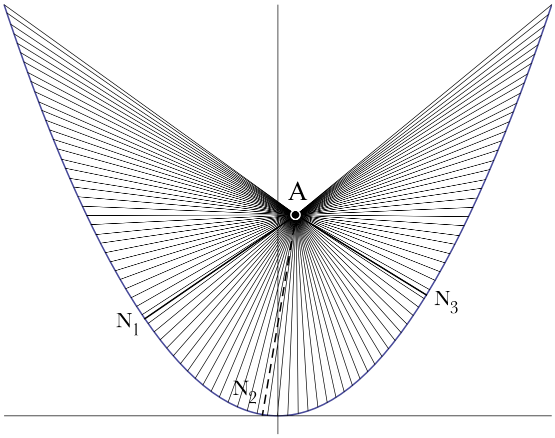

Fig. 3. Apollonius extremal lines. AN_1, AN_2 and AN_3 are the extremal lines from the point A. N_1 and N_3 are minima while N_2 is a maximum.

This curve turns out to be the envelope of the normals to the conic, or the evolute of the conic, i.e. the set of the centers of curvature of the conic [6]. In the following we discuss the case of the parabola.

Let P be a parabola whose parametric equation is given by

$$x = s, \qquad y = s^2$$

Let $A = (X, Y)$ be a point in the plane. The distance between A and an arbitrary point $M = (x, x^2)$ on the parabola is given by

$$\Delta = \left((X - s)^2 + \left(Y - s^2\right)^2\right)^{\frac{1}{2}}$$

Extrema of Δ are such that the derivative of Δ with respect to s vanishes.

$$\Delta' = \frac{d\Delta}{ds} = \frac{2s(s^2 - Y) + (s - X)}{\sqrt{(s^2 - Y)^2 + (s - X)^2}} = 0$$

This condition leads to the cubic equation

$$\Phi(s; X, Y) \equiv 2s^3 + (1 - 2Y)s - X = 0$$

In his analysis Apollonius first considers the case where the point A belongs to the axis of the conic. In our case $A = (0, Y)$. The equation becomes

$$s\left(2s^2 + 1 - 2Y\right) = 0$$

In addition with the trivial solution $s = 0$, two other solutions exist when $Y > \frac{1}{2}$

$$s_{\pm} = \pm\frac{\sqrt{2Y - 1}}{\sqrt{2}}$$

These non-symmetric solutions bifurcate from the symmetric one ($s = 0$) when $Y = \frac{1}{2}$. Are they maxima or minima? When $Y > \frac{1}{2}$ the only extremum is AS where S is the vertex of the parabola and it is clearly a minimum. The qualitative changes not only break the symmetry but also lead to a change of the nature of the trivial extremum, which becomes a maximum, with the appearance of two minima, symmetric one of the other.

When A is not on the axis, the equation is a general cubic equation which can have one or three real solutions. Two solutions appear or disappear through a tangent bifurcation, i.e. when the cubic at the point where it vanishes has a zero derivative, i.e. $\Phi' = 0$. In the plane the two equations ($\Phi = 0$ and $\Phi' = 0$) lead to a curve $X(s), Y(s)$, the locus of a bifurcation where the number of extrema changes from 1 to 3.

$$X(s) = -4s^3, \qquad Y(s) = \frac{1}{2}\left(6s^2 + 1\right)$$

This curve, first studied by Huygens, is the semi-cubic parabola

$$y = \frac{1}{2} + \frac{3}{4}(2x)^{\frac{2}{3}}$$

This curve turns out to be the locus of the center of curvature of the parabola (see Fig. 4). Let us also remark that the equation $\Delta' = 0$ is equivalent to

