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Averaging in variational inequalities with nonlinear restrictions along manifolds ☆

Homogénéisation de inégalités variationnelles avec restrictions non linéaires sur une variété

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ABSTRACT

We consider variational inequalities for the Laplace operator in a domain Ω of $\mathbb{R}^n$ periodically perforated along a manifold, with nonlinear restrictions for the flux on the boundary of the cavities. We assume that the perforations are balls of radius $O(\varepsilon^\alpha)$ distributed along a $(n-1)$-dimensional manifold γ with period ε. Here $\varepsilon > 0$ is a small parameter, $\alpha > 0$ and $n \geqslant 3$. On the boundary of the perforations, we have the restrictions for the solution $u_\varepsilon \geqslant 0$, $\partial_\nu u_\varepsilon \geqslant -\varepsilon^{-\kappa}\sigma(x,u_\varepsilon)$ and $u_\varepsilon(\partial_\nu u_\varepsilon + \varepsilon^{-\kappa}\sigma(x,u_\varepsilon)) = 0$, where $\kappa \geqslant 0$ and σ is a certain smooth function. For $\alpha \geqslant 1$ and $\kappa = (\alpha-1)(n-2)$, we characterize the asymptotic behavior of u_ε as $\varepsilon \to 0$ providing the homogenized problems. A critical size of the cavities is found when $\alpha = \kappa = (n-1)/(n-2)$ for which the corrector in the energy norm is constructed.

RÉSUMÉ

Nous considèrons inégalités variationnelles pour l'opérateur de Laplace dans une domaine Ω de $\mathbb{R}^n$ périodiquement perforé, et avec des restrictions pour le flux sur la frontière des trous. On suppose que les perforations sont des boules de rayon $O(\varepsilon^\alpha)$ distribuées sur une variété de dimension $(n-1)$, γ, de période ε. Ici $\varepsilon > 0$ est une petite paramètre, $\alpha > 0$ et $n \geqslant 3$. Sur la frontière des trous nous avons des restrictions pour la solution $u_\varepsilon \geqslant 0$, $\partial_\nu u_\varepsilon \geqslant -\varepsilon^{-\kappa}\sigma(x,u_\varepsilon)$ et $u_\varepsilon(\partial_\nu u_\varepsilon + \varepsilon^{-\kappa}\sigma(x,u_\varepsilon)) = 0$, où $\kappa \geqslant 0$ et σ est une certaine fonction régulière. Pour $\alpha \geqslant 1$ and $\kappa = (\alpha-1)(n-2)$, nous caractérisons le comportement asymptotique de u_ε pour $\varepsilon \to 0$. On trouve les problèmes homogéneisés et une taille critique des trous pour $\alpha = \kappa = (n-1)/(n-2)$. Pour cette taille on construit le correcteur dans la norme de l'énergie.

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1. Introduction

In this Note, we consider the solution u_ε of a variational inequality for the Laplace operator in a domain Ω_ε perforated along a $(n-1)$-dimensional manifold with a nonlinear adsorption rate on the boundary S_ε of the cavities G_ε. Ω_ε denotes the perforated domain $\Omega \setminus G_\varepsilon$, Ω a domain of $\mathbb{R}^n$ with $n \geqslant 3$, and the nonlinear term involves a large parameter and a continuously differentiable function $\sigma = \sigma(x,u)$ defined in $\bar{\Omega} \times \mathbb{R}$, which is strictly monotonic with respect to u. We assume that the perforations G_ε are the unions of balls of radius $C_0\varepsilon^\alpha$ with $C_0 > 0$ and α ranges in $[1,\infty)$. These perforations are periodically distributed along the manifold $\gamma = \Omega \cap \{x_1 = 0\} \neq \emptyset$ with period ε. Here, $\varepsilon > 0$ denotes a parameter that we shall make converge towards zero. On the boundary of the cavities S_ε (the union of the boundaries of the balls), we consider the nonlinear restrictions (5) involving the parameter $\varepsilon^{-\kappa}$ with $\kappa = (\alpha-1)(n-1)$. We study the asymptotic behavior of the solution u_ε as $\varepsilon \to 0$ of the problem, namely of problem (3)–(4) for a given data $f \in L^2(\Omega)$. Note that, among the possible values of α and κ, here we consider those such that the parameter $\varepsilon^{-\kappa}$ multiplied by the area of S_ε is of order $O(1)$. We also emphasize that the restrictions on S_ε are different from those considered in previous homogenization problems in the literature of applied mathematics. The problem arises in the framework of the modeling of the diffusion of substances in porous media: see [1] and [2] for more precise models.

