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A transformation approach for the derivation of boundary conditions between a curved porous medium and a free fluid

Sur une approche de transformation pour le calcul des conditions à l'interface entre un milieu poreux courbé et un fluide libre

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A B S T R A C T

The behaviour of a free fluid flow above a porous medium, both separated by a curved interface, is investigated. By carrying out a coordinate transformation, we obtain the description of the flow in a domain with a straight interface. Using periodic homogenisation, the effective behaviour of the transformed partial differential equations in the porous part is given by a Darcy law with non-constant permeability matrix. Then the fluid behaviour at the porous-liquid interface is obtained with the help of generalised boundary-layer functions: Whereas the velocity in normal direction is continuous across the interface, a jump appears in tangential direction. Its magnitude seems to be related to the slope of the interface. Therefore the results indicate a generalised law of Beavers and Joseph.

R É S U M É

On considère le comportement d'un fluid libre au-dessus d'un milieu poreux avec une interface courbée. Utilisant une transformation des coordonnées, on obtient la description de l'écoulement dans un domaine avec une frontière plane. En limite à deux échelles, le comportement du fluid en milieu poreux est donné par une loi de Darcy avec une matrice de perméabilité non-constante. Ensuite, on obtient le comportement du fluid à l'interface: La vitesse est continue à travers l'interface dans le sens normal, mais une discontinuité apparaît en sens tangentiel. Par conséquent, les résultats indiquent une loi généralisée de Beavers et Joseph.

1. Introduction

The interface conditions coupling a free fluid flow and a flow inside a porous medium are of great interest in various fields, e.g. mathematical modelling or soil chemistry. For incompressible fluids, the continuity of the normal velocity has to be assured. However, other conditions are not clearly available. Beavers ans Joseph investigated the flow behaviour experimentally in [1] and derived a jump boundary condition in tangential direction. A first mathematical verification of this condition was available with the work of Saffman [2], who used a statistical approach. Starting in 1996, Willi Jäger and

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Andro Mikelić used the theory of homogenisation to obtain a rigorous proof. In the preparatory paper [3], they developed a mathematical boundary layer together with several corrector terms, which allowed them to justify a jump boundary condition. These constructions were used in [4] to give a mathematical proof of the Saffman modification of the boundary condition of Beavers and Joseph, yielding the condition

$$\sqrt{k^\varepsilon}\,(\nabla v_F \nu)\cdot\tau = \alpha v_F\cdot\tau + \mathcal{O}\big(k^\varepsilon\big) \tag{1}$$

where v_F denotes the velocity of the free fluid at the interface; $k^\varepsilon = k\varepsilon^2$ is the (scalar) permeability of the porous medium (where ε denotes its characteristic length); and ν and τ are the unit normal and unit tangential vector, resp. The slip-coefficient α can be calculated explicitly.

However, all the results above suffer from one drawback: Only a planar interface in the form of a line or a plane is considered. Therefore, the effect of a possible curvature of the interface is not known.

Generalizations of the boundary layers in [3] were developed by Maria Neuss-Radu in [5]. However, applications only treat reaction–diffusion systems without flow, and explicit results can only be obtained in the case of a layered medium, see [6]. In this note we propose a different approach to deal with the case of a curved interface: The main idea is to transform a reference geometry with a straight interface to a domain with a curved interface. It is assumed that the porous part in the reference geometry consists of a periodic array of a scaled reference cell and that the flow in the transformed geometry is governed by the stationary Stokes equation. Therefore one obtains a set of transformed differential equations in the reference configuration. Boundary layer functions for these equations are constructed such that – due to the straight interface – their exponential decay can be assured. The complete analysis can be found in [7] and will be presented in forthcoming publications.

2. Overview of the involved geometries

Let $L > 0$ be given. We consider a fluid flowing in the semi-infinite strip $[0, L]\times\mathbb{R}$, being divided into two parts $\Omega_1 = [0, L]\times\mathbb{R}_{>0}$, corresponding to the free fluid domain, and $\Omega_2 = [0, L]\times\mathbb{R}_{<0}$, corresponding to the porous medium. Both parts are separated by the interface $\Sigma = [0, L]\times\{0\}$.

