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Boundary conditions for an axisymmetric circular cylinder

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Through generalizing the method of a decay analysis technique determining the interior solution developed by Gregory and Wan, two necessary conditions on the edge-data of an axisymmetric circular cylinder for the existence of a rapidly decaying solution are established. By accurate solutions for auxiliary regular state, and using the reciprocal theorem and Boussinesq solution, these necessary conditions for the edge-data to induce only a decaying elastostatic state will be translated into appropriate boundary conditions for the circular cylinder with axisymmetric deformations. The results of the present Note extend the known results to circular cylinder's deformation problems, which enable us to establish two correct boundary conditions with stress and mixed edge-data. For the stress data, our boundary conditions coincide with those obtained in conventional forms of elastic theories. More importantly, the appropriate boundary condition with mixed edge-data is obtained for the first time.

1. Introduction

Axisymmetric deformation is a fundamental deformation in engineering. As a solution, Saint-Venant's solution of the deformation was obtained using Saint-Venant's assumption. Some other authors gave various axisymmetric deformation theories different from that of Saint-Venant. Purser and Dougall gave a nonlinear solution by using series expansion. Robert and Keer [1], Stephen and Wang [2] studied the semi-infinite circular cylinder and boundary conditions under a general load. Birsan [3,4] studied cylindrical Cosserat elastic shells and boundary conditions.

It is generally known that the exact solution of linear elastostatic problems for slender and thin elastic bodies consists of an interior component and a boundary layer component (in a decaying form). Near a lateral edge, the interior solution is supplemented by boundary layer solution component which becomes insignificant away from the edge. The admissible boundary conditions can be satisfied only by a combination of these components. However, the boundary layer solution, even just a leading term approximation, needed to fit the edge-data is rather intractable except for cases with simple geometries and load symmetries. This, and the fact that the solution behavior near the edges is often not needed from a practical viewpoint, have driven people to make efforts to formulate the interior solution, by assigning an appropriate portion of the prescribed edge-data to it, without any reference to the boundary layer solution.

By an application of the Betti–Rayleigh reciprocal theorem, Gregory and Wan developed a decay analysis technique determining the interior solution successfully and effectively and provided the results for several plate problems, and derived a set of correct boundary conditions for arbitrarily prescribed admissible edge-data [5–8]. From these results, they now have explicit examples showing that the higher order accuracy offered by the governing differential equations of a higher order

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plate theory may not be attained unless commensurate boundary conditions are developed and used for these equations. These general results also show that, to be strictly correct, Saint-Venant's principle should be applied only to the leading term outer solution, i.e. the classical plate theory.

Recently, relevant boundary conditions for elastic beams and piezoelectric beams have been tried [9,10]. Moreover, Gao et al. [11] extended the model and method suggested by Gregory and Wan [6] for elastic plates to piezoelectric plates, a set of necessary conditions on the mixed data for the existence of a rapidly decaying solution of the bending and stretching problems were obtained. Through generalizing the method developed by Gregory and Wan [6] and by invoking the general solution of equilibrium equations for hexagonal quasicrystals, these necessary conditions of are translated into the desired set of boundary conditions for the interior expansion. Moreover, an analytical solution of the decaying state has been formulated to verify the validity of these boundary conditions for plate bending in hexagonal quasicrystals [12,13].

It is the purpose of this Note to extend our previous work. We introduce two definitions for two equilibrium states, and the necessary conditions for a decaying state will be obtained in the next section. By accurate solutions for the auxiliary regular state, these necessary conditions for the edge-data to induce only a decaying elastostatic state will be translated into appropriate boundary conditions for the circular cylinder with axisymmetric deformations in Section 3. Our results extend the known results to a circular cylinder in axisymmetric deformation problems, which enable us to formulate the correct edge conditions for circular cylinder in axisymmetric deformation theories with stress and mixed edge-data. For the stress data, our boundary condition on edge-data for a decaying state is consistent with conventional boundary conditions of axially loaded theories. The appropriate boundary condition with mixed edge-data is obtained for the first time.

