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Parallelization of the algorithm of asymptotic partial domain decomposition in thin tube structures

Parallélisation de l'algorithme de décomposition asymptotique de domaine pour des cylindres minces

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A B S T R A C T

The method of asymptotic partial domain decomposition for thin tube structures (finite unions of thin cylinders) is revisited. Its application to the Newtonian and non-Newtonian flows in great systems of vessels is considered. The possibility of a parallelization of its algorithm is discussed for linear and non-linear models.

R É S U M É

La méthode de décomposition asymptotique de domaine pour des structures minces (une réunion des cylindres minces) est revisitée. Son application aux écoulements newtoniens et non newtoniens est considérée. La possibilité d'une parallélisation de son algorithme est discuté pour des modèles linéaires ainsi que non linéaires.

1. Introduction. Tube structures. Setting the problem. Description of the asymptotic behavior of solutions for the Stokes and Navier–Stokes equations

A computational strategy of partial asymptotic decomposition for thin domains with complex geometry (tube structures or finite rod structures) is discussed. These structures have been studied in some previous papers (see for example, [1–4]) and they can be considered as some idealized geometrical model of the blood circulation system. The method of asymptotic partial domain decomposition (MAPDD) (see [1]), reduces considerably the volume of computations with respect to the direct numerical simulations. The MAPDD generates a multi-scale model of hybrid dimension with one-dimensional

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description of the flow at some small distance from the ends of the cylinders and with a three- (or two-) dimensional description ("zooms") at some small domains around the bifurcations of vessels. It was formulated and justified for the Stokes and Navier–Stokes equations with homogeneous no-slipping boundary condition of vanishing velocity. However, this method was not studied in the case of the Navier–Stokes or Stokes equations with non-homogeneous boundary conditions, which are more natural for the fluid motion with some inflows and outflows on some parts of the boundary. In the present paper we formulate the version of the MAPDD in this case and give a theorem (Theorem 2.1) which justifies an error estimate for this method. Another important question is discussed in Section 3: it is the possibility of the parallelization of the algorithm of MAPDD. This parallelization plays an important role in practical multi-processor implementation of the MAPDD for great systems of tubes or channels. In the fourth section the applicability of the MAPDD and its parallelization in case of non-Newtonian flows are discussed.

Let us remind the definition of a tube structure. Define first a tube structure containing one bundle. We consider here two possible dimensions of the space: two and three.

Let $e_1, \dots, e_n$ be n closed segments in $\mathbb{R}^s$ ($s = 2, 3$), which have a single common point O (i.e. the origin of the coordinate system), and let it be the common end point of all these segments. Changing variables (by rotations) we pass to the local coordinate system, associated to the segment e_j: the new axis x_1 is denoted $x_1^{e_j}$ and it contains the segment e_j, while the axes $x_2^{e_j}, \dots, x_s^{e_j}$ are orthogonal to the segment e_j. Define the graph

$$B = \bigcup_{j=1}^{n} e_j$$

Let $b_1, \dots, b_n$ be n bounded $(s-1)$-dimensional domains in $\mathbb{R}^{s-1}$, which contain the origin O'. Let β_j be the set of points, such that in the local coordinate system associated to the segment e_j it has a form $\beta_j = \{x \in \mathbb{R}^s\colon\ x_1^{e_j} = 0,\ (x_2^{e_j}, \dots, x_s^{e_j}) \in b_i\}$; it belongs to a hyperplan orthogonal to e_j. Let ε be a small positive parameter. Let β_j^ε be the image of β_j obtained by a homothetic contraction in $1/\varepsilon$ times with the center O. Denote B_j^ε the open cylinders with the bases β_j^ε and with the heights e_j:

$$B_j^\varepsilon = \left\{x \in \mathbb{R}^s\colon\ x_1^{e_j} \in \big(0, |e_j|\big),\ \left(\frac{x_2^{e_j}}{\varepsilon}, \dots, \frac{x_2^{e_j}}{\varepsilon}\right) \in b_j\right\}$$

Denote $\hat{\beta}_j^\varepsilon$ the second base of each cylinder B_j^ε and let O_j be the end of the segment e_j which belongs to the base $\hat{\beta}_j^\varepsilon$, $O_j \in \hat{\beta}_j^\varepsilon$.