For $\alpha \in [1, (n-1)/(n-2)]$, we obtain the weak convergence of the solution u_ε, when $\varepsilon \to 0$, as stated in Theorems 3.1 and 3.3, to the solution u of a problem for the Laplace operator in Ω with a certain *homogenized transmission condition* on γ. This transmission condition contains a nonlinear function of u which represents the macroscopic contribution of the nonlinear law on the boundary of the microscopic cavities. The nonlinear term is obtained from the function σ depending on the value of α in (1): see (14) for $\alpha \in [1, (n-1)/(n-2))$ and (7) for $\alpha = (n-1)/(n-2)$. Note that the case where $\alpha = (n-1)/(n-2)$ differs from the rest of the cases since we obtain a boundary value problem and the nonlinear term is different: it involves a new function $H(x,u)$ defined implicitly by the nonlinear equation (8), which proves to have similar properties to the given function σ (cf. (2)). This value for parameters α and κ, namely $\alpha = \kappa$ in (1), provides a *critical size* of the balls G_ε. See Remark 1 in this respect. In the case where $\alpha > (n-1)/(n-2)$ the homogenized problem is the Dirichlet problem (15).

Similar geometrical configurations for linear and nonlinear boundary value problems have been considered in many previous papers: let us mention [1–9] for some of these problems and for further references. Also let us mention [10,11] for the homogenization of variational inequalities. We refer to [3] as the closest problem to the problem here considered. In [3] a nonlinear boundary condition on S_ε has been considered, namely $\partial_\nu u_\varepsilon + \varepsilon^{-\kappa}\sigma(x,u_\varepsilon) = 0$ for the value $\kappa = \alpha$. [1] considers the same boundary condition but with the cavities periodically distributed on the whole volume and with $n = 3$. In this connection, let us mention [4] for non-homogeneous boundary conditions, [2] and [5] for $\alpha = 1$, and [6] for evolution problems.

It is worth mentioning that for different homogenization problems, with different homogenized equations in Ω and on γ, the kind of nonlinear equation (8) also appears in [1] and [3] respectively. The change of type of nonlinearity was first noticed in [1] for spatially distributed cavities and in [3] for the cavities along γ. This recalls the so-called *strange term* arising in many papers on homogenization problems with critical sizes: see, e.g., [8] for different linear problems and further references, and [1] and [3] for nonlinear boundary value problems. In the present paper, we highlight the phenomena for problems with strong nonlinear restrictions on the boundary of the microscopic cavities.

It should be noted that, since we are dealing here with homogenization of variational inequalities, and nonlinear restrictions on the boundary of the perforations, proofs rely on extension operators, on transformations of surface integral on S_ε into volume integrals in Ω_ε, on convergence of measures, and on the appropriate choice of positive test functions (cf. (12) and Remark 1) which allows us to pass to the limit in the weak formulations. The main convergence results are stated in Theorems 3.1, 3.3 and 3.4. Furthermore, an improved approximation for the macroscopic solution is constructed when $\alpha = \kappa$, and more accurate results are obtained with respect to the energy norm (cf. Theorem 3.2 and [12] for other values of α and κ). For the sake of brevity, we only provide a sketch of the proofs involving the critical size, leaving the technical and laborious computations, and the rest of the proofs, to be performed in a forthcoming publication (cf. [12]). Finally, the structure of the paper is as follows: Section 2 contains the setting of the ε-dependent problem while Section 3 contains the homogenized problems and the corrector result.

