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Singular layers for transmission problems in thin shallow shell theory: Elastic junction case

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ABSTRACT

In this Note we study two-dimensional transmission problems for the linear Koiter's model of an elastic multi-structure composed of two thin shallow shells with the same thickness $\varepsilon \ll 1$, in the elastic junction case. We suppose that the loading is singular, that the elastic coefficients are of different order on each part ($O(\varepsilon^{-1})$ and $O(1)$ respectively) and that the elastic stiffness coefficient of the hinge is $k^\varepsilon = O(\varepsilon)$. The formal limit problem fails to give a solution satisfying all boundary and transmission conditions; it gives only the outer solution. We derive the inner limit problem which allows us to describe the transmission layer.

RÉSUMÉ

Cette Note s'intéresse à des problèmes de transmission bidimensionnels pour le modèle linéaire de Koiter d'une multi-structure élastique mince composée de deux coques peu profondes de même épaisseur $\varepsilon \ll 1$, pour le cas de jonction élastique avec chargements singuliers. On suppose que les coefficients d'élasticité sont d'ordre très différent sur chaque partie (respectivement $O(\varepsilon^{-1})$ et $O(1)$) et que le coefficient de rigidité de la charnière $k^\varepsilon = O(\varepsilon)$. Le problème limite formel ne donne pas une solution vérifiant toutes les conditions au bord et de transmission; il donne seulement la solution « extérieure ». Nous identifions le problème limite « intérieur » qui nous permet de déterminer la couche limite de transmission.

Version française abrégée

Dans cette Note, on étudie un problème de transmission bidimensionnel, il s'agit d'un problème simplifié du modèle linéaire de Koiter avec les conditions de jonction élastique (voir [1, III.3]). On considère deux surfaces elliptiques S et $\tilde{S}$ représentant les surfaces moyennes de deux coques peu profondes minces C et $\tilde{C}$ de même épaisseur ε. La surface moyenne est elliptique totalement encastrée présentant une arête Γ. Les coefficients d'élasticité dans chaque partie de la multi-structure sont très différents, c'est-à-dire $O(\varepsilon^{-1})$, $O(\varepsilon)$ et $O(1)$ respectivement.

Soient ω et $\tilde{\omega}$ deux domaines plans, ϕ et $\tilde{\phi}$ deux bijections telles que

$$\omega = (0,\pi)_{x_1} \times (0,1)_{x_2}, \qquad \tilde{\omega} = (0,\pi)_{x_1} \times (0,1)_{\tilde{x}_2}, \qquad \Sigma = \left\{(x_1, 0, 0) \,\middle|\, 0 < x_1 < \pi\right\} \tag{1}$$

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$$\Omega = \omega \cup \Sigma \cup \tilde{\omega}, \qquad S = \phi(\omega), \qquad \tilde{S} = \tilde{\phi}(\tilde{\omega})$$
$$\Gamma = \phi(\Sigma) = \tilde{\phi}(\Sigma), \qquad \Gamma_0 = \partial\Omega \cap \partial\omega \quad \text{et} \quad \tilde{\Gamma}_0 = \partial\Omega \cap \partial\tilde{\omega} \tag{2}$$

La formulation variationnelle du problème étudié est donnée par (4). Si les chargements sont dans l'espace $H^{-1} \times H^{-1} \times H^{-2}$ et ε est fixé, l'existence et l'unicité de la solution du problème (4) est une conséquence du lemme de Lax–Milgram. Dans cette Note, on suppose que la donnée sur ω est de la forme $f(x_1, x_2) = (0, 0, F(x_1)\delta(x_2))$ et que la donnée (sur $\tilde{\omega}$) $\tilde{f}$ est suffisament régulière, ici δ désigne la masse de Dirac et $F \in L^2(0, \pi)$. Pour $\varepsilon > 0$ fixé le problème est bien posé, mais le problème limite formel, i.e., $\varepsilon = 0$ est mal posé. Les conditions au bord (6) et de transmissions (7) ne peuvent être satisfaites. Une couche limite ordinaire sur le bord et une couche limite de transmission le long de l'interface se développent. En effectuant le changement de variables (11) et d'inconnues (12), on obtient un nouveau problè me limite, dont la solution est une approximation valable au voisinage de la zone de transmission.