We assume an ε-periodic geometry in Ω_2: Define a reference cell as $Y = [0, 1]^2$, containing a connected open set Y_S (corresponding to the solid part of the cell). Its boundary ∂Y_S is assumed to be of class $\mathcal{C}^\infty$ with $\partial Y_S \cap \partial Y = \emptyset$. Let $Y^* = Y\backslash\overline{Y}_S$ be the fluid part of the reference cell.

For given $\varepsilon > 0$ such that $\frac{L}{\varepsilon}\in\mathbb{N}$, let χ be the characteristic function of Y^*, extended by periodicity to the whole $\mathbb{R}^2$. Set $\chi^\varepsilon(x) := \chi(\frac{x}{\varepsilon})$ and define the fluid part of the porous medium as $\Omega_2^\varepsilon = \{x\in\Omega_2 \mid \chi^\varepsilon(x) = 1\}$. The fluid domain is then given by $\Omega^\varepsilon = \Omega_1\cup\Sigma\cup\Omega_2^\varepsilon$.

In order to be able to obtain the effective fluid behaviour near Σ, we have to define a number of so-called boundary layer problems. Therefore we introduce the following setting: We consider the domain $[0, 1]\times\mathbb{R}$, subdivided as follows: $Z^+ = [0, 1]\times(0,\infty)$ corresponds to the free fluid region, whereas the union of translated reference cells $Z^- = \bigcup_{k=1}^\infty\{Y^* - \binom{0}{k}\}\backslash S$ is considered to be the void space in the porous part. Here $S = [0, 1]\times\{0\}$ denotes the interface between Z^+ and Z^-. Finally, let $Z = Z^+\cup Z^-$ and $Z_{\mathrm{BL}} = Z^+\cup S\cup Z^-$ be the fluid domain without and with interface.

3. Coordinate transformation of the Stokes flow

3.1. General formulas

We recall some facts about coordinate transformations: Let $\tilde\Omega\subset\mathbb{R}^n$ with $n\in\mathbb{N}$ be a Lipschitz domain; let $\tilde c : \tilde\Omega\to\mathbb{R}$ be a scalar function, $\tilde j : \tilde\Omega\to\mathbb{R}^n$ a vector field and $\tilde M : \tilde\Omega\to\mathbb{R}^{n\times n}$ a matrix function. They are assumed to be sufficiently smooth.

Definition 3.1. The gradient of a vector field is defined as $(\nabla\tilde j)_{ik} = \frac{\partial\tilde j_k}{\partial x_i}$, with $i, k = 1,\dots,n$ (i.e. $\nabla\tilde j$ is the *transpose* of the Jacobian matrix of $\tilde j$); the divergence of a matrix-valued function is defined column-wise, thus $(\mathrm{div}(\tilde M))_k = \sum_{i=1}^n\frac{\partial\tilde M_{ik}}{\partial x_i}$, $k = 1,\dots,n$; and the Laplacian of a vector field is given by $\Delta\tilde j = \mathrm{div}(\nabla\tilde j)$.

Let $\Omega\subset\mathbb{R}^n$ be another Lipschitz domain and let $\psi : \Omega\to\tilde\Omega$ be a regular orientation-preserving $\mathcal{C}^k$-coordinate transformation. We will indicate coordinates in Ω by $z = (z_1,\dots,z_n)$ and those in $\tilde\Omega$ by $x = (x_1,\dots,x_n)$. Define $c(z) := \tilde c(\psi(z))$, $j(z) := \tilde j(\psi(z))$, $M(z) := \tilde M(\psi(z))$. Then

Lemma 3.2. *Let ψ be a volume-preserving $\mathcal{C}^1$-coordinate transformation and denote by F the Jacobian matrix of ψ. The operators from Definition* 3.1 *transform according to*

(i) $\nabla_x\tilde c = F^{-T}\nabla_z c$,
(ii) $\Delta_x(\tilde c) = \mathrm{div}_z(F^{-1}F^{-T}\nabla_z c)$ *and* $\Delta_x(\tilde j) = \mathrm{div}_z(F^{-1}F^{-T}\nabla_z j)$,
(iii) $\mathrm{div}_x(\tilde j) = \mathrm{div}_z(F^{-1}j)$ *and* $\mathrm{div}_x(\tilde M) = \mathrm{div}_z(F^{-1}M)$.