Denote below $O_0 = O$. Let γ_0 be a bounded domain containing O, let γ_j be bounded domains containing O_j. For the sake of simplicity assume that the diameters of all these domains γ_0, γ_j are less than 2. Let γ_j^ε, $j = 0, 1, \dots, n$, be the images of the bounded domains γ_j (such that $\bar{\gamma}_j$ contain the ends of the segments O_j and independent of ε) obtained by a homothetic contraction in $1/\varepsilon$ times with the center O_j:

$$\gamma_j^\varepsilon = \left\{x \in \mathbb{R}^s\colon\ \frac{x - O_j}{\varepsilon} + O_j \in \gamma_j,\ j = 0, \dots, n\right\}$$

Define the tube structure associated with the graph B as

$$B^\varepsilon = \left(\bigcup_{j=1}^{n} B_j^\varepsilon\right) \cup \left(\bigcup_{j=0}^{n} \gamma_j^\varepsilon\right)$$

We suppose it be a domain with C^2-smooth boundary ∂B^ε.

We add the domains γ_j^ε, $j = 0, 1, \dots, n$, to make the boundary of the tube structure C^2-smooth surface.

In a more general case consider a finite set of the tube structures $B_{\varepsilon 1}, B_{\varepsilon 2}, \dots, B_{\varepsilon m}$, associated to the graphs consisting of the segments

$$\begin{aligned}
&e_{11}, \dots, e_{1n_1} && \text{for } (B_{\varepsilon 1})\\
&e_{21}, \dots, e_{2n_2} && \text{for } (B_{\varepsilon 2})\\
&\vdots\\
&e_{m1}, \dots, e_{mn_m} && \text{for } (B_{\varepsilon m})
\end{aligned}$$

Assume that any two segments of this list may have not more than one common point, that this point is an end point for both segments and that the graph is a connected set. The union

$$B_\varepsilon = B_{\varepsilon 1} \cup B_{\varepsilon 2} \cup \dots \cup B_{\varepsilon m}$$

of all these one bundle structures is called a multi-bundle structure. We assume that it is a connected domain with the C^2-smooth boundary. Its graph B is defined as a union of all segments e_{jl} of the above list. A vertex O_{jl} of the graph is called single iff it is an end point of the only one segment e_{jl} of the above list.

Denote $\Gamma_i, i = 1, \dots, r$, some smooth parts $\partial\gamma_i^\varepsilon$ of domains γ_i^ε, which are at the same time some parts of ∂B_ε, and such that the vertices $\hat{x}_i$ belonging to γ_i^ε, are single. Moreover, $\Gamma_i = \partial\gamma_i^\varepsilon \cap B(\hat{x}_i, r_i\varepsilon)$. Here and below, r_i are independent of ε, and $B(a,r)$ is an open ball with the center a and radius r. Consider the Stokes equation in B_ε

$$\begin{aligned} &\operatorname{div}\left(\frac{\nu}{2}\left(\nabla u_\varepsilon + (\nabla u_\varepsilon)^T\right) - p_\varepsilon I\right) = 0 \\ &\operatorname{div} u_\varepsilon = 0 \end{aligned} \tag{1}$$

where the divergence is applied to every string of the matrix $\frac{\nu}{2}(\nabla u_\varepsilon + (\nabla u_\varepsilon)^T) - p_\varepsilon I$, and ν is a positive number (viscosity).