2. Setting of the ε-dependent problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, with a smooth boundary $\partial\Omega$. Assume that $\gamma = \Omega \cap \{x_1 = 0\} \neq \emptyset$ is a domain on the hyperplane $\{x_1 = 0\}$. We denote by G_0 the ball of radius 1 centered at the origin of coordinates. For a set B, and $\delta > 0$, we denote by $\delta B = \{x \mid \delta^{-1}x \in B\}$. We set

$$\tilde{G}_\varepsilon = \bigcup_{z' \in \mathbb{Z}'} \left(a_\varepsilon G_0 + \varepsilon z'\right) = \bigcup_{j \in \mathbb{Z}'} G_\varepsilon^j$$

where $\mathbb{Z}'$ is the set of vectors of the form $z' = (0, z_2, \ldots, z_n)$ with integer components z_l, $l = 2, \ldots, n$, $a_\varepsilon = C_0\varepsilon^\alpha$, C_0 is a positive number, ε is a small positive parameter that we shall make converge towards zero, and α is a parameter, $\alpha \geqslant 1$. If no confusion arises, we identify $z \in \mathbb{Z}'$ with $j \in \mathbb{Z}'$, and we define

$$G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j, \quad \text{where } \Upsilon_\varepsilon = \big\{z\in\mathbb{Z}'\colon\ G_\varepsilon^z \subset \tilde G_\varepsilon, \bar G_\varepsilon^z \subset \Omega, \rho\big(\partial\Omega, \bar G_\varepsilon^z\big) \geqslant 2\varepsilon\big\}$$

As is self-evident, the number of G_ε^z with index $z\in\Upsilon_\varepsilon$ is $|\Upsilon_\varepsilon| \cong d\varepsilon^{1-n}$ for a certain $d>0$.

In what follows, we set

$$\Omega_\varepsilon = \Omega\setminus \bar G_\varepsilon, \qquad S_\varepsilon = \partial G_\varepsilon, \qquad \partial\Omega_\varepsilon = \partial\Omega\cup S_\varepsilon$$

Also, let us consider $f\in L^2(\Omega)$ and the space $H^1(\Omega_\varepsilon,\partial\Omega)$ to be the completion with respect to norm $H^1(\Omega_\varepsilon)$ of the set of infinitely differentiable functions on $\bar\Omega_\varepsilon$, vanishing in the neighborhood $\partial\Omega$. Let ω_n denote the area of the unit sphere in $\mathbb{R}^n$ and κ denote a positive parameter depending on α and n. In particular, here we consider

$$\alpha\in\left[1,\frac{n-1}{n-2}\right] \quad \text{and} \quad \kappa = (\alpha-1)(n-1) \tag{1}$$

and, $\alpha > (n-1)/(n-2)$ for any real κ.

Let us consider $\sigma(x,u)$ a continuously differentiable function of variables $(x,u)\in\bar\Omega\times\mathbb{R}$ satisfying: $\sigma(x,0)=0$, and there exist two constants $k_1>0$ and $k_2>0$ such that

$$k_1 \leqslant \frac{\partial\sigma}{\partial u}(x,u) \leqslant k_2, \quad x\in\bar\Omega,\ u\in\mathbb{R} \tag{2}$$

Note that, for any fixed $x\in\Omega$, $\sigma(x,u)\geqslant 0$ if $u\geqslant 0$, $\sigma(x,u)\leqslant 0$ if $u\leqslant 0$, and $k_1u^2\leqslant u\sigma(x,u)\leqslant k_2u^2$.