Remark 1. Let $\nu(x)$ be the unit normal vector at $x \in \partial\Omega$. Then the corresponding transformed normal vector is given by $\tilde{\nu}(x) = \|F^{-T}(x)\nu(x)\|^{-1} F^{-T}(x)\nu(x)$. If $n = 2$, then for the unit tangential vectors it holds $\tilde{\tau}(x) = \|F(x)\tau(x)\|^{-1} F(x)\tau(x)$. ($\|\cdot\|$ indicates the chosen norm in $\mathbb{R}^n$.)

3.2. Application to the Stokes equation

Let $L > 0$ and define $\tilde{\Omega} = [0, L] \times \mathbb{R}$. Let $g \in \mathcal{C}^\infty(\mathbb{R})$ be a given function such that $g(x + L) = g(x)$ for all $x \in \mathbb{R}$. We consider the graph $\{(x_1, g(x_1)) \mid x_1 \in [0, L]\} \subset \mathbb{R}^2$ to describe an interface $\tilde{\Sigma}$ in $\tilde{\Omega}$, dividing it into two parts, $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$:

$$\tilde{\Omega}_1 := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [0, L], x_2 > g(x_1)\}$$
$$\tilde{\Omega}_2 := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [0, L], x_2 < g(x_1)\}$$
$$\tilde{\Sigma} := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [0, L], x_2 = g(x_1)\}$$

In $\tilde{\Omega}_2$ let there be a closed set $\tilde{S} \subset \tilde{\Omega}_2$ (corresponding to a solid part in the domain) with $\partial\tilde{S} \cap \partial\tilde{\Omega} = \emptyset$.

We want to transform the domain $\Omega = [0, L] \times \mathbb{R}$ with a straight boundary $\Sigma = [0, L] \times \{0\}$, with parts $\Omega_1 = [0, L] \times \mathbb{R}_{>0}$ and $\Omega_2 = [0, L] \times \mathbb{R}_{<0}$ to the above situation. We will indicate coordinates in Ω by $z = (z_1, z_2)$ and those in $\tilde{\Omega}$ by $x = (x_1, x_2)$. Therefore define the transformation

$$\psi : \Omega \longrightarrow \tilde{\Omega}$$
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 + g(z_1) \end{pmatrix}$$

Then the Jacobian matrix F of ψ is given by

$$F(z) = \begin{bmatrix} 1 & 0 \\ g'(z_1) & 1 \end{bmatrix}$$

Since $\det F = 1$, ψ is a volume preserving $\mathcal{C}^\infty$-coordinate transformation.

Let $(\tilde{u}, \tilde{p})$ be the solution of the Stokes equation in $\tilde{\Omega}_F$ with no slip condition on $\partial\tilde{S}$ and periodic boundary conditions on $(\{0\} \cup \{L\}) \times \mathbb{R}$. By using the transformation formulas from Lemma 3.2, we obtain for $u(z) = \tilde{u}(\psi(z))$, $p(z) = \tilde{p}(\psi(z))$ and $f(z) = \tilde{f}(\psi(z))$:

$$-\mathrm{div}_z\big(F^{-1}(z)F^{-T}(z)\nabla_z u(z)\big) + F^{-T}(z)\nabla_z p(z) = f(z) \quad \text{in } \Omega_F \tag{2a}$$
$$\mathrm{div}_z\big(F^{-1}(z)u(z)\big) = 0 \quad \text{in } \Omega_F \tag{2b}$$
$$u(z) = 0 \quad \text{on } \partial S \tag{2c}$$
$$u, p \text{ are } L\text{-periodic in } z_1 \tag{2d}$$

with $S = \psi^{-1}(\tilde{S})$ and $\Omega_F = \Omega \backslash S$.

