C. R. Mecanique 340 (2012) 702–705

Contents lists available at SciVerse ScienceDirect

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Mixed boundary value problem in Potential Theory: Application to the hydrodynamic impact (Wagner) problem

Yves-Marie Scolan [a,*], Alexander A. Korobkin [b]

[a] ENSTA-Bretagne, LBMS/DFMS, 2, rue François-Verny, 29806 Brest cedex 9, France
[b] University of East Anglia, School of Mathematics, Norwich, United Kingdom

ARTICLE INFO

Article history:
Received 6 June 2012
Accepted after revision 26 September 2012
Available online 18 October 2012

Keywords:
Dynamical systems
Potential Theory
Mixed Neumann–Dirichlet problem

ABSTRACT

A three-dimensional solution of the mixed boundary value problem posed in Potential Theory is proposed. The support of the Neumann condition is conformally mapped onto a unit disk. On that disk, the solution is broken down as Fourier series of azimuthal angle and linear combinations of known functions of the radial coordinate. It is shown that the whole problem reduces highly nonlinear equations for the coefficients of the mapping function. The present method of solution is to be applied to hydrodynamic impact problem.

1. Introduction

A mixed Neumann–Dirichlet boundary value problem is posed in Potential Theory. The Laplace equation is solved in a half three-dimensional space. This configuration is relevant in several domains of the physics; among them a charged disk in electrostatics, or the hydrodynamic impact problem known as the linearized Wagner problem [1]. This latter problem is even more complicated since the line which bounds the two supports of the two boundary conditions is a part of the solution.

We propose here a new mathematical solution of that Wagner problem. A semi-analytical solution of the problem is hence established as a series of known functions whose properties are checked and analyzed. The proposed method has not been studied for practical cases yet. However some physical properties of the solution, like hydrodynamic loads are formulated. It is also expected that the present solution will help to propose new criteria regarding the stability of the solution in connection with the regularity of the shape at the initial contact point.

2. Method of solution

An illustration of the linearized three-dimensional Wagner problem is shown in Fig. 1. A Neumann condition is prescribed on a closed area D. A homogeneous Dirichlet condition is prescribed on the complementary surface F. Following Korobkin [2] and Howison et al. [3] among others, the corresponding boundary value problem is formulated in terms of the displacement potential ϕ as follows:

$$\begin{cases} \phi_{,xx}+\phi_{,yy}+\phi_{,zz}=0, & z<0 \\ \phi=0, & z=0,\ (x,y)\in F \\ \phi_{,z}=f^\star(x,y,t), & z=0,\ (x,y)\in D \\ \phi\to 0, & \left(x^2+y^2+z^2\right)\to\infty \end{cases} \tag{1}$$

* Corresponding author.
E-mail address: yves-marie.scolan@ensta-bretagne.fr (Y.-M. Scolan).

1631-0721/$ – see front matter
http://dx.doi.org/10.1016/j.crme.2012.09.006

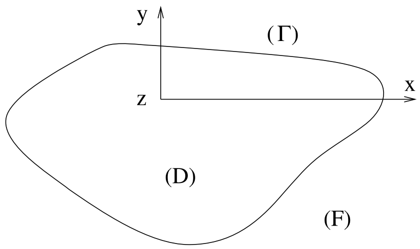

Fig. 1. Illustration of the linearized three-dimensional Wagner problem.

where $f^\star$ is a regular enough function of the Cartesian coordinates (x, y) and time t. By using a theorem by Zaremba [4] applied to the vertical displacement $V = -\phi_{,z}$, it is shown that the field V is induced by a unique distribution of sources over the surface D. The source intensities identify with the planar Laplacian $\Delta_2\phi = \phi_{,xx} + \phi_{,yy} = S(x, y, t)$. In addition $V(x, y, z, t)$ is continuous throughout the fluid domain $z \leqslant 0$ including its boundary $z = 0$ and the contact line Γ. After some algebra of the integral representation of field V, the following integral equation is obtained (see Korobkin [2]):

$$\frac{1}{2\pi}\int_D \frac{S(x_0, y_0, t)\,\mathrm{d}x_0\,\mathrm{d}y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = f^\star(x, y, t) \tag{2}$$

Additional conditions are prescribed along the contact line. This condition implies that not only the displacement potential $\phi(x, y, 0, t)$ and the vertical displacement $\phi_{,z}(x, y, 0, t)$ but also the horizontal displacements $\phi_{,x}(x, y, 0, t)$ and $\phi_{,y}(x, y, 0, t)$ are continuous through Γ. Those conditions are necessary in order to determine the contact line Γ which is unknown.