1. Introduction

This Note is devoted to a transmission problem for the linear Koiter's shell model set on an elastic multi-structure composed of two thin shallow shells for the elastic junction conditions (see [1, III.3]). The common middle surface is piecewise elliptic with a simply fixed edge (in the terminology of [2]). We restrict ourselves to the case when the coefficient of the elastic stiffness of the hinge becomes very small. Nevertheless, in the shell theory, it appears a very interesting feature when the loading is singular (for example δ-like singularity along a curve C, see [3]), the limit problem becomes ill-posed in the original energy space, leading to the outbreak of internal layers. In this paper we restrict our attention to the limit problem, when the shell is linked to an other shell with very different elastic coefficients and singular loading along the interface between the two shells. Hence an appropriate dilation of the normal coordinate leads to an equivalent problem whose solution is a correct approximation of the original solution in a neighborhood of the transmission zone.

Throughout we use the convention of summation of repeated indices, which run from 1 to 2 when they are Greek. The notation $(.)^{\{c\}}$ means that the property holds true for both $\omega, \tilde{\omega}, u, \tilde{u}, x, \tilde{x}, \ldots$, i.e., on both shells.

2. The model problem and its formal limit

Let us consider two thin elliptic shallow shells C and $\tilde{C}$ with the same thickness ε, whose middle surfaces S and $\tilde{S}$ intersect along a common boundary Γ. Namely we assume that the domains are given by (1)–(2), i.e., $S^{\{c\}}$ is the image through the bijection $\phi^{\{c\}}$ of the bounded open subset $\omega^{\{c\}}$ of the plane of variables $x_1^{\{c\}}$, $x_2^{\{c\}}$ which represent the local coordinates in the basis:

$$a_\alpha^{\{c\}} = \partial_\alpha^{\{c\}}\phi^{\{c\}} = \frac{\partial \phi^{\{c\}}}{\partial x_\alpha^{\{c\}}}, \quad \alpha = 1, 2 \quad \text{and} \quad a_3^{\{c\}} = \left|a_1^{\{c\}} \times a_2^{\{c\}}\right|^{-1}\left(a_1^{\{c\}} \times a_2^{\{c\}}\right)$$

We suppose that the common middle surface is clamped along the whole boundary and that the angle between the two shells along the edge Σ is equal to $\pi/2$.

We consider a simplified problem, when the coefficients of the second fundamental form of the middle surface $S^{\{c\}}$ satisfy $b_{11}^{\{c\}} = b_{22}^{\{c\}} = 1$, $b_{12}^{\{c\}} = 0$ and the elasticity coefficients $A^{\alpha\beta\gamma\lambda,\{c\}}$ satisfy

$$A^{\alpha\beta\gamma\lambda,\{c\}}\gamma_{\gamma\lambda}\left(v^{\{c\}}\right) = T^{\alpha\beta}\left(v^{\{c\}}\right) = \gamma_{\alpha\beta}\left(v^{\{c\}}\right) = \frac{1}{2}\left(\partial_\alpha^{\{c\}} v_\beta^{\{c\}} + \partial_\beta^{\{c\}} v_\alpha^{\{c\}}\right) - b_{\alpha\beta}^{\{c\}} v_3^{\{c\}}$$
$$\frac{A^{\alpha\beta\gamma\lambda,\{c\}}}{12}\varrho_{\gamma\lambda}\left(v^{\{c\}}\right) = M^{\alpha\beta,\{c\}}\left(v^{\{c\}}\right) = \partial_{\alpha\beta}^{\{c\}} v_3^{\{c\}}, \qquad \tilde{A}^{\alpha\beta\gamma\lambda} = \varepsilon^{-1} A^{\alpha\beta\gamma\lambda}$$

The rigidity coefficient of the hinge is $k^\varepsilon = \varepsilon$.