4. Homogenisation of the transformed Stokes equation

In this section we carry out the homogenisation procedure for the transformed Stokes equation (2). As we do not need results on an unbounded strip, but rather are interested in the effective equation for the velocity and pressure, we change the geometrical setting *in this chapter*: Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. We denote by $Y = [0, 1]^2$ the reference cell. Let Y_S be a strictly included subset of Y (the solid part) and set $Y^* := Y \backslash \overline{Y}_S$ (the fluid part). Let $\Gamma = \partial Y_S$ be the boundary of the solid part. It is assumed to belong to the class $\mathcal{C}^\infty$.

Define for $M \subset Y$ and $k \in \mathbb{Z}^2$ the shifted subset $M^k = M + \sum_{j=1}^{2} k_j e_j$, with e_j denoting the j-th unit vector. For given $\varepsilon > 0$ we define the following ε-periodic domains:

$$\Omega^\varepsilon = \bigcup_{k \in \mathbb{Z}^n} \big(\varepsilon(Y^*)^k \cap \Omega\big) \quad \text{the fluid part}$$
$$\Omega_S^\varepsilon = \bigcup_{k \in \mathbb{Z}^n} \big(\varepsilon Y_S^k \cap \Omega\big) \quad \text{the solid part}$$
$$\Gamma^\varepsilon = \bigcup_{k \in \mathbb{Z}^n} \big(\varepsilon \Gamma^k \cap \Omega\big) \quad \text{the boundary of the solid part}$$

In order to avoid technical difficulties, we assume that $\Gamma^\varepsilon \cap \partial\Omega = \emptyset$ for all ε. Changing the name of the variables from z back to x in Eq. (2), we consider the problem

$$
\begin{aligned}
-\operatorname{div}\big(F^{-1}(x)F^{-T}(x)\nabla u^\varepsilon(x)\big)+F^{-T}(x)\nabla p^\varepsilon(x)&=f(x) && \text{in } \Omega^\varepsilon\\
\operatorname{div}\big(F^{-1}(x)u^\varepsilon(x)\big)&=0 && \text{in } \Omega^\varepsilon\\
u^\varepsilon(x)&=0 && \text{on } \Gamma^\varepsilon\\
u^\varepsilon(x)&=0 && \text{on } \partial\Omega
\end{aligned}
$$

with a given volume force $f\in L^2(\Omega)^2$. Existence- and uniqueness-results follow by an adaption of the functional analytic treatment of the Stokes equation. Extensions of u^ε and p^ε to the whole of Ω can also be defined, which lead to the following estimates:

Lemma 4.1. *There exists a constant C, such that for the extended velocity u^ε and pressure p^ε it holds $\|u^\varepsilon\|_{L^2(\Omega)^2}+\varepsilon\|\nabla u^\varepsilon\|_{L^2(\Omega)^4}\leqslant C$ and $\|p^\varepsilon\|_{L^2(\Omega)/\mathbb{R}}\leqslant C$, independent of ε.*

Using the theory of two-scale convergence, there exist a $u_0\in L^2(\Omega,H^1_\#(Y))$ and a $p\in L^2(\Omega\times Y)$ such that along a subsequence u^ε two-scale converges against u_0, $\varepsilon\nabla u^\varepsilon$ two-scale converges against $\nabla_y u_0$ and p^ε two-scale converges against p. These limits are characterised in the following theorem:

Theorem 4.2. *The two-scale limits u_0, p_0 are solutions of the homogenised problem*

$$
\begin{aligned}
&F^{-T}(x)\nabla_y p_1(x,y)+F^{-T}(x)\nabla_x p(x)-\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\nabla_y u_0(x,y)\big)=f(x) && \textit{in } \Omega\times Y^*\\
&\operatorname{div}_y\big(F^{-1}(x)u_0(x,y)\big)=0 && \textit{in } \Omega\times Y^*\\
&\operatorname{div}_x\bigg(F^{-1}(x)\int_Y u_0(x,y)\,\mathrm{d}y\bigg)=0 && \textit{in } \Omega\\
&u_0(x,y)=0 && \textit{in } \Omega\times Y_S\\
&\bigg(\int_Y F^{-1}(x)u_0(x,y)\,\mathrm{d}y\bigg)\cdot\nu=0 && \textit{on } \partial\Omega\\
&u_0(x),\ p_1(x)\ \textit{are } Y\textit{-periodic in } y
\end{aligned}
$$