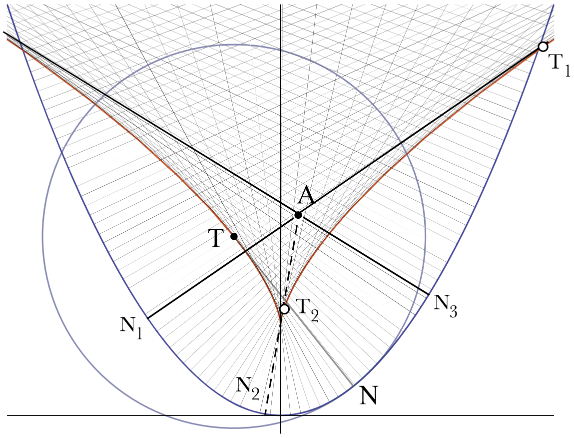

Fig. 4. Evolute of the parabola. From the point A, we search for the tangents to the evolute (AT_1, AT_2, AT_3). T_iAN_i, for $i = 1, 2, 3$ are the normals. The dashed line corresponds to the maximum. The circle centered in T with radius TN is the osculating circle at the point N.

$$\vec{t}.\vec{AM} = 0$$

where $\vec{t} = (1, 2s)$ is the vector tangent to the parabola. This shows that extrema of the distance are normals to the parabola. Book V of Apollonius is indeed dealing with the general problem of normals. Before him minimal lines were used to construct the tangents to the conic. Apollonius has a very different approach, considering the tangent as the limit of a secant in book I. Thus he devotes book V to study not only minimal but also maximal lines.

The way that normals change as one moves along the curve is the key to understand its curvature. Let a general curve Γ be given parametrically by

$$x = f(s), \qquad y = g(s)$$

and a point $A = (X, Y)$ not located on Γ. The equation which gives the extremal lines is then given by

$$\Phi(s; X, Y) = f'(f - X) + g'(g - Y) = 0$$

where f' and g' are the derivatives of f and g respectively, showing explicitly that the extremal lines are normals of Γ. At the bifurcation where the number of extrema changes, $\Phi' = 0$. The resulting curve E, the envelope of the normals is given

$$x = \frac{ff''g' - ff'g'' + f'^2g' + g'^3}{f''g' - f'g''}$$
$$y = \frac{-gf''g' + f'(gg'' + g'^2) + f'^3}{f'g'' - f''g'}$$

The distance R of the normal from the envelope E to the corresponding point of the curve Γ, given by

$$R = \frac{(f'^2 + g'^2)^{3/2}}{(f''g' - f'g'')}$$

is the radius of curvature of Γ. It shows that E is the locus of the centers of curvature of Γ. It is also called the evolute of Γ.

4. Distorting mirrors

The ancients mathematicians were fascinated by the curved mirrors. On the one hand they realized that the curved mirrors could light a fire at a distance and on the other hand they were puzzled by the deformation of their own image on such mirrors. In the "Timée", Platon describes the reversal of image when the curvature of the mirror is either horizontal or vertical. His explanation relies on the visual rays emanating from the eyes which reflect on the surface of the mirror. Although the physics of the visual rays is wrong, thanks to the principle of reversibility of light, its geometrical analysis is correct. The theory of visual rays imposed itself until Alhazen. The question of the multiple images observed in curved mirror can be reduced to the simple mathematical question of finding the rays which, from a point A, a punctual source, pass by a given point B, the observer, after a single reflection on the mirror (see Fig. 5). This problem is also known as the "Alhazen" billiard problem. For A and B given the problem reduces to a quartic equation, that Alhazen succeeds to solve using the intersection of two conic sections, a circle and a hyperbola.

The mirror M is given by the parametric equation

$$M = (\sin s, -\cos s)$$

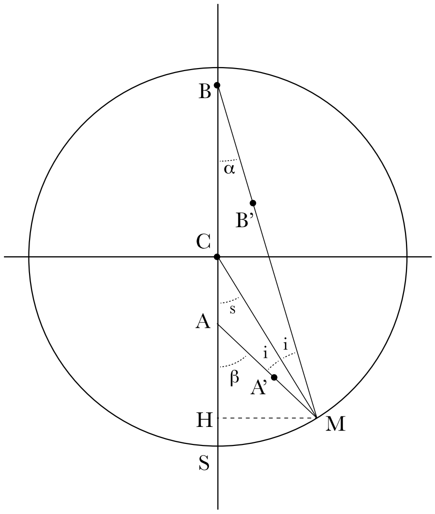

Fig. 5. Geometry of the Alhazen problem.

Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$. We first consider the simple case where A and B are on the same diameter ($x_A = y_A = 0$) (see Fig. 5). A given ray AM reflects on the mirror and cut the vertical axe in B. From the figure we see that

$$s = (\alpha + \beta)/2$$

and

$$\tan\alpha = \frac{HM}{HA}, \qquad \tan\beta = \frac{HM}{HB}, \qquad \tan s = \frac{HM}{HC}$$
$$HM = \sin s, \qquad HA = \cos s + y_A, \qquad HB = \cos s + y_B, \qquad HC = \cos s$$

For s close to 0 we can linearize the previous expressions (Gauss paraxial approximation). It leads to the Descartes conjugacy relation for a spherical mirror

$$\frac{1}{1+y_A} + \frac{1}{1+y_B} = 2$$

This relation allows us to compute y_B for a given y_A in the paraxial approximation. It does not solve the Alhazen problem in which one searches for rays, that is values of s, for y_A and y_B given. For s close to zero, at the quadratic leading order approximation ($\cos s = 1 - \frac{s^2}{2} + O(s^4)$ and $\tan s = s + O(s^3)$, $\tan\alpha = \alpha + O(\alpha^3)$, $\tan\beta = \beta + O(\beta^3)$), the Descartes relation becomes

$$\frac{1}{1 - \frac{s^2}{2} + y_A} + \frac{1}{1 - \frac{s^2}{2} + y_B} = \frac{2}{1 - \frac{s^2}{2}}$$

or equivalently

$$(y_A + y_B)s^2 = 2(y_A + 2y_A y_B + y_B)$$

This quadratic equation has solutions $s_\pm$

$$s_\pm = \pm\sqrt{\frac{2(y_A + 2y_A y_B + y_B)}{(y_A + y_B)}}$$

which exist if $y_B > -\frac{y_A}{1+2y_A+}$ and $(y_A + y_B) > 0$ or $y_B < -\frac{y_A}{1+2y_A}$ for $(y_A + y_B) < 0$. The bifurcation occurs when $y_B = y_c = -\frac{y_A}{1+2y_A}$. When $y_B < y_c$, for $(y_A + y_B) > 0$, the only ray which solves the Alhazen problem, close to $s = 0$ is $s = 0$. When $y_B > y_c$, for $(y_A + y_B) > 0$, two other solutions appear. They break the symmetry of the problem. From the observer B three images of A are seen on the mirror $(0, s_-, s_+)$. Bifurcation occurs when $y_B < y_c$, for $(y_A + y_B) < 0$. For paraxial rays close to $s = \pi$, the critical value of y_B is $y_c = -\frac{y_A}{1-2y_A}$.

When A and B are not on a same diameter, the problem reduces to a quartic equation. Alhazen and later Huygens succeeded to construct the solution as the intersection of a hyperbola and the circle of the mirror. $A' = (x'_A, y'_A)$ and $B' = (x'_B, y'_B)$ (see Fig. 5) now represent the points inside the circle. To make the notation simpler, we will suppress the prime on letters A' and B'. We could let $x_A = 0$ without loss of generality, but the price to pay is to get a formula which

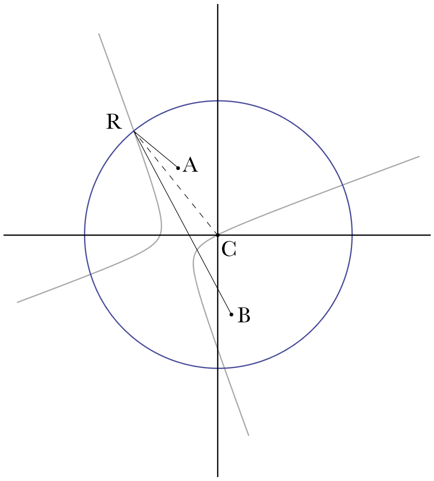

Fig. 6. The Alhazen–Huygens hyperbola and the solutions of the Alhazen problem for $A = (-0.3, 0.5)$ and $B = (0.1, -0.6)$.