At any point P of the interface, the elastic behavior only insures the continuity of the displacements. The tangential component M_n of the moment M is proportional to the jump of the tangential component of the rotations, i.e.,

$$\begin{cases} u(P) = \tilde{u}(P) \\ M_n = \varepsilon\left[\dfrac{\partial u_3^\varepsilon}{\partial x_2} - \varepsilon^{-1}\dfrac{\partial \tilde{u}_3^\varepsilon}{\partial \tilde{x}_2}\right] \end{cases} \tag{3}$$

According to [1, III.3], the variational formulation of the problem is

$$\begin{cases} \text{Find } \left(u^\varepsilon, \tilde{u}^\varepsilon\right) \in H \quad \text{s.t. } \forall(v, \tilde{v}) \in H \\ a\left(u^\varepsilon, v\right) + \varepsilon^2 b\left(u^\varepsilon, v\right) + \varepsilon^{-1}\left(\tilde{a}\left(\tilde{u}^\varepsilon, \tilde{v}\right) + \varepsilon^2 \tilde{b}\left(\tilde{u}^\varepsilon, \tilde{v}\right)\right) + \varepsilon c^\varepsilon\left(\left(u^\varepsilon; \tilde{u}^\varepsilon\right), (v; \tilde{v})\right) = \langle f, v\rangle + \langle \tilde{f}, \tilde{v}\rangle \end{cases} \tag{4}$$

where

$$H=\{(v,\tilde{v})\in V\times\tilde{V}\mid v_1=\tilde{v}_1,\ v_2=\tilde{v}_3,\ v_3=-\tilde{v}_2 \text{ on } \Sigma\}$$

$$V^{\{c\}}=\left\{v^{\{c\}}\in H^1(\omega^{\{c\}})^2\times H^2(\omega^{\{c\}})\ \middle|\ v^{\{c\}}_{|\Gamma_0^{\{c\}}}=\left(\frac{\partial v_3^{\{c\}}}{\partial n^{\{c\}}}\right)_{|\Gamma_0^{\{c\}}}=0\right\}$$

$$a^{\{c\}}(u,v)=\int_{\omega^{\{c\}}}\gamma_{\alpha\beta}(u)\gamma_{\alpha\beta}(v)\,\mathrm{d}x,\qquad b^{\{c\}}(u,v)=\int_{\omega^{\{c\}}}\varrho_{\alpha\beta}(u)\varrho_{\alpha\beta}(v)\,\mathrm{d}x$$

$$c^{\varepsilon}\big((u;\tilde{u}),(v;\tilde{v})\big)=\int_{\Sigma}\left(\frac{\partial u_3}{\partial x_2}-\varepsilon^{-1}\frac{\partial\tilde{u}_3}{\partial\tilde{x}_2}\right)\left(\frac{\partial v_3}{\partial x_2}-\varepsilon^{-1}\frac{\partial\tilde{v}_3}{\partial\tilde{x}_2}\right)\mathrm{d}\Sigma$$

For a fixed ε, the left-hand side of (4) is strongly coercive on H and its right-hand side defines a linear form on H, hence the existence and the uniqueness of a solution $(u^\varepsilon,\tilde{u}^\varepsilon)\in H$ are ensured by the Lax–Milgram lemma. At least formally, it is also a solution of the transmission system (which is a system of total order 16):

$$\begin{cases}-\partial_\alpha T^{\alpha\beta}(u^\varepsilon)=f^\beta\\ -b_{\alpha\beta}T^{\alpha\beta}(u^\varepsilon)+\varepsilon^2\partial^4_{\alpha\beta\lambda\mu}u_3^\varepsilon=f^3\end{cases}\text{in } \omega \quad\text{and}\quad \begin{cases}-\partial_\alpha T^{\alpha\beta}(\tilde{u}^\varepsilon)=\varepsilon\tilde{f}^\beta\\ -b_{\alpha\beta}T^{\alpha\beta}(\tilde{u}^\varepsilon)+\varepsilon^2\partial^4_{\alpha\beta\lambda\mu}\tilde{u}_3^\varepsilon=\varepsilon\tilde{f}^3\end{cases}\text{in } \tilde{\omega} \tag{5}$$

with the boundary conditions

$$u_\alpha^\varepsilon=u_3^\varepsilon=\frac{\partial u_3^\varepsilon}{\partial n}=\tilde{u}_\alpha^\varepsilon=\tilde{u}_3^\varepsilon=\frac{\partial\tilde{u}_3^\varepsilon}{\partial\tilde{n}}=0\quad\text{on } \partial\Omega \tag{6}$$