In Ω_ε we consider the problem: Find $u_\varepsilon\in K_\varepsilon$, such that the following variational inequality

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon\nabla(v-u_\varepsilon)\,\mathrm{d}x + \varepsilon^{-\kappa}\int_{S_\varepsilon}\sigma(x,u_\varepsilon)(v-u_\varepsilon)\,\mathrm{d}s \geqslant \int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x \tag{3}$$

is satisfied for all $v\in K_\varepsilon$. Here set K_ε is defined by

$$K_\varepsilon = \big\{g\in H^1(\Omega_\varepsilon,\partial\Omega)\colon\ g\geqslant 0 \text{ a.e. on } S_\varepsilon\big\} \tag{4}$$

Problem (3)–(4) is the variational formulation of the problem

$$\begin{aligned} &-\Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega \\ &u_\varepsilon \geqslant 0, \qquad \partial_\nu u_\varepsilon \geqslant -\varepsilon^{-\kappa}\sigma(x,u_\varepsilon), \qquad u_\varepsilon\big(\partial_\nu u_\varepsilon + \varepsilon^{-\kappa}\sigma(x,u_\varepsilon)\big) = 0 \quad \text{for } x\in S_\varepsilon \end{aligned} \tag{5}$$

and the existence and uniqueness of solution u_ε of (3)–(4) follows from the monotonicity of the function $\sigma(x,u)$ with respect to u (see, e.g., Section II.8.2 in [13], and [3] for further references). Above, ∂_ν denotes the derivative along the unit outward normal vector ν to $\partial\Omega_\varepsilon$.

Setting $v=0$ in (3) we get the estimate $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\leqslant C$. Let $\tilde u_\varepsilon$ be an H^1-extension of u_ε to Ω with the following properties

$$\|\tilde u_\varepsilon\|_{H^1(\Omega)} \leqslant C\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}, \qquad \|\nabla\tilde u_\varepsilon\|_{L^2(\Omega)} \leqslant C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

Here and in what follows C denotes a constant which does not depend on ε. See Lemma 1 in [7] for the construction of $\tilde u_\varepsilon$. Hence, $\|\tilde u_\varepsilon\|_{H^1(\Omega)}\leqslant C$.

Thus, we have that for each sequence of ε we can extract a subsequence (still denoted by ε) such that

$$\tilde u_\varepsilon \rightharpoonup u \quad \text{in } H_0^1(\Omega)\text{-weak} \quad \text{and} \quad \tilde u_\varepsilon \to u \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon\to 0 \tag{6}$$

for a certain function u which we identify in Section 3 with the solution of (7) ((14), (15), respectively) when $\alpha = (n-1)/(n-2)$ ($\alpha\in[1,(n-1)/(n-2))$, $\alpha>(n-1)/(n-2)$, respectively), and (6) holds for the whole sequence.

3. The homogenized problems and the correctors

Theorem 3.1. *Let α be $\alpha=\kappa=\frac{n-1}{n-2}$. Then, the limit function u in* (6) *is the weak solution of the problem*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega^-\cup\Omega^+ \\ u = 0, & \text{on } \partial\Omega \\ [u] = 0, & \text{on } \gamma \\ \left[\dfrac{\partial u}{\partial x_1}\right] = \mathcal{A}_n\big(H\big(x,u^+\big)+u^-\big), & \text{on } \gamma \end{cases} \tag{7}$$

where $\phi^+(x) = \sup(\phi(x), 0)$, $\phi^-(x) = \phi(x) - \phi^+(x)$, $\mathcal{A}_n$ *is the constant* $\mathcal{A}_n = (C_0)^{n-2}\omega_n(n-2)$, *the brackets mean* $[g]|_{P\in\gamma} = \lim_{p\to P, p\in\Omega^+} g(p) - \lim_{p\to P, p\in\Omega^-} g(p)$ *for any point* $P \in \gamma$, *and* $H(x,u)$ *is the solution of the functional equation*

$$\frac{(n-2)}{C_0} H = \sigma(x, u - H) \tag{8}$$

Sketch of the proof. First, let us show that (7) has a unique weak solution $u \in H_0^1(\Omega)$.

Note that inequality (2) and the implicit function theorem provide the existence of a unique solution $H \equiv H(x,u)$, which is continuously differentiable on $(x,u) \in \bar{\Omega} \times \mathbb{R}$ and satisfies $\tilde{k}_1 < \partial_u H(x,u) < \tilde{k}_2$, for certain constants $\tilde{k}_i > 0, i = 1, 2$ (see (2), and [3] for more details of this proof).