2. Decaying state in circular cylinder in axisymmetric deformations problems

The axisymmetric deformation is fundamental deformation of engineering. A circular cylinder in a fixed cylindrical coordinate system occupies the region

$$\Omega = \left\{(r,\theta,z) \,\middle|\, z \in D,\ |r| \leqslant a\right\} \tag{1}$$

where D is the axes of the circular cylinder which has diameter a, and z is slewing axis.

The displacement filed of axisymmetric deformation is

$$\mathbf{u} = \mathbf{r}^0 u_r(r,z) + \mathbf{z}^0 u_z(r,z) \tag{2}$$

where $\mathbf{r}^0$ and $\mathbf{z}^0$ are the unit vectors of radial direction and axis direction, respectively, and

$$\begin{cases} \left(\nabla^2 - \dfrac{1}{r^2}\right)u_r + \dfrac{1}{1-2\nu}\dfrac{\partial}{\partial r}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{u_r}{r} + \dfrac{\partial u_z}{\partial z}\right) = 0 \\ \nabla^2 u_z + \dfrac{1}{1-2\nu}\dfrac{\partial}{\partial z}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{u_r}{r} + \dfrac{\partial u_z}{\partial z}\right) = 0 \end{cases} \tag{3}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial z^2} \tag{4}$$

where, μ is shear modules, and ν is Poisson's ratio.

By taking advantage of Boussinesq solution [14] for the axisymmetric deformation problem, the expressions of displacements and stresses can be obtained as

$$\begin{aligned} &u_r(r,z) = -\frac{1}{2}\frac{\partial}{\partial r}(B_0 + zB), \qquad u_z(r,z) = 2(1-\nu)B - \frac{1}{2}\frac{\partial}{\partial z}(B_0 + zB) \\ &\sigma_{rr}(r,z) = \mu\left(2\nu\frac{\partial B}{\partial z} - \frac{\partial^2 B_0}{\partial r^2} - z\frac{\partial^2 B}{\partial r^2}\right), \qquad \sigma_{zz}(r,z) = \mu\left[2(1-\nu)\frac{\partial B}{\partial z} - \frac{\partial^2 B_0}{\partial z^2} - z\frac{\partial^2 B}{\partial z^2}\right] \\ &\sigma_{zr}(r,z) = \mu\left[(1-2\nu)\frac{\partial B}{\partial r} - \frac{\partial^2 B_0}{\partial r\partial z} - z\frac{\partial^2 B}{\partial r\partial z}\right], \qquad \sigma_{\theta\theta}(r,z) = \mu\left(2\nu\frac{\partial B}{\partial z} - \frac{\partial B_0}{r\partial r} - z\frac{\partial B}{r\partial r}\right) \end{aligned} \tag{5}$$

where B and B_0 are harmonic functions in cylindrical coordinate system.

In order to satisfy the homogeneous boundary conditions on the circular cylinder surface, we set

$$\sigma_{rr}(a,z) = 0, \qquad \sigma_{rz}(a,z) = 0 \tag{6}$$

The presence of any body or surface loads may be removed by a particular solution. Then the only forcing terms in the problem are prescribed on the end $z = 0$ in terms of stress or displacement edge-data in the form of one of the following four admissible combinations, one of the following sets of edge-data is prescribed:

$$\text{Case (A):} \quad \sigma_{zr}(r,0)=\bar{\sigma}_{zr}(r), \qquad \sigma_{zz}(r,0)=\bar{\sigma}_{zz}(r) \tag{7}$$

$$\text{Case (B):} \quad u_r(r,0)=\bar{u}_r(r), \qquad \sigma_{zz}(r,0)=\bar{\sigma}_{zz}(r) \tag{8}$$

$$\text{Case (C):} \quad \sigma_{zr}(r,0)=\bar{\sigma}_{zr}(r), \qquad \bar{u}_z(r,0)=\bar{u}_z(r) \tag{9}$$

$$\text{Case (D):} \quad u_r(r,0)=\bar{u}_r(r), \qquad \bar{u}_z(r,0)=\bar{u}_z(r) \tag{10}$$