the elastic junction conditions and the action–reaction principle giving (eight) transmission conditions:

$$\begin{cases}\tilde{u}_2^\varepsilon=-u_3^\varepsilon\\ \tilde{u}_3^\varepsilon=u_2^\varepsilon\\ u_1^\varepsilon=\tilde{u}_1^\varepsilon\end{cases}\text{on } \Sigma\quad\text{and}\quad\begin{cases}\varepsilon\left[\dfrac{\partial u_3^\varepsilon}{\partial x_2}-\varepsilon^{-1}\dfrac{\partial\tilde{u}_3^\varepsilon}{\partial\tilde{x}_2}\right]=M_n\\ \varepsilon T^{12}=\tilde{T}^{12}\\ T^{22}=0 \text{ and } \tilde{T}^{22}=0\\ \varepsilon M^{22}=\tilde{M}^{22}\end{cases}\text{on } \Sigma \tag{7}$$

The formal limit problem by taking $\varepsilon=0$ is the "membrane problem" (which is a system of total order 8) given by

$$\begin{cases}-\partial_\alpha T^{\alpha\beta}(u)=f^\beta\\ -b_{\alpha\beta}T^{\alpha\beta}(u)=f^3\\ u_{\alpha|\Gamma_0}=0\end{cases}\text{in } \omega\quad\text{and}\quad\begin{cases}-\partial_\alpha T^{\alpha\beta}(\tilde{u})=0\\ -b_{\alpha\beta}T^{\alpha\beta}(\tilde{u})=0\\ \tilde{u}_{\alpha|\tilde{\Gamma}_0}=0\end{cases}\text{in } \tilde{\omega} \tag{8}$$

with four transmission conditions on Σ:

$$u_1=\tilde{u}_1,\qquad \tilde{T}^{12}=0,\qquad T^{22}=\tilde{T}^{22}=0\quad\text{on } \Sigma \tag{9}$$

Proposition 2.1. *The formal limit problem is not elliptic in the Agmon, Douglis and Nirenberg sense, as the transmission conditions do not satisfy the Shapiro–Lopatinski condition.*

Proof. We can see that from the system (8) the $\tilde{T}^{\alpha\beta}$ vanish but this does not mean that the $\tilde{u}_i$ are nulls (see [4, p. 238]) (we take the symmetry in order to obtain $u_1=\tilde{u}_1$). □

Remark 1. The previous proposition asserts that even if $f^{i,\{c\}}$ are smooth, the formal limit problem is ill-posed. Any solution $(u,\tilde{u})$ does not satisfy the transmission conditions (7) at Σ and the boundary conditions on $\partial\Omega$. Therefore, we may expect that $(u^\varepsilon,\tilde{u}^\varepsilon)$ will develop boundary layers (transmission layers) along Σ and (standard) boundary layers along $\partial\Omega$.

Remark 2. If the loading $f^3\notin L^2(\omega)$, the limit problem fails to be variational as the loading is out of the dual of the energy space of the limit problem.

3. The case of singular loadings

Suppose now that the loading $\tilde{f}$ on $\tilde{\omega}$ is sufficiently smooth and that the loading on ω is of the form:

$$f(x_1,x_2)=\big(0,0,F(x_1)\delta(x_2)\big) \tag{10}$$

where δ is the Dirac mass and $F\in L^2(0,\pi)$. Note that they are not in the dual of the energy space of the limit problem. Hence to study the limit procedure we perform a dilatation of the coordinate normal to the layer

$$y_1 = x_1, \qquad y_2^{\{c\}} = \frac{x_2^{\{c\}}}{\eta(\varepsilon)} \quad \text{for } \left(x_1^{\{c\}}, x_2^{\{c\}}\right) \in \omega^{\{c\}} \tag{11}$$

The domain $\omega^{\{c\}}$ then becomes $B_\eta^{\{c\}} =]0, \pi[_{y_1} \times]0, \frac{1}{\eta}[_{y_2^{\{c\}}}$.