Also, we observe that the weak solution of problem (7) is the solution in $H_0^1(\Omega)$ of the integral equation

$$\int_\Omega \nabla u \nabla v \,\mathrm{d}x + \mathcal{A}_n \int_\gamma \big(H\big(x, u^+\big) + u^-\big) v \,\mathrm{d}\hat{x} = \int_\Omega f v \,\mathrm{d}x, \quad \forall v \in H_0^1(\Omega) \tag{9}$$

From the strict monotonicity of the function H with respect to the second argument, the existence and uniqueness of solution of (9) holds (cf. [13] and [3]).

We show that the solution of (9) coincides with the weak limit in $H^1(\Omega)$ of $\tilde{u}_\varepsilon$; we use the energy method. Below, we construct the test functions (cf. (12)) which allow us to take limits in (3)–(4) from the solution of the local problem (10).

Let P_ε^j be the center of the ball G_ε^j and we denote by T_ε^j the ball of radius $\varepsilon/4$ with center P_ε^j. Let us consider the functions w_ε^j $(j \in \Upsilon_\varepsilon)$ as the solutions of the following problems

$$\begin{cases} \Delta w_\varepsilon^j = 0, & \text{in } T_\varepsilon^j \setminus \overline{G_\varepsilon^j} \\ w_\varepsilon^j = 1, & \text{on } \partial G_\varepsilon^j \\ w_\varepsilon^j = 0, & \text{on } \partial T_\varepsilon^j \end{cases} \tag{10}$$

We define the function $W_\varepsilon \in H^1(\mathbb{R}^n)$ by setting

$$W_\varepsilon(x) = w_\varepsilon^j(x) \quad \text{for } x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j},\ j \in \Upsilon_\varepsilon, \qquad W_\varepsilon(x) = 1 \quad \text{for } x \in \overline{G_\varepsilon} \tag{11}$$

and extending $W_\varepsilon(x)$ with the value 0 for $x \in \mathbb{R}^n \setminus \bigcup_{j\in\Upsilon_\varepsilon} T_\varepsilon^j$.

As it is well known, the solution of (10) can be constructed explicitly, this being an essential fact for the proof of the statement in the theorem. Also, the weak convergence $W_\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$, as $\varepsilon \to 0$, holds.

Let us consider the function

$$v^\varepsilon = \phi^+(x) - W_\varepsilon(x) H\big(x, \phi^+(x)\big) + \big(1 - W_\varepsilon(x)\big)\phi^-(x) \tag{12}$$

where ϕ is an arbitrary function from $C_0^\infty(\Omega)$. Let us prove that $v^\varepsilon \geqslant 0$ on S_ε, and thus it belongs to K_ε. Since $W_\varepsilon = 1$ on S_ε, we show that $\phi^+ - H(x, \phi^+) \geqslant 0$ on S_ε. This is clear if $H(x, \phi^+(x)) \leqslant 0$ on S_ε. Suppose that for some point $x \in S_\varepsilon$ we have that $H(x, \phi^+(x)) > 0$ and $\phi^+(x) - H(x, \phi^+(x)) < 0$. Then, we get that $\sigma(x, \phi^+ - H(x, \phi^+(x))) \leqslant 0$ and $H(x, \phi^+(x)) = \frac{C_0}{(n-2)}\sigma(x, \phi^+(x) - H(x, \phi^+(x))) \leqslant 0$. Thus we obtain a contradiction.

Using (2) the left hand side of (3) can be written as: $\int_{\Omega_\varepsilon} \nabla v \nabla(v - u_\varepsilon)\,\mathrm{d}x + \varepsilon^{-\kappa} \int_{S_\varepsilon} \sigma(x, v)(v - u_\varepsilon)\,\mathrm{d}s$. Then, we take $v = v^\varepsilon$ defined in (12), and we pass to the limit when $\varepsilon \to 0$ (cf. Theorems 1 and 2 in [3] for the technique, and [12] for details of the proof) to obtain the following inequality for $u \in H_0^1(\Omega)$

$$\int_\Omega \nabla\phi \nabla(\phi - u)\,\mathrm{d}x + \mathcal{A}_n \int_\gamma \big(H\big(x, \phi^+\big) + \phi^-\big)(\phi - u)\,\mathrm{d}\hat{x} \geqslant \int_\Omega f(\phi - u)\,\mathrm{d}x, \quad \forall \phi \in H_0^1(\Omega) \tag{13}$$