We define the following parameter-dependent cell problems: For fixed $x\in\Omega$ let $(w^i_x,\pi^i_x)\in\{\phi\in H^1(Y)^2;\ \phi$ is Y-periodic$\}\times\{\psi\in L^2(Y^*)/\mathbb{R};\ \psi$ is Y-periodic$\}$ be a solution of

$$
\begin{aligned}
-\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\nabla_y w^i_x(y)\big)+F^{-T}(x)\nabla_y\pi^i_x(y)&=e_i && \text{in } Y^*\\
\operatorname{div}_y\big(F^{-1}(x)w^i_x(y)\big)&=0 && \text{in } Y^*\\
w^i_x(y)&=0 && \text{in } Y_S\\
w^i_x,\ \pi^i_x\ &\text{are } Y\text{-periodic}
\end{aligned}
$$

Defining

$$
u_0(x,y)=\sum_{i=1}^2\big(f(x)-F^{-T}(x)\nabla p(x)\big)_i w^i_x(y)\quad\text{as well as}\quad p_1(x,y)=\sum_{i=1}^2\big(f(x)-F^{-T}(x)\nabla p(x)\big)_i\pi^i_x(y),
$$

one obtains a transformed Darcy's law with non-constant permeability matrix: Set $u(x):=\int_Y u_0(x,y)\,\mathrm{d}y$ and the matrix $A(x)$ by $(A(x))_{ij}=\int_Y w^i_{x,j}(y)\,\mathrm{d}y$, then $u^\varepsilon\rightharpoonup u$ in $L^2(\Omega)$ weakly with

$$
\begin{aligned}
u(x)&=A(x)\big(f(x)-F^{-T}\nabla p(x)\big) && \text{in } \Omega\\
\operatorname{div}\big(F^{-1}(x)u(x)\big)&=0 && \text{in } \Omega\\
u(x)\cdot F^{-T}(x)\nu(x)&=0 && \text{on } \partial\Omega
\end{aligned}
$$

Remark 2. The cell problem is a parameter-dependent partial differential equation. By using the implicit function theorem for Banach spaces, one can derive differentiability properties of w^i_x and π^i_x in x-direction. In the sequel, we will write $w^i(x,y):=w^i_x(y)$. Furthermore, one can show that the matrix $A(x)$ is symmetric positive definite.

5. Fluid behaviour at the interface

Let $l \in C_0^\infty(\Omega)$ be a given force, $l \neq 0$ on Σ. The fluid flow in the semi-infinite strip Ω^ε is assumed to be governed by the following equations: Find $u^\varepsilon \in \{\phi \in L^2_{\mathrm{loc}}(\Omega^\varepsilon);\ \nabla\phi \in L^2(\Omega^\varepsilon)^4, \phi \in L^2(\Omega_2^\varepsilon)^2, \operatorname{div}(F^{-1}\phi) = 0$ a.e. in $\Omega^\varepsilon, \phi = 0$ on $\partial\Omega_2^\varepsilon \backslash \partial\Omega, \phi$ is L-periodic in $x_1\}$ and $p^\varepsilon \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)/\mathbb{R}$ with

$$-\operatorname{div}\big(F^{-1}(x)F^{-T}(x)\nabla u^\varepsilon(x)\big) + F^{-T}(x)\nabla p^\varepsilon(x) = L^\varepsilon(x) \quad \text{in } \Omega^\varepsilon \tag{3a}$$

$$\operatorname{div}\big(F^{-1}(x)u^\varepsilon(x)\big) = 0 \quad \text{in } \Omega^\varepsilon \tag{3b}$$

$$u^\varepsilon(x) = 0 \quad \text{on } \partial\Omega^\varepsilon \backslash \partial\Omega \tag{3c}$$

$$u^\varepsilon, p^\varepsilon \text{ are } L\text{-periodic in } x_1 \tag{3d}$$

where

$$L^\varepsilon = \begin{cases} \varepsilon^2 l & \text{in } \Omega_1 \\ l & \text{in } \Omega_2^\varepsilon \end{cases}$$

In [3], the more general scaling $L^\varepsilon = \varepsilon^\gamma l$ in Ω_1 was considered. The case $\gamma \geqslant 2$ – which has been the guide to our derivations – is the most simplest one, while $\gamma < 2$ leads to even more auxiliary constructions. As we are interested in developing a generalisation of the results of Jäger and Mikelić, we chose to restrict our considerations to one particular scaling.