Two classes of exact states are investigated for the equations of axisymmetric circular cylinder with free faces. An elastostatic state in the axisymmetric circular cylinder is said to be a **decaying state**:

$$[u,\sigma]=O\left(M_1 e^{\frac{-\gamma l}{a}}\right) \tag{11}$$

or a **regular state**:

$$[u,\sigma]=O\left(M_2 a^{\alpha}\right) \tag{12}$$

as $a\to 0$, where u and σ are the displacement fields and stress fields, l is the minimum distance of the observation point from the edge of the circular cylinder, M_1 is the maximum modulus of the prescribed edge-data for the decaying state, M_2 is the maximum modulus for the regular state, and γ and α are positive constants.

Supposing that the stress data does give rise to the decaying state in the circular cylinder, we now apply the reciprocal theorem for a circular cylinder, which takes the form

$$\oint\!\!\!\int_S \left(\sigma_{ij}^{(1)}u_j^{(2)}-\sigma_{ij}^{(2)}u_j^{(1)}\right)n_i\,\mathrm{d}S=0 \tag{13}$$

where S is the surface of the circular cylinder, n_i is the direction cosine of the outward normal to S. With the foregoing two definitions in mind, now we take the state with a superscript "(1)" to be the exact solution of axisymmetric problem of circular cylinder, and the decaying state induced by the prescribed edge-data, $\bar{\sigma}_{zr}$, $\bar{\sigma}_{zz}$, $\bar{u}_r$, $\bar{u}_z$. For the auxiliary state, denoted by a superscript "(2)", we take any regular state which fulfills load-free conditions on S. Similar to the derivation of necessary conditions for a decaying state in the plate, generalizing Gregory and Wan's decay analysis technique to an axisymmetric circular cylinder, we finally obtain the necessary conditions for a decaying state.

$$\text{Case (A):} \quad \int_0^a r\left[\bar{\sigma}_{zr}(r)u_r^{(2)}+\bar{\sigma}_{zz}(r)u_z^{(2)}\right]_{z=0}\mathrm{d}r=0 \tag{14}$$

$$\text{Case (B):} \quad \int_0^a r\left[\bar{u}_r(r)\sigma_{zr}^{(2)}-\bar{\sigma}_{zz}(r)u_z^{(2)}\right]_{z=0}\mathrm{d}r=0 \tag{15}$$

$$\text{Case (C):} \quad \int_0^a r\left[\bar{u}_z(r)\sigma_{zz}^{(2)}-\bar{\sigma}_{zr}(r)u_r^{(2)}\right]_{z=0}\mathrm{d}r=0 \tag{16}$$

$$\text{Case (D):} \quad \int_0^a r\left[\bar{u}_r(r)\sigma_{zr}^{(2)}+\bar{u}_z(r)\sigma_{zz}^{(2)}\right]_{z=0}\mathrm{d}r=0 \tag{17}$$

These necessary conditions for Cases (A), (B) and (C) to induce only a decaying elastostatic state will be translated into appropriate boundary conditions for axisymmetric circular cylinder in the next section.

3. Boundary conditions for the axisymmetric circular cylinder

The main difficulty in performing the preceding process lies in obtaining suitable regular states which satisfy the appropriate boundary conditions. However, for the case of axisymmetric circular cylinder, the necessary regular states can be explicitly determined as follows, at least for edge-data in Cases (A), (B) and (C).