For the consistency of forthcoming developments we shall choose $\eta = \sqrt{\varepsilon}$ as is usual in elliptic shell layers. We shall further make the change of unknowns

$$\begin{cases} u_\alpha^\varepsilon(x) = u_\alpha^\eta(y) \\ u_3^\varepsilon(x) = \eta^{-1} u_3^\eta(y) \end{cases} \quad \text{and} \quad \begin{cases} \tilde{u}_\alpha^\varepsilon(\tilde{x}) = \eta^2 \tilde{u}_\alpha^\eta(\tilde{y}) \\ \tilde{u}_3^\varepsilon(\tilde{x}) = \eta \tilde{u}_3^\eta(\tilde{y}) \end{cases} \tag{12}$$

Hence for the test functions the continuity conditions on Σ becomes

$$v_1(y_1, 0) = \eta^2 \tilde{v}_1(y_1, 0), \qquad v_2(y_1, 0) = -\eta \tilde{v}_3(y_1, 0), \qquad v_3(y_1, 0) = \eta^3 \tilde{v}_2(y_1, 0) \tag{13}$$

As in [3] we shall introduce the expressions

$$\ell_0^{\{c\}}\left(v^{\{c\}}\right) = \left[\ell_0^1, \ell_0^2, \ell_0^3, \ell_0^4\right]^{\{c\}} = \left[-v_3, \frac{1}{2}\partial_2 v_1, \partial_2 v_2 - v_3, \partial_2^2 v_3\right]^{\{c\}}$$

$$\ell_1^{\{c\}}(u) = \left[\ell_1^1, \ell_1^2, \ell_1^3\right]^{\{c\}} = \left[\partial_1 u_1, \frac{1}{2}\partial_1 u_2, \partial_1 \partial_2 u_3\right]^{\{c\}}$$

With this choice of loadings (10) and the changes of unknowns and variables, the variational formulation (4) becomes the new problem

$$\begin{cases} \text{Find } \left(u^\eta, \tilde{u}^\eta\right) \in H_\eta \text{ s.t. } \forall (v, \tilde{v}) \in H_\eta \\ a^\eta\left(u^\eta, v\right) + \tilde{a}^\eta\left(\tilde{u}^\eta, \tilde{v}\right) + c\left(u^\eta; \tilde{u}^\eta, v; \tilde{v}\right) = \displaystyle\int_0^\pi F(y_1) v_3(y_1, 0)\,\mathrm{d}y_1 + \eta^2 \int_{\tilde{B}_0} \tilde{f}_3 \tilde{v}_3(y_1, \tilde{y}_2)\,\mathrm{d}\tilde{y} \end{cases} \tag{14}$$

where

$$H_\eta = \left\{(v, \tilde{v}) \in V_\eta \times \tilde{V}_\eta \text{ fulfilling (13)}\right\}$$

$$V_\eta^{\{c\}} = \left\{ v^{\{c\}} \in H^1\left(B_\eta^{\{c\}}\right)^2 \times H^2\left(B_\eta^{\{c\}}\right) \;\Big|\; v^{\{c\}}_{|\Gamma_0^{\eta,c}} = \left(\frac{\partial v_3^{\{c\}}}{\partial n}\right)_{|\Gamma_0^{\eta,c}} = 0 \right\}$$

$$\Gamma_0^\eta = \partial B_\eta \setminus \{y_2 = 0\} \quad \text{and} \quad \tilde{\Gamma}_0^\eta = \partial \tilde{B}_\eta \setminus \{\tilde{y}_2 = 0\}$$

$$a^\eta\left(u^\eta, v\right) = a_{00}\left(u^\eta, v\right) + \eta\left(a_{01}\left(u^\eta, v\right) + a_{10}\left(u^\eta, v\right)\right) + \eta^2 a_{11}\left(u^\eta, v\right) + \eta^4 a_{22}\left(u^\eta, v\right)$$

$$a_{00}(u, v) = \int_{B_0} \left[\ell_0^1(u)\ell_0^1(v) + \ell_0^2(u)\ell_0^2(v) + \ell_0^3(u)\ell_0^3(u) + \ell_0^4(u)\ell_0^4(v)\right]\mathrm{d}y$$