will not be unchanged when we permute A and B. We first solve the intermediate problem to find the curve $M = (x, y)$ such that MC bisects the angle AMB, for A, B and C given. For the moment C represents an arbitrary point situated between A and B such that the angle AMC is smaller than the angle AMB. We have

$$\tan\alpha = \frac{x - x_A}{y - y_A}, \qquad \tan\beta = \frac{x - x_B}{y - y_B}, \qquad \tan s = \frac{x}{y}$$

The condition of bisection, $\alpha + \beta = 2s$, leads to the relation between x and y. After some elementary algebraic manipulations we end up with the cubic equation

$$\left(y^2 - x^2\right)(x_A y_B + x_B y_A) + \left(y^2 + x^2\right)\left(x(y_A + y_B) - y(x_A + x_B)\right) + 2(x_A x_B - y_A y_B)xy = 0$$

The solutions of the Alhazen problem are then the intersections of this curve with the circle

$$x^2 + y^2 = 1$$

or equivalently the intersections of the circle with a hyperbola

$$\left(y^2 - x^2\right)(x_A y_B + x_B y_A) + 2(x_A x_B - y_A y_B)xy + x(y_A + y_B) - y(x_A + x_B) = 0$$

In the case where both A and B are on the y axis, the hyperbola is degenerated and becomes two lines $x = 0$ and $y = \frac{y_A + y_B}{2 y_A y_B}$. The intersections of these lines with the circle give the solutions (see Fig. 6). The intersection with the vertical line leads to $s = 0$ and $s = \pi$ as expected. The intersection with the horizontal line exists if

$$-1 < \frac{y_A + y_B}{2 y_A y_B} < 1$$

as shown before.

We next show that the floating body problem and the Alhazen problem can be reduced to the problem of finding the normals of some curve and their envelope.

5. The metacentric curve: the evolute of the buoyancy curve

In the limit $\rho = 0$, the parabolic body floats at the fluid surface. This limit corresponds to a physical situation where the body is lying on a hard and flat surface. The Archimedes thrust becomes, in that case, the reaction of the support. The reaction is vertical, normal to the parabolic profile. At equilibrium, the center of mass of the body must be on a normal to the parabola. The question of equilibria then reduces to the Apollonius problem. The parametric equation of the parabola reads

$$M = \left(t, \frac{H}{R^2} t^2\right)$$

The Cartesian equation of the envelope of the normals of the parabola is given by

$$y = \frac{R^2}{2H} + \frac{3}{4H}(2HRx)^{\frac{2}{3}}$$

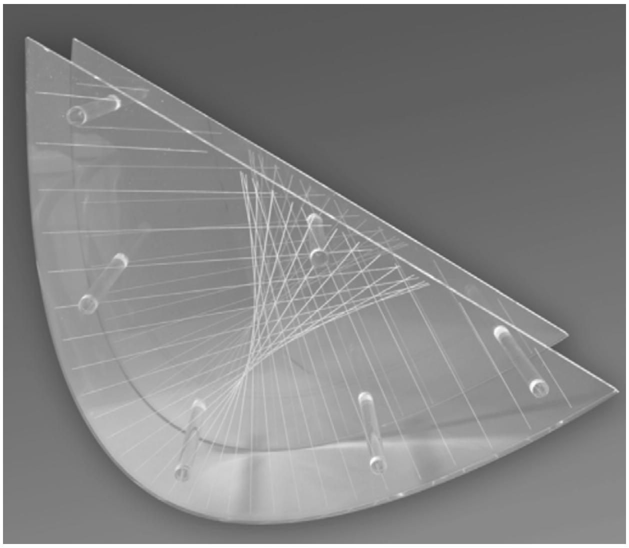

Fig. 7. A "floating body" in the limit $\rho = 0$. The object has been made by Jean-Luc Filippi.