By using the Riemann mapping theorem (see Nehari [5, pp. 173–174]), we introduce the conformal mapping function g which turns the surface D onto a unit disk C_1

$$Z = x + iy = g(\omega), \quad \omega = \xi + i\eta \tag{3}$$

where Z denotes a complex number in the physical plane $z = 0$. As function g is conformal (hence analytical), its derivative never vanishes and the following formula is helpful to express the Green function of the problem.

$$\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \frac{1}{|g(\omega)-g(\omega_0)|} = \frac{1}{|g'(\omega)|}\frac{1}{|\omega-\omega_0|} + \frac{T(\omega,\omega_0)}{|\omega-\omega_0|} \tag{4}$$

The function $T(\omega, \omega_0)$ is regular since the leading order term behaves as $O(|\omega - \omega_0|)$. By substituting (4) in the integral equation (2), it can be shown that S is a solution of

$$\frac{1}{2\pi}\int_{C_1} \frac{S(\omega_0)\,\mathrm{d}\sigma_0}{|\omega-\omega_0|} = Q(\rho, \alpha, t), \qquad \phi = 0, \quad \text{and} \quad \phi_{,n} = 0, \quad \text{at } |\omega| = 1 \tag{5}$$

where the right-hand side Q depends on the solution S, but it is regular enough all over D. The integral equation in combination with boundary conditions (5) proves that ϕ is the solution of the following boundary value problem now posed on the unit disk

$$\begin{cases} \phi_{,\rho^2} + \dfrac{1}{\rho}\phi_{,\rho} + \phi_{,\tilde{z}^2} + \dfrac{1}{\rho^2}\phi_{,\alpha^2} = 0, & \tilde{z} < 0 \\ \phi = 0, & \tilde{z} = 0,\ \rho > 1 \\ \phi_{,\tilde{z}} = Q(\rho, \alpha, t), & \tilde{z} = 0,\ 0 < \rho < 1 \\ \phi \to 0, & (\tilde{z}^2 + \rho^2 \to \infty) \end{cases} \tag{6}$$

where the vertical coordinate $\tilde{z}$ is formal, its connection with the original variable z is complicated and is not discussed here. The method for solving the problem (6) is described by Korobkin and Scolan in [6]. Provided that Q has a Fourier transform of the azimuthal variable α and a polynomial expansion with ρ, say

$$Q(\rho, \alpha, t) = \Re \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q_k^{(n)} \rho^k e^{in\alpha}, \quad \rho \in [0:1] \tag{7}$$

after some algebra, it is shown that the displacement potential ϕ reads

$$\phi(\rho, \alpha, 0, t) = \frac{2}{\pi}\Re \sum_{n=0}^{\infty} \rho^n \sum_{k=0}^{\infty} q_k^{(n)} z_{k+n}\big[\sqrt{1-\rho^2} + (k-n)D_{k-n}(\rho)\big] e^{in\alpha} \tag{8}$$

where

$$z_\kappa = \int_0^{\frac{\pi}{2}} (\sin\beta)^{\kappa+1}\,d\beta, \qquad D_\kappa(\rho) = \int_\rho^1 \nu^{\kappa-1}\sqrt{\nu^2-\rho^2}\,d\nu \tag{9}$$

The functions D_κ are defined on the interval $\rho \in [0:1]$, they are determined from recursion formulæ. It is also proved that $D_\kappa(\rho)$ vanishes as $(1-\rho^2)^{3/2}$ at the boundary $\rho = 1^-$. At the contact line Γ, the normal displacement vanishes $(\partial\phi/\partial\rho)(1,0,t) = 0$ leading to the so-called Wagner condition

$$\sum_{k=0}^{\infty} q_k^{(n)} z_{k+n} = 0 \tag{10}$$

The square root term in Eq. (8) hence disappears. We get the expected result that the displacement potential vanishes as $(1-\rho^2)^{3/2}$ at the boundary $\rho = 1^-$. Accordingly the planar Laplacian S is calculated

$$S(\rho,\alpha,t) = \frac{2}{\pi}\Re\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} e^{in\alpha}\rho^n q_k^{(n)} z_{k+n} \times\left[\frac{k}{\sqrt{1-\rho^2}} - k(k+1)\sqrt{1-\rho^2} + (k-n)(k+n+1)(k-n-2)D_{k-n-2}(\rho)\right] \tag{11}$$

We thus obtain a general solution of problem (1). It should be noted that the solution (11) is singular as $(1-\rho^2)^{-1/2}$; that is another proof of the results by Stephan [7].