5.1. Some auxiliary problems

In this subsection we present the auxiliary problems which are needed in Theorem 5.1. However, note that in order to proof this theorem, more constructions of the same form are necessary. They can be found in [7], together with existence results and estimates.

Let the cell problem and the pressure p be defined as above. Let H denote the Heaviside function and set $D^i(x) = (l - F^{-T}\nabla p)_i(x)$ as well as $D^i_\delta(x) = D^i(x_1, -0)e^{-\delta x_2}$ for given $\delta > 0$.

A first approximation of the velocity and pressure in Ω_1 is given by $(u_0, \pi_0) \in \{\phi \in L^2_{\mathrm{loc}}(\Omega_1)^2;\ \nabla\phi \in L^2(\Omega_1)^4$, $\operatorname{div}(F^{-1}\phi) = 0$ a.e. in $\Omega_1,\ \phi = 0$ on Σ, ϕ is L-periodic in $x_1\} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$, solution of

$$\begin{aligned}
&-\operatorname{div}\big(F^{-1}(x)F^{-T}(x)\nabla u_0(x)\big) + F^{-T}(x)\nabla\pi_0(x) = l && \text{in } \Omega_1\\
&\operatorname{div}\big(F^{-1}(x)u_0(x)\big) = 0 && \text{in } \Omega_1\\
&u_0(x) = 0 && \text{on } \Sigma\\
&u_0, \pi_0 \text{ are } L\text{-periodic in } x_1
\end{aligned}$$

Next, for $i = 1, 2$ we need the following family of so-called boundary layer functions $(w^{i,\mathrm{bl}}(x), \pi^{i,\mathrm{bl}}(x)) \in \{\phi \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})^2;$ $\nabla\phi \in L^2(Z)^4, \phi \in L^2(Z^-)^2, \phi = 0$ on $\bigcup_{k=1}^\infty \{\partial Y_S - \binom{0}{k}\}$, ϕ is 1-periodic in $y_1\} \times L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})/\mathbb{R}$ defined by

$$\begin{aligned}
&-\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\nabla_y w^{i,\mathrm{bl}}(x,y)\big) + F^{-T}(x)\nabla_y\pi^{i,\mathrm{bl}}(x,y) = 0 && \text{in } \Omega \times Z\\
&\operatorname{div}_y\big(F^{-1}(x)w^{i,\mathrm{bl}}(x,y)\big) = 0 && \text{in } \Omega \times Z\\
&\big[w^{i,\mathrm{bl}}(x)\big]_S(y) = w^i(x,y) && \text{on } \Omega \times S\\
&\big[\big(F^{-1}(x)F^{-T}(x)\nabla_y w^{i,\mathrm{bl}}(x) - F^{-1}(x)\pi^{i,\mathrm{bl}}(x)\big)e_2\big]_S(y)\\
&\quad = \big(F^{-1}(x)F^{-T}(x)\nabla_y w^i(x) - F^{-1}(x)\pi^i(x)\big)e_2(y) && \text{on } \Omega \times S\\
&w^{i,\mathrm{bl}}(x,y) = 0 && \text{on } \Omega \times \textstyle\bigcup_{k=1}^\infty\big\{\partial Y_S - \binom{0}{k}\big\}\\
&w^{i,\mathrm{bl}}(x), \pi^{i,\mathrm{bl}}(x) \text{ are 1-periodic in } y_1
\end{aligned}$$

Here $[w]_S$ denotes the jump of w across S. There exist constants $\gamma_0 > 0$, $y^* > 0$, $C^{i,\mathrm{bl}}(x)$ and $C^i_\pi(x)$ such that

$$\begin{aligned}
&\big|w^{i,\mathrm{bl}}(x,y) - C^{i,\mathrm{bl}}(x)\big| \leqslant Ce^{-\gamma_0|y_2|}, \quad y_2 > y^*\\
&\big|\pi^{i,\mathrm{bl}}(x,y) - C^i_\pi(x)\big| \leqslant Ce^{-\gamma_0|y_2|}, \quad y_2 > y^*
\end{aligned}$$