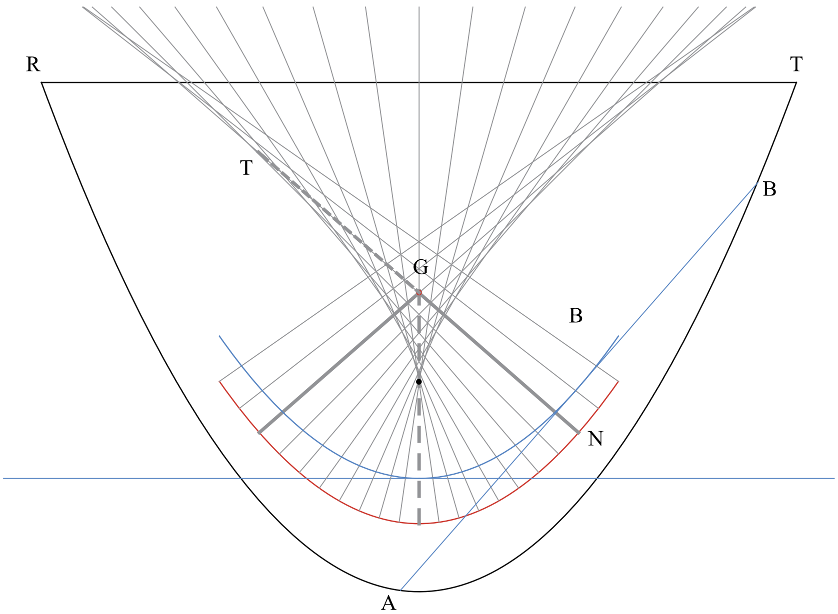

Fig. 8. In red on the figure we have represented the buoyancy line. In blue, the envelope of the flotation line. From the center of mass G one looks for the tangent to the evolute of the buoyancy line (TA). TAN is normal to the buoyancy line and represents the vertical direction at this equilibrium position. Its perpendicular, tangent to the envelope of the flotation line AB represents the flotation line corresponding to this equilibrium.

If the parabolic body is not homogeneous, its center of mass is an arbitrary point $G = (X_G, Y_G)$. When G is above the envelope of the normals, we do have three equilibria, one unstable, corresponding to the largest distance from the horizontal plane and two others stable. Bifurcation occurs when the center of mass is on the evolute of the parabola (see Fig. 8). Below the envelop we have a single, stable equilibrium. If the parabola is homogeneous, $G = (0, \frac{3}{5}H)$. Stability is insured if the center of mass is below the minimum of the semi-parabola $y_M = \frac{R^2}{2H}$, that is when $\eta = \frac{H}{R} < \sqrt{\frac{5}{6}}$, as we have shown before.

This idea extends when $\rho \neq 0$ (see Fig. 8), as was first shown by Bouguer [14], Euler [15] and more generally by Charles Dupin [16]. The buoyancy center describes a curve as the parabolic body tilt