3.1. Cases (A) and (B)

Now we look for the auxiliary regular state with the use of Boussinesq solution for axisymmetric problem. According to the characteristics of axisymmetric problem, we assume

$$B=0, \qquad B_0=-2Cz \tag{18}$$

which corresponds to a rigid body translation in the z-direction and it is easy to prove that the rigid body translation belongs to regular state. We can take it as the auxiliary regular state, so that Eqs. (14) and (15) give the condition for the stress data

$$\int_0^a r\bar{\sigma}_{zz}\,\mathrm{d}r = 0 \tag{19}$$

3.2. Case (C)

Now we look for another auxiliary regular state with the use of Boussinesq solution for axisymmetric deformation problem, we take the harmonic functions as

$$B_0 = D_1\left(2z^2 - r^2\right), \qquad B = D_2 z \tag{20}$$

where D_1 and D_2 are unknown constants yet to be determined. By noting that

$$\sigma_{rr}^{(2)} = \sigma_{zr}^{(2)} = 0 \quad (r = a), \qquad \sigma_{zr}^{(2)} = 0, \qquad u_z^{(2)} = 0 \quad (z = 0) \tag{21}$$

we have the relationship among these unknown constants

$$D_1 = -\nu D_2 \tag{22}$$

With the help of Eqs. (20) and (22), the necessary condition for a decaying state is obtained from Eq. (16) when $\bar{\sigma}_{zr}$ and $\bar{u}_z$ are prescribed

$$\int_0^a r\left[\bar{u}_z + \frac{\nu r\bar{\sigma}_{zr}}{2\mu(1+\nu)}\right]\mathrm{d}r = 0 \tag{23}$$

4. Discussion and conclusions

These aforementioned necessary conditions for a decaying state (boundary layer solution) can then be converted into the boundary conditions appropriate for the interior solution or its various approximate elastic theories, which do not involve the boundary layer solution components. The difference between the exact solution and the interior one is a decaying state. Then the above necessary conditions applied to the edge-data at $z = 0$ of a circular cylinder

$$\begin{aligned} &\bar{u}_r = \left[u_r - u_r^I\right]_{z=0}, \qquad \bar{u}_z = \left[u_z - u_z^I\right]_{z=0} \\ &\bar{\sigma}_{zr} = \left[\sigma_{zr} - \sigma_{zr}^I\right]_{z=0}, \qquad \bar{\sigma}_{zz} = \left[\sigma_{zz} - \sigma_{zz}^I\right]_{z=0} \end{aligned} \tag{24}$$

where u_r^I, u_z^I, σ_{zr}^I, σ_{zz}^I are interior solutions, give

$$\int_0^a r\left[\sigma_{zz}^I\right]_{z=0}\mathrm{d}r = \int_0^a r\hat{\sigma}_{zz}\,\mathrm{d}r \tag{25}$$

$$\int_0^a r\left[u_z^I + \frac{\nu r\sigma_{zr}^I}{2\mu(1+\nu)}\right]_{z=0}\mathrm{d}r = \int_0^a r\left[\hat{u}_z + \frac{\nu r\hat{\sigma}_{zr}}{2\mu(1+\nu)}\right]\mathrm{d}r \tag{26}$$

where $\hat{u}_z$, $\hat{\sigma}_{zr}$ and $\hat{\sigma}_{zz}$ are the actual prescribed edge-data. By superposition, the boundary conditions for axisymmetric deformation of the circular cylinder are formed. Thus a portion of the edge-data is effectively allocated to the interior solution, which is analogous to the assignment of edge-data in the form of resultant force by Saint-Venant's principle. Similar to those pointed out by Gregory and Wan [2], the results reveal that indiscriminate extension of Saint-Venant's principle is not justified in general, which may lead to an erroneous solution for the circular cylinder's deformation even away from the circular cylinder edge.

The results of the present Note extend the known results to circular cylinder's deformation problems, which enable us to establish two correct boundary conditions with stress and mixed edge-data. However, attempts to derive similar results on boundary conditions for pure displacement edge-data case have not been successful.

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