$$a_{01}(u, v) = \int_{B_0} \left[\ell_0^1(u)\ell_1^1(v) + 2\ell_0^2(u)\ell_1^2(v)\right]\mathrm{d}y, \qquad a_{10}(u, v) = \int_{B_0} \left[\ell_1^1(u)\ell_0^1(v) + 2\ell_1^2(u)\ell_0^2(v)\right]\mathrm{d}y$$

$$a_{11}(u, v) = \int_{B_0} \left[\ell_1^1(u)\ell_1^1(v) + 2\ell_1^2(u)\ell_1^2(v) + \ell_1^3(u)\ell_1^3(v)\right]\mathrm{d}y, \qquad a_{22}(u, v) = \int_{B_0} \partial_1^2 u_3 \partial_1^2 v_3\,\mathrm{d}y$$

$$c\left((u; \tilde{u}), (v; \tilde{v})\right) = \int_\Sigma \left(\frac{\partial u_3}{\partial y_2} - \frac{\partial \tilde{u_3}}{\partial \tilde{y_2}}\right)\left(\frac{\partial v_3}{\partial y_2} - \frac{\partial \tilde{v_3}}{\partial \tilde{y_2}}\right)\mathrm{d}\Sigma \tag{15}$$

Analogous expressions hold for $\tilde{a}_{00}$, $\tilde{a}_{11}$, $\tilde{a}_{10}$, $\tilde{a}_{01}$, $\tilde{a}_{22}$ with integration over $\tilde{B}_0$.

Since we may extend the elements of $V^{\eta,\{c\}}$ by zero for $y_2^{\{c\}} \geqslant \frac{1}{\eta}$, we have the inclusion $H_\eta \subset H_{\eta'}$ for $\eta > \eta'$. This extension by zero is also used in the definition of $a_{00}^{\{c\}}$, $a_{01}^{\{c\}}$, $a_{10}^{\{c\}}$, $a_{11}^{\{c\}}$ and $a_{22}^{\{c\}}$.

3.1. A priori estimates

The equivalence between (4) and (14) allows us to deduce the next estimates for the solution $(u^\eta, \tilde{u}^\eta) \in H_\eta$ of (14).

Lemma 3.1. *For a fixed η, if F is in $L^2(0,\pi)$ and $(u^\eta, \tilde{u}^\eta) \in H_\eta$ is solution of* (14) *then*

$$\|u_3^\eta\|_{L^2((0,\pi)_{y_1};H^2(0,\infty))} + \|\tilde{u}_3^\eta\|_{L^2((0,\pi)_{y_1};H^2(0,\infty))} + \eta^2\|\partial_1^2 u_3^\eta\|_{L^2(B_0)} + \eta^2\|\partial_1^2 \tilde{u}_3^\eta\|_{L^2(\tilde{B}_0)} \leqslant C$$

$$\|\ell_0(u^\eta)\|_{(L^2(B_0))^4} + \|\tilde{\ell}_0(\tilde{u}^\eta)\|_{(L^2(\tilde{B}_0))^4} + \eta\|\ell_1(u^\eta)\|_{(L^2(B_0))^3} + \eta\|\tilde{\ell}_1(\tilde{u}^\eta)\|_{(L^2(\tilde{B}_0))^3} \leqslant C$$

for some C independent of η.

3.2. Limit problem and convergence theorem

We are now in position to construct the energy space for the limit problem. Let

$$E_0 = \left(L^2(B_0)\right)^4, \qquad \tilde{E}_0 = \left(L^2(\tilde{B}_0)\right)^4, \qquad E_1 = \left(L^2(B_0)\right)^3 \quad \text{and} \quad \tilde{E}_1 = \left(L^2(\tilde{B}_0)\right)^3 \tag{16}$$

We may consider the space $\bigcup_{\eta\searrow 0} H_\eta$ with the norm:

$$\|(v,\tilde{v})\|_{V_0\times\tilde{V}_0} = \left(\|\ell_0(v)\|_{E_0}^2 + \|\ell_1(v)\|_{E_1}^2 + \|\ell_0(\tilde{v})\|_{\tilde{E}_0}^2 + \|\ell_1(\tilde{v})\|_{\tilde{E}_1}^2\right)^{\frac{1}{2}} \tag{17}$$

H_0 is the completion of $\bigcup_{\eta\searrow 0} H_\eta$ with this norm.