The boundary conditions are:

$$u_\varepsilon = 0, \quad x \in \partial B_\varepsilon \Big\backslash \left(\bigcup_{t=1}^{r} \Gamma_t\right) \tag{2}$$

$$u_\varepsilon = g_t(x) = G\left(\frac{x - x_{b_t}}{\varepsilon}\right), \quad x \in \Gamma_t \tag{3}$$

Here x_{b_t} is an end point of the segment e_j such that $x_{b_t} \in \gamma_t$, n is an outer normal vector, $G \in C_0^2(\bar{\Gamma}_t)$,

$$\sum_t \int_{\Gamma_t} G\left(\frac{x - x_{b_t}}{\varepsilon}\right) n \,\mathrm{d}s = 0$$

Along with the Stokes equation consider the Navier–Stokes equation

$$\begin{aligned} &\operatorname{div}\left(\frac{\nu}{2}\left(\nabla u_\varepsilon + (\nabla u_\varepsilon)^T\right) - p_\varepsilon I\right) - (u_\varepsilon, \nabla) u_\varepsilon = 0 \\ &\operatorname{div} u_\varepsilon = 0 \end{aligned} \tag{4}$$

with the same boundary conditions.

A solution u_ε is defined as a vector-valued function from the space $H_{\mathrm{div}=0}(B_\varepsilon)$, that is a subspace of $(H^1(B_\varepsilon))^s$ such that its elements are divergence free vector-valued functions vanishing at the boundary of B_ε everywhere except for the parts Γ_t, and such that $u_\varepsilon - g \in H^0_{\mathrm{div}=0}(B_\varepsilon)$ and for any test function $\varphi \in H^0_{\mathrm{div}=0}(B_\varepsilon)$,

$$-\nu \int_{B_\varepsilon} \sum_{i=1}^{s} \left(\frac{\partial u_\varepsilon}{\partial x_i}, \frac{\partial \varphi}{\partial x_i}\right) \mathrm{d}x + \int_{B_\varepsilon} \sum_{i=1}^{s} u_{i,\varepsilon} \left(u_\varepsilon, \frac{\partial \varphi}{\partial x_i}\right) \mathrm{d}x = 0$$

Here $g \in H_{\mathrm{div}=0}(B_\varepsilon)$ is an extension of function G defined on Γ_t, and $H^0_{\mathrm{div}=0}(B_\varepsilon)$ is a subspace of vector-valued functions of $H_{\mathrm{div}=0}(B_\varepsilon)$, vanishing at the boundary. In the case of the Stokes equation the second integral of this variational formulation should be omitted.

A complete asymptotic expansions of solutions to these two problems are constructed and rigorously justified in [1,2,4]. In particular, the error estimates are proved for the difference of the asymptotic solution and the exact solution of these problems. The asymptotic expansions have a form of the Poiseuille flows at some small distance from the vertices of the graph glued near the vertices by some boundary layer correctors depending on the fast variable $\frac{x-O_j}{\varepsilon}$. The technique is close to the matching. This structure of the solution allows to justify the method of asymptotic partial domain decomposition (MAPDD), projecting the variational formulation on the subspace of $H_{\mathrm{div}=0}(B_\varepsilon)$ of functions exactly equal to the Poiseuille flows at some distance from the vertices greater than some δ.