Now, we consider $\phi = u \pm \lambda v$ in (13) with $\lambda > 0$, and v an arbitrary function of $H_0^1(\Omega)$, and passing to the limit as $\lambda \to +0$, we obtain the integral identity (9) for the limit function u. □

Theorem 3.2. *Let* α *be* $\alpha = \kappa = \frac{n-1}{n-2}$. *Let* u_ε *be the solution of the variational inequality* (3)–(4), $u \in H_0^1(\Omega)$ *the weak solution of the boundary value problem* (7), *with the additional regularity* $u \in C^1(\bar{\Omega}^+)$ *and* $u \in C^1(\bar{\Omega}^-)$, *and* W_ε *defined by* (11). *Then, we have*

$$\big\|\nabla\big(u_\varepsilon - u + W_\varepsilon H\big(x, u^+\big) + W_\varepsilon u^-\big)\big\|_{L_2(\Omega_\varepsilon)}^2 + \varepsilon^{-\alpha}\big\|u_\varepsilon - u^+ + H\big(x, u^+\big)\big\|_{L_2(S_\varepsilon)}^2 \leqslant C\sqrt{\varepsilon}$$

Sketch of the proof. Let us consider (3) and take $v = \mathrm{v}^\varepsilon$ with $\mathrm{v}^\varepsilon = v^\varepsilon$ defined by (12) for $\phi \equiv u$. Let us consider (9) and take $v = \mathrm{v}^\varepsilon - \tilde{u}_\varepsilon$ for $\tilde{u}_\varepsilon$ arising in (6). Subtracting both equations, we use among other tools, Lemma 2 in [7] and Lemma 1 in [8] to obtain the inequality in the statement (see [12] for details of the proof). □

Theorem 3.3. *Let α be $\alpha \in [1, \frac{n-1}{n-2})$ and $\kappa = (\alpha - 1)(n - 1)$. Then, the limit function u in* (6) *is the solution of the problem*: *find $u \in K_0 = \{g \in H_0^1(\Omega):\ g \geqslant 0$ a.e. on $\gamma\}$, such that inequality*

$$\int_\Omega \nabla u \nabla (v - u)\,\mathrm{d}x + \mathcal{B}_n \int_\gamma \sigma(x, u)(v - u)\,\mathrm{d}\hat{x} \geqslant \int_\Omega f(v - u)\,\mathrm{d}x \tag{14}$$

is satisfied for all $v \in K_0$. The constant $\mathcal{B}_n$ is defined by $\mathcal{B}_n = C_0^{n-1}\omega_n$.

Theorem 3.4. *Let α be $\alpha > \frac{n-1}{n-2}$ and $\kappa \in \mathbb{R}$. Then, the limit function u in* (6) *is the weak solution of the Dirichlet problem*:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{15}$$

Remark 1. We observe that we need to construct test functions different from (12) to show Theorems 3.3 and 3.4 (cf. [12]). Note that $\alpha = 1$ means $\kappa = 0$; namely, the size of the cavities and the periodicity are of the same order of magnitude. For $\alpha \in [1, (n-1)/(n-2))$, in order to prove Theorem 3.3, we must introduce new local problems on the unit cell $\varepsilon(-1/2, 1/2)^n \setminus a_\varepsilon G_0$ (cf. [8] and [9] for related problems) and obtain precise bounds for their solutions. The proofs differ depending on whether $\alpha = 1$ or $\alpha > 1$. Also, note that the asymptotic behavior of the solution u_ε, as $\varepsilon \to 0$, is described by a variational inequality for the Laplacian with a nonlinear restriction for the flux transmission on γ. In contrast, for $\alpha \geqslant (n-1)/(n-2)$, the asymptotic behavior of the solution is described by boundary value problems (cf. (7) and (15)).

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