By noting P the function

$$P(\omega,\omega_0) = \frac{|\omega-\omega_0|}{|g(\omega)-g(\omega_0)|} \tag{12}$$

we can now turn the original integral equation (5) into

$$\frac{1}{2\pi}\int_{C_1} \frac{S(\rho_0,\alpha_0,t)P(\omega,\omega_0)\,d\sigma_0}{|\omega-\omega_0|} = f^\star(x,y,t), \quad |\omega| < 1 \tag{13}$$

If the conformal mapping function g is expressed as an integer series of ω on C_1, say $g(\omega) = \sum_{n=1}^{\infty} b_n\omega^n$, the function P can hence be broken down as follows (see Scolan and Korobkin [8]),

$$P = \sum_{n,m=0}^{\infty} \rho_0^n\rho^m \sum_{p=0}^{n}\sum_{q=0}^{m} c_{(p,n,q,m)} e^{i(n-2p)\alpha_0} e^{i(m-2q)\alpha} \tag{14}$$

where the coefficients $c_{(p,n,q,m)}$ only depend on the coefficients b_n ranged in the vector $\mathbf{b}$. In Eq. (13), the integrals in α_0 appear as Copson's integrals (see Sneddon [9, pp. 69–71]). By using expansions (11) and (14), all integrals in (13) can be performed analytically, yielding

$$\int_{C_1} \frac{S(\omega_0)P(\omega,\omega_0)\,d\sigma_0}{|\omega-\omega_0|} = \Re\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} q_k^{(n)} H_{kn}(\rho,\alpha) \tag{15}$$

where $H_{kn}(\rho,\alpha)$ are polynomials of ρ and Fourier series of α. If the function $f^\star$ is broken down in the same way, the integral equation reduces to a linear system for the coefficients $q_k^{(n)}$ ranged in the vector $\mathbf{q}$, say

$$H(\mathbf{b})\mathbf{q} = F(\mathbf{b}) \tag{16}$$

By using Wagner condition (10), denoted $C\mathbf{q} = 0$, the vector $\mathbf{q}$ can be eliminated, yielding a nonlinear system for $\mathbf{b}$. In Wagner problem, the function $f^\star$ is precisely $h(t) - f(x,y)$ where $h(t)$ is the penetration depth at time t of the body whose shape is described by the equation $z = f(x,y)$ at initial time $t = 0$, that is to say when the body hits a liquid free surface. By differentiating Eqs. (10) and (16) in time, it is shown that $\frac{d\mathbf{b}}{dt}$ is proportional to the velocity $U(t) = \frac{dh}{dt}$.

We can assess the hydrodynamic loads whose vertical component, at the leading order, follows from

$$F(t) = -\mu\frac{d^2}{dt^2}\iint_D \phi(x,y,0,t)\,dx\,dy = -\mu\frac{d^2}{dt^2}\int_{C_1}\phi(\rho,\alpha,0,t)\left|g'(\omega)\right|^2\rho\,d\rho\,d\alpha \tag{17}$$

where μ is density of the liquid. The function $|g'(\omega)|^2$ can be expanded as a Fourier series. By using the expression (8) of ϕ and the Wagner condition (10), the force finally reads

$$F(t) = -2\mu \frac{\mathrm{d}^2}{\mathrm{d}t^2} \Re\big[\mathbf{b}^t A(\mathbf{q})\bar{\mathbf{b}}\big] \tag{18}$$

where $A(\mathbf{q})$ is a triangular matrix whose coefficients are linear combinations of $\mathbf{q}$. $\mathbf{b}^t$ denotes the transposed of vector $\mathbf{b}$ and the overline is the complex conjugate. Once $F(t)$ calculated, Newton's law can be time integrated yielding explicitly the penetration in terms of $\mathbf{b}$ and $\mathbf{q}$.

3. Conclusion

A solution of a mixed boundary value problem is derived in Potential Theory. The method of solution provides a way to solve the linearized three-dimensional Wagner problem. Basically a conformal mapping is used to turn the closed support of the Neumann condition onto a unit disk. The problem hence reduces to highly nonlinear equations for the parameters of the conformal mapping function. The expected behaviors of the solution at the intersection of Neumann and Dirichlet condition are obtained.

The present formulation offers a way to analyze the stability of the solution in terms of the convexity of the closed support of the Neumann condition. Other aspects of the mechanical problem can be investigated as the conservation of energy and, depending on the kinematics of the body, the distribution of energy either in the liquid jet or in the bulk of the fluid.

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