$$\left(e, \frac{3}{5}H\rho^{\frac{2}{3}} + \frac{H}{R^2}e^2\right)$$

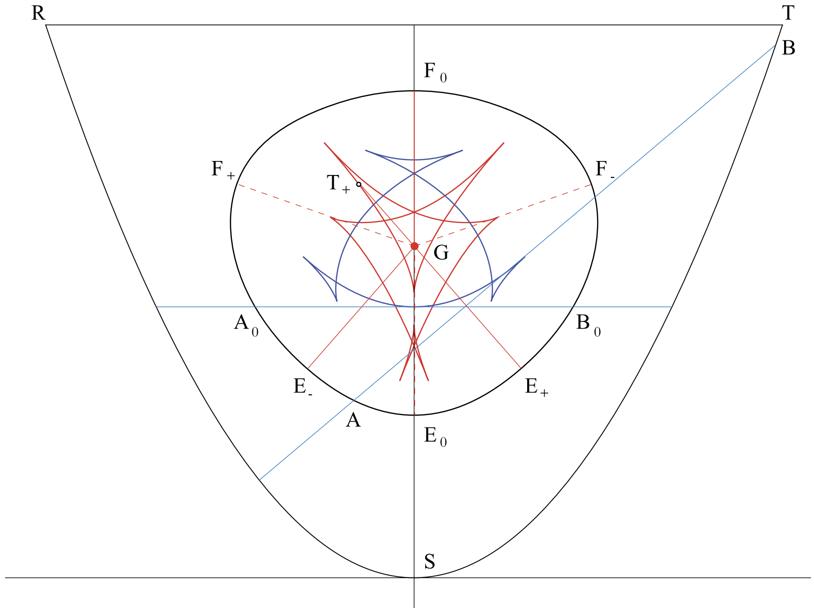

Fig. 9. Equilibria of the parabolic body for $\rho = 0.34$ and $\eta = 1.4$. The black curve represents the buoyancy line, the red curve represents the metacentric curve and the blue line is the envelope of flotation. E_i and F_i represents the equilibria. We have explained the construction of one of the symmetry breaking solution E_+. From the center of mass G one draws the tangent GT_+ to the evolute of the metacentric curve. Its perpendicular, tangent to the envelope of the flotation line, represents the actual flotation line AB for this equilibrium.

This curve is simply the floating body parabola translated vertically. We check that its tangent at point P (see Fig. 1) is parallel to the flotation line by computing the slope of this curve, $\tan\theta = \frac{2eH}{R^2}$. We conclude from this that equilibria are such that the center of mass of the body should be on a normal of the buoyancy center line. This brings us back to the problem of Apollonius. The envelope of the normals of the buoyancy center line, the so-called metacentric curve is then given by the semi-cubic parabola

$$x = -\frac{4e^3H^2}{R^4}$$
$$y = \frac{3e^2H}{R^2} + \frac{3}{5}H\rho^{2/3} + \frac{R^2}{2H}$$

Multiple solutions occur when the center of mass of the body is above the minimum of this curve $y_C = \frac{3}{5}H\rho^{2/3} + \frac{R^2}{2H}$, that is when

$$\frac{H}{R} > \sqrt{\frac{6}{5}}\sqrt{1-\rho^{2/3}}$$

as it was shown in the first section. Another curve of interest is the envelope of the flotation line

$$\left(e, H\rho^{2/3} + \frac{e^2H}{R^2}\right)$$

It also turns out to be the same parabola vertically shifted. The two envelopes are then used to graphically solve the problem of equilibria of the floating body when the flotation line does not touch the base of the parabolic floating body. When the body is not homogeneous, as it was assumed before, let $G = (X_G, Y_G)$ be its center of mass. From G we search for the tangents to the metacentric curve. They correspond to equilibria and they indicate the vertical directions. The tangents of the envelope of the flotation lines, normal to these directions, are then the actual flotation lines of the body at equilibrium. A similar analysis can be done when the body is upside down, that is when its base is completely immersed. The buoyancy line is again a parabola. When the flotation line touches the base, the buoyancy curve has a higher degree. The complete buoyancy curve is a closed differentiable curve (see Fig. 9).

This previous analysis, summarized in Fig. 9, solves the problem for all inclinations of the body. P. Bouguer and L. Euler introduced the basic idea of the metacenter but their theory was restricted to symmetric bodies and small inclinations. Charles Dupin in 1822 gave a very general theory of the stability of arbitrary three dimensional body, by introducing the surface of buoyancy and its evolute.