2. Method of asymptotic partial domain decomposition for Newtonian flows

Let us describe the algorithm of the MAPDD for the Stokes and the Navier–Stokes problems set in a tube structure B_ε. This method for the homogeneous boundary condition $u_\varepsilon = 0$ all over the whole boundary ∂B_ε and with some right-hand side in the equations has been introduced and justified in [1,3,4]. Here we consider the homogeneous equation and the non-homogeneous boundary conditions (1)–(4). Let δ be a small positive number much greater than ε (it will be chosen of order $O(\varepsilon|ln\varepsilon|)$). For any segment e_{ij} of the graph of the structure introduce two hyperplanes orthogonal to this segment and crossing this segment at the distance δ from its ends. Enumerate the vertices of the graph $O_1, \dots, O_M$. Let O_{j_1} and O_{j_2} be the ends of the segment e_{ij}, Denote the cross-sections of the cylinder containing e_{ij} by these two hyperplanes respectively, S_{j_1,j_2} at the distance δ from O_{j_1}, and S_{j_2,j_1} at the distance δ from O_{j_2}, and denote the part of the cylinder between these two cross-sections by $B_{ij}^{dec,\varepsilon}$. Denote $B_i^{\varepsilon,\delta}$ the connected truncated by the cross-sections $S_{i,j}$, part of B_ε containing the vertex O_i. Define subspaces $H_{\mathrm{div}=0}(B_{\varepsilon,\delta})$ of the space $H_{\mathrm{div}=0}(B_\varepsilon)$, and $H^0_{\mathrm{div}=0}(B_{\varepsilon,\delta})$ of space $H^0_{\mathrm{div}=0}(B_\varepsilon)$, such that the elements of these subspaces (vector-valued functions) are the Poiseuille flows on every truncated cylinder $B_{ij}^{dec,\varepsilon}$. Here the Poiseuille flow is a vector-valued function u_P, such that in the local coordinates $x^{e_{ij}}$ associated to segment e_{ij}, its "first" (longitudinal) component $\tilde{u}_{1,P}(x^{e_{ij}}/\varepsilon)$ is a solution to the problem

$$\Delta_\xi \tilde{u}_{1,P}(\xi) = c_{ij}, \quad \xi \in b_{e_{ij}}, \qquad \tilde{u}_{1,P}(\xi) = 0, \quad \xi \in \partial b_{e_{ij}}$$

where c_{ij} is a "free" constant, and $\partial b_{e_{ij}}$ is the boundary of the domain b_i, corresponding to the segment e_{ij}; all other (normal) components of the Poiseuille function u_P in local coordinates are equal to zero.

The method of asymptotic partial domain decomposition (MAPDD) replaces the problem (1)–(4) by its projection on $H_{\mathrm{div}=0}(B_{\varepsilon,\delta})$:

Find $\hat{U}_{\varepsilon,\delta}$ from $H_{\mathrm{div}=0}(B_{\varepsilon,\delta})$, such that $\hat{U}_{\varepsilon,\delta} - g \in H^0_{\mathrm{div}=0}(B_{\varepsilon,\delta})$ and for any test function $\varphi \in H^0_{\mathrm{div}=0}(B_{\varepsilon,\delta})$,

$$-\nu \int_{B_\varepsilon} \sum_{i=1}^{2} \left(\frac{\partial \hat{U}_{\varepsilon,\delta}}{\partial x_i}, \frac{\partial \varphi}{\partial x_i} \right) \mathrm{d}x + \int_{B_\varepsilon} \sum_{i=1}^{2} \hat{U}_{i,\varepsilon,\delta} \left(\hat{U}_{\varepsilon,\delta}, \frac{\partial \varphi}{\partial x_i} \right) \mathrm{d}x = 0$$

Let g be an extension of function G from the boundary to the domain B_ε, belonging to $H_{\mathrm{div}=0}(B_{\varepsilon,\delta})$. Such an extension exists. Indeed, applying the asymptotic expansion for problem (4), (2), (3) constructed in Section 4.5 of [1] with the small additional correctors from Section 6.3 of [1], we see that this asymptotic approximation gives one of such extensions. Then following the arguments of Section 6.3 of the same book, using the construction of the asymptotic expansion of Section 4.5 of [1] and applying the arguments of Section 5.1 of [6], we get the following

Theorem 2.1. *For any integer K there exists a constant C_K, independent of ε, such that for $\delta \geqslant C_K \varepsilon |\ln \varepsilon|$ the estimate holds*:

$$\|\hat{U}_{\varepsilon,\delta} - u_\varepsilon\|_{H^1(B_\varepsilon)} = O\left(\varepsilon^K\right)$$

This estimate justifies the MAPDD for problem (4), (2), (3). In a similar way we can prove the same estimate for the Stokes problem (1)–(3).