We have to correct the influence of the boundary layer functions in Ω_1 by defining a family of so-called counterflows: For $i, j = 1, 2$, let $(u^{ik}, \pi^{ik}) \in \{\phi \in L^2_{\mathrm{loc}}(\Omega_1)^2;\ \nabla\phi \in L^2(\Omega_1)^4, \phi$ is L-periodic in x_1, $\operatorname{div}(F^{-1}\phi) = 0$ a.e. in $\Omega_1\} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$ be the solution of

$$
\begin{aligned}
&-\operatorname{div}\bigl(F^{-1}(x)F^{-T}(x)\nabla u^{ik}(x)\bigr)+F^{-T}(x)\nabla\pi^{ik}(x)=0 && \text{in } \Omega_1\\
&\operatorname{div}\bigl(F^{-1}(x)u^{ik}(x)\bigr)=0 && \text{in } \Omega_1\\
&u^{ik}(x_1,+0)=\bigl(C_k^{i,\mathrm{bl}}D_\delta^i\bigr)e_k(x_1,0) && \text{on } \Sigma\\
&u^{ik},\pi^{ik} \text{ are } L\text{-periodic in } x_1
\end{aligned}
$$

Note that the function u^{ik} corresponds to the construction $C_k^{i,\mathrm{bl}}u^{ik}$ in [3], and π^{ik} corresponds to $C_k^{i,\mathrm{bl}}\pi^{ik}$.
Moreover, we need the following assumptions:

Remark 3.

(i) Let $\rho^{i,\mathrm{bl}}(x,y)$ be a boundary layer function which stabilizes exponentially in y towards some constant $C_\rho^\pm(x)$ in $Z^\pm$. We assume that $\rho^{i,\mathrm{bl}}$ and $C_\rho^\pm$ are differentiable in x, that also $\nabla_x\rho^{i,\mathrm{bl}}$ decays exponentially in y and that the corresponding stabilizing constant is the matching derivative of $C_\rho^\pm$. Thus especially
– $\nabla_x\rho^{i,\mathrm{bl}}(x,y)$ stabilizes exponentially in y towards $\nabla_x C_\rho^\pm(x)$;
– $\operatorname{div}_x(\nabla_x\rho^{i,\mathrm{bl}}(x,y))$ stabilizes exponentially in y towards $\operatorname{div}_x(\nabla C_\rho^\pm(x))$.

(ii) We assume that all the stabilizations are uniform in x, i.e. there exist constants $C,\gamma_0>0$ independent of x such that

$$
\begin{aligned}
&\bigl|\rho^{i,\mathrm{bl}}(x,y)-C_\rho^\pm(x)\bigr|\leqslant Ce^{-\gamma_0|y_2|}\\
&\bigl|\nabla_x\rho^{i,\mathrm{bl}}(x,y)-\nabla_x C_\rho^\pm(x)\bigr|\leqslant Ce^{-\gamma_0|y_2|}
\end{aligned}
$$

etc., in $Z^\pm$.

(iii) Finally, we assume that for the function Ψ below it holds $\int_\Sigma F^{-1}F^{-T}\nabla\Psi\cdot e_2\,\mathrm{d}\sigma_x=0$. A sufficient condition for this assumption to hold is that the function g defining the boundary (cf. Section 3.2) is point-symmetric with respect to the point $\frac{L}{2}$ on $[0,L]$. Here $\Psi\in\{z\in L^2_{\mathrm{loc}}(\Omega_1);\ \nabla z\in L^2(\Omega_1)\}$ is a solution of

$$
\begin{aligned}
&\operatorname{div}\bigl(F^{-1}F^{-T}\nabla\Psi\bigr)=\frac{1}{L(1+x_2)^2} && \text{in } \Omega_1\\
&\Psi=0 && \text{on } \Sigma\\
&\Psi \text{ is } L\text{-periodic in } x_1
\end{aligned}
$$

5.2. Main results

Theorem 5.1. *Let u^ε and p^ε be given by the system* (3)*. Under the assumptions given in Remark* 3*, it holds*

$$
\frac{u^\varepsilon}{\varepsilon^2}\rightharpoonup H(x_2)\Bigl[u_0+\sum_{i,k}u^{ik}\Bigr]+H(-x_2)A\bigl(l-F^{-T}\nabla p\bigr)\quad \textit{weakly in } L^2(K)^2
$$

and

$$
p^\varepsilon\rightharpoonup H(-x_2)p\quad \textit{weakly in } L^2(K)
$$

for all $K\subset\Omega$ such that K is precompact.