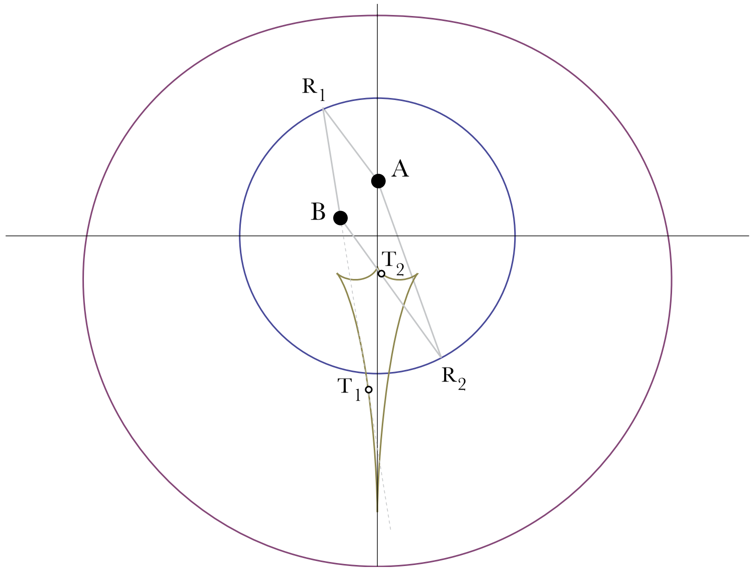

Fig. 10. The mirror is drawn in blue, the orthotomic wavefront in red. We have sketched the construction of one ray ($R1$), solution of the Alhazen problem for $y_A = 0.4$.

6. The caustic: the evolute of the wavefront

The analogy between light and sound waves has been proposed by Robert Hooke in his "Micrographia" in 1665 [17]. In particular he introduced there the important idea of wavefronts but he failed to explain the sine law. He indeed assumes that the rays and the wavefront in the refracted medium make a particular angle, computed in order to satisfy the refraction law and furthermore was led to suppose, as Descartes and Newton later, that the velocity of light was higher in the medium denser than the air. In 1672 a Jesuit, Ignace-Gaston Pardies was able to show his works on refraction to Huygens [18]. Huygens wrote in his notes "Refraction, how explained by Pardies ... compared with water waves ... light travels in circles and not instantaneously ... This explanation fits the experiments on sines ...". Pardies died in 1673 at the age of 39 and his manuscript has been lost. In 1673 Huygens again expressed his admiration for Pardies treatise in a letter addressed to Oldenburg, the secretary of the Royal Society. Huygens was indeed looking for a more general principle which would naturally lead to the orthogonality of the wavefront and the rays in the refracted medium. Furthermore he was puzzled by the phenomenon of double refraction, in which he knew that one of the two wavefronts was not orthogonal to the rays. He discovers around 1677 that the wavefront at any instant conforms to the envelope of wavelets emanating from every point on the wavefront at the prior instant. In the ordinary case the wavelets are ordinary spheres, while they are ellipsoids of revolution in the case of extraordinary waves. Let $w = (x(s), y(s))$ be an ordinary wavefront in two dimensions and a point B outside W. The problem to find the values of s which lead to rays passing through B reduces to the Apollonius problem of extremal lines. It is obviously equivalent to the Fermat principle. Normals of the front which cross B are the solutions searched. In order to solve the Alhazen problem, we need to know the wavefront generated by the reflection on the mirror of a circular wave emitted by the source A. A very elegant solution to this problem has been proposed by Adolphe Quetelet [19] in 1827. Consider a point $M = (\sin(s), -\cos(s))$ on the circular mirror and its tangent $\vec{t}(s)$. The virtual image of the point A on the mirror at point s is a point $w(s)$ symmetric of A with respect of the tangent. As s varies, $w(s)$ describes a virtual wavefront of the reflected ordinary circular waves emitted by A. This curve is known, after Adolphe Quetelet [19] as the orthotomic curve of the circle with respect to A.

$$\vec{OW} = \vec{OA} - 2\big((\vec{OA} - \vec{OM}).\hat{n}\big)\hat{n}$$

where $\vec{OW} = (x(s), y(s))$, $\vec{OA} = (x_A, y_A)$, $\vec{OM} = (\sin(s), -\cos(s))$ and $\hat{n}$ is the normal $\hat{n} = (-\sin(s), \cos(s))$.