Thus, the method reduces considerably the computational cost of the numerical solution of the Stokes or the Navier–Stokes equation set in a thin tube structure and it keeps a high order accuracy. This approach can be applied for the computations of flows in great system of tubes or channels, such as the blood circulation system. In this case a parallelization of computations and multi-processor strategy becomes a very important issue.

3. Parallelization of the algorithm of the method of asymptotic partial domain decomposition

1. In the case of a linear problem such parallelization can be done with help of the principle of superposition. Actually, consider the Stokes problem (1)–(3). Consider a truncated part $B_i^{\varepsilon,\delta}$. Integrating by parts the variational formulation of the problem for $\hat{U}_{\varepsilon,\delta}$ we obtain that on every cross-section $S_{i,j}$ the flux rate is conserved; moreover, the trace of function $\hat{U}_{\varepsilon,\delta}$ on $S_{i,j}$ is equal to some Poiseuille flow corresponding to some constant c_{ij}. Let the boundary of $B_i^{\varepsilon,\delta}$ contains M_i sections $S_{i,j}$: $S_{i,j_1}, \dots, S_{i,j_{M_i}}$. Assume that $\partial B_i^{\varepsilon,\delta}$ does not contain parts Γ_t. Then on $B_i^{\varepsilon,\delta}$ we have

$$\hat{U}_{\varepsilon,\delta} = \sum_{r=1}^{M_i} c_{ij_r} \hat{U}^{j_r}_{\varepsilon,\delta}$$

where $\hat{U}^{j_r}_{\varepsilon,\delta}$ is a solution of the homogeneous Stokes equation (1) with the boundary condition $\hat{U}^{j_r}_{\varepsilon,\delta} = 0$ on $\partial B_i^{\varepsilon,\delta}$ except for S_{i,j_r}, and on S_{i,j_r}, $\hat{U}^{j_r}_{\varepsilon,\delta} = u_P^0$, where u_P^0 is the Poiseuille function corresponding to constant $c_{ij} = 1$.

In the case if $\Gamma_t \subset \partial B_i^{\varepsilon,\delta}$, then $B_i^{\varepsilon,\delta}$ contains a single vertex $O_i = x_{b_t}$ and its boundary $\partial B_i^{\varepsilon,\delta}$ contains only one section of truncation S_{i,j_1}. Then

$$\hat{U}_{\varepsilon,\delta} = \hat{U}^0_{\varepsilon,\delta} + c_{ij_1} \hat{U}^{j_1}_{\varepsilon,\delta},$$

where $\hat{U}^0_{\varepsilon,\delta}$ is a solution of the homogeneous Stokes equation (1) with the boundary condition $\hat{U}^{j_r}_{\varepsilon,\delta} = 0$ on $\partial B_i^{\varepsilon,\delta}$ except for Γ_t, and on Γ_t, $\hat{U}^{j_r}_{\varepsilon,\delta} = G(\frac{x - x_{b_t}}{\varepsilon})$, and $\hat{U}^{j_1}_{\varepsilon,\delta}$ is a solution of the homogeneous Stokes equation with the boundary condition $\hat{U}^{j_1}_{\varepsilon,\delta} = 0$ on $\partial B_i^{\varepsilon,\delta}$ except for S_{i,j_1}, and on S_{i,j_1}, $\hat{U}^{j_1}_{\varepsilon,\delta} = u_P^0$.

We see that the solution $\hat{U}_{\varepsilon,\delta}$ of the partially decomposed problem is some linear combination of the problems for $\hat{U}^{j_r}_{\varepsilon,\delta}$ and these problems are completely independent each of the other. So, they can be solved in parallel. The only one problem which binds all these auxiliary problems