Thus the velocity u_F of the free fluid in Ω_1 is given by

$$
u_F=u_0+\sum_{i,k=1}^{2}u^{ik}
$$

whereas for the filtration velocity u_D in the porous medium Ω_2 it holds

$$
u_D=A\bigl(l-F^{-T}\nabla p\bigr)
$$

Now the question arises which conditions hold at the interface Σ, coupling u_F and u_D. We have the following result:

Corollary 5.2. *At the interface Σ it holds*

$$
u_F(x)\cdot F^{-T}(x)e_2=u_D(x)\cdot F^{-T}(x)e_2 \tag{4}
$$

as well as

$$\big(u_F(x) - u_D(x)\big) \cdot F(x)e_1 = \left[\sum_i \big(C^{i,\mathrm{bl}}(x) - A_i(x)\big) D^i(x) \right] \cdot F(x)e_1 \tag{5}$$

where A_i denotes the i-th column of the permeability matrix A. Moreover, the constant $C^{i,\mathrm{bl}}(x) = \begin{pmatrix} C_1^{i,\mathrm{bl}}(x) \\ C_2^{i,\mathrm{bl}}(x) \end{pmatrix}$ is the solution of the following system of equations:

$$\begin{bmatrix} 1 & g'(x_1) \\ -g'(x_1) & 1 \end{bmatrix} \begin{pmatrix} C_1^{i,\mathrm{bl}}(x) \\ C_2^{i,\mathrm{bl}}(x) \end{pmatrix} = \begin{pmatrix} \int_0^1 w^{i,\mathrm{bl}}(x, y_1, +0) \cdot F(x)e_1 \, \mathrm{d}y_1 \\ \int_0^1 w^i(x, y_1, 0) \cdot F^{-T}(x)e_2 \, \mathrm{d}y_1 \end{pmatrix}$$

For the pressure we have

$$p(x_1, -0) = 0 \quad \text{on } \Sigma \tag{6}$$

The first equation of the preceding corollary is a consequence of the conservation of mass of the incompressible fluid: In the continuity condition (4), the vector $F^{-T}(x)e_2$ corresponds to the direction of the transformed normal vector of Σ. Thus, after transformation, the velocity is continuous across the curved interface $\tilde{\Sigma}$ in normal direction. Furthermore, condition (5) indicates a jump across Σ in the direction of the transformed tangential vector. The magnitude of the jump is given by the value of the involved constants, which can be calculated explicitly by solving the associated auxiliary problems. Since the auxiliary functions depend on the function g describing the interface, the local geometry of the porous surface influences this jump condition.

For the choice $g \equiv 0$, the matrix F is given by the identity matrix, and the auxiliary problems and constants match those described in [3]. Thus – in the special case of a planar interface – we recover the interface conditions obtained by Jäger and Mikelić in the paper cited above.

We can compare our results with those obtained by Beavers, Joseph and Saffman in [1] and [2]. They proposed the continuity of the normal velocity together with a condition of the form (1). The jump condition in Corollary 5.2 does not have this specific form; however, the results do indicate the validity of a Beavers-and-Joseph-type law for curved interfaces. In addition, the geometry of the interface seems to have an effect on the magnitude of the jump. In order to obtain more precise conditions, one could try to use our constructions for a generalisation of the results obtained in [4]. In the papers [8] and [9], Levi, Ene and Sanchez-Palencia also considered boundary effects. They distinguished the case when the pressure gradient on the side of the porous medium is normal to the porous surface or not. In the latter case, using a specific boundary layer approach, it was found that the pressure is continuous across the interface. In the former case, they deduced that the velocity is continuous in normal direction and the pressure is constant on the interface; a condition which resembles (4) and (6).

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