Lemma 3.2. *The quantity $\|(v,\tilde{v})\|_{\bar{H}_0} := (\|\ell_0(v)\|_{E_0}^{\frac{1}{2}} + \|\tilde{\ell}_0(\tilde{v})\|_{\tilde{E}_0}^{\frac{1}{2}})^2$ is a norm on H_0.*

Hence we can define the larger space $\bar{H}_0$ as the completion of H_0 with respect to the norm $\|\cdot\|_{\bar{H}_0}$.

Theorem 3.3. *Let $(u^\eta, \tilde{u}^\eta) \in H_\eta$ be the solution of* (14)*, then $(u^\eta, \tilde{u}^\eta) \to (u, \tilde{u})$ weakly in $\bar{H}_0$, where $(u,\tilde{u}) \in \bar{H}_0$ is the solution of*:

$$a_{00}(u,v) + \tilde{a}_{00}(\tilde{u},\tilde{v}) + c\big((u;\tilde{u}),(v;\tilde{v})\big) = \int_0^\pi F(y_1)v_3(y_1,0)\,\mathrm{d}y_1 \quad \forall (v,\tilde{v}) \in \bar{H}_0 \tag{18}$$

The corresponding limit problem in the inner variables y takes the following form. Choosing appropriate test functions in (18), we obtain the partial differential equations

$$\begin{cases} -\partial_2 T_0^{12}(u) - \partial_2 T_0^{22}(u) = 0 \\ -\partial_2 T_0^{12}(u) = 0 \\ -\partial_2 u_2 + 2u_3 + \partial_2^4 u_3 = F \end{cases} \quad \text{and} \quad \begin{cases} -\tilde{\partial}_2 T_0^{12}(\tilde{u}) - \tilde{\partial}_2 T_0^{22}(\tilde{u}) = 0 \\ -\tilde{\partial}_2 \tilde{T}_0^{12}(\tilde{u}) = 0 \\ -\tilde{\partial}_2 \tilde{u}_2 + 2\tilde{u}_3 + \tilde{\partial}_2^4 \tilde{u}_3 = 0 \end{cases} \tag{19}$$

with the transmission conditions on Σ:

$$\begin{cases} u_3 = 0 \\ \tilde{u}_3 = 0 \\ \partial_2^3 u_3 = \tilde{\partial}_2^3 \tilde{u}_3 = 0 \end{cases} \text{on } \Sigma \qquad \begin{cases} \partial_2^2 u_3 = \tilde{\partial}_2^2 \tilde{u}_3 = \partial_2 u_3 - \tilde{\partial}_2 \tilde{u}_3 \\ \partial_2 u_1 = \tilde{\partial}_2 \tilde{u}_1 = 0 \\ \partial_2 u_2 - u_3 = \tilde{\partial}_2 \tilde{u}_2 - \tilde{u}_3 = 0 \end{cases} \text{on } \Sigma \tag{20}$$

Remark 3. The conditions (20) satisfied by the solution of the limit problem are the formal limit of the conditions (7) (in the inner variables y) as $\varepsilon \to 0$.

4. Concluding remarks

We have shown that for the case of transmission problems of two thin shells the formal limit problem is ill-posed as the transmission conditions do not satisfy the Shapiro–Lopatinski condition. An appropriate dilation of the normal coordinate leads to an equivalent problem whose solution is a correct approximation of the original solution in a neighborhood of the transmission zone. The scaling (12) is consistent with the fact that the layer appears in the part where the elasticity coefficients are small. The immediate extension apart from more general data (the rigidity coefficient of the hinge and the elasticity coefficients) is to consider arbitrary k^ε and $\tilde{A}^{\alpha\beta\lambda\mu}(\varepsilon) = O(\varepsilon^d)$, $d \in \mathbb{R}$, then the behavior of the limit problem depends on the limit of k^ε as $\varepsilon \to 0$. Using a naive formal asymptotic expansion of the inner solution, we can show that if $k^\varepsilon = O(\varepsilon^l)$, $l < 1$, then the inner solution of the elastic case converges to the inner solution of the rigid one, i.e., the solution satisfies in addition to (20) the relation $\partial_2 u_3 = \tilde{\partial}_2 \tilde{u}_3$ on Σ.

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