Without loss of generality we can choose $x_A = 0$. The orthotomic of the circle with respect to A is given by (see Fig. 10),

$$x = 2\big(\sin(s) + y_A \sin(2s)\big)$$
$$y = 2\big(\cos(s) + y_A \cos(2s)\big)$$

The wavefront is a closed convex curve for $|y_A| < 1/2$. It looses its convexity at $s = 0$ for negative y_A and at $s = \pi$ for positive y_A when $|y_A| > 1/2$. Inflexion points of the wavefront then lead to the phenomenon of caustic at infinity (see Fig. 11). The bifurcation set in the plane is given by the evolute of the wavefront

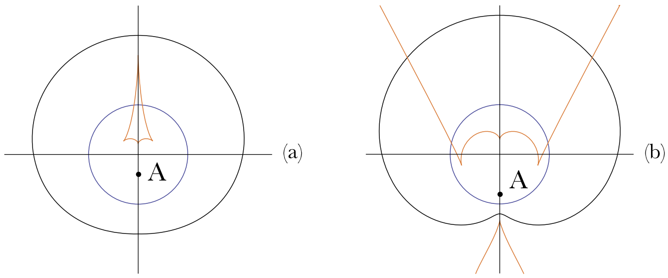

Fig. 11. For $y_A = -0.4$ the wavefront is convex. For $y_A = -0.8$ it has inflection point which leads to caustic asymptotic to lines. This phenomena is well known and has been terms as "caustic to infinity". It occurs for example in the case of the rainbow formation.

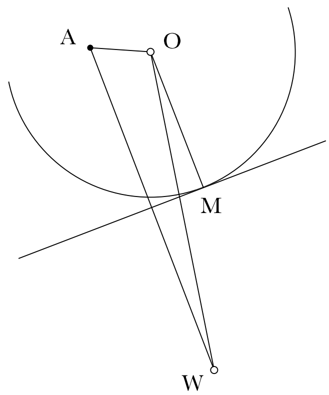

Fig. 12. The Quetelet construction of the orthotomic.

$$x = \frac{y_A^2(3\sin(s) - \sin(3s))}{2(1 + 2y_A^2 + 3y_A\cos(s))}$$
$$y = -\frac{2y_A + y_A^2(3\cos(s) - \cos(3s))}{2(1 + 2y_A^2 + 3y_A\cos(s))}$$

In the case of point B on the vertical axe, the bifurcation takes place for $s \approx 0$ or $s \approx \pi$. For $s \approx 0$, a the leading order in s the evolute becomes

$$x = O\left(s^3\right)$$
$$y = -\frac{y_A}{2y_A + 1} + O\left(s^2\right)$$

The bifurcation then occurs when $y_B = -\frac{1+y_A}{2y_A}$. Similarly when $s \approx \pi$, the bifurcation occurs for $y_B = -\frac{y_A}{1-2y_A}$. (See Fig. 12.)

7. Conclusion

We have reviewed in this article two classical physical bifurcation problems, the instability of axisymmetric floating bodies and the multiplicity of the images seen in a curved mirror. We have tried to keep some flavors of the original works although our analysis uses elementary calculus. We have shown that these problems could be reduced to the calculation of the evolute of a particular curve, the buoyancy curve in the case of a floating body and a particular virtual wavefront in the case of optics. The analogy between the two problems is related to their variational nature. In hydrostatics, at equilibrium, the floating body extremalizes its potential energy. In optics, the rays which solve the Alhazen problem extremalize the optical path.

Acknowledgements

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Fig. 13. Laurie Chiara is photographed in front of a hyperbolic mirror. We see its three images on the mirror. Note that the central image is reversed. On the ground we clearly see two caustics. The yellow one is due to a light spot in the room. The white one is produced by the flash of the camera. The position of Laurie inside the white caustic explains the existence of the three images.

paper. Laurie Chiara (see Fig. 13) is a scientific writer; the photographer here, Jean-Luc Filippi, is a master of the Mechanical Engineering (see Fig. 7). I want to thank them. Finally I want to thank the CNRS for allowing me to make this work, thanks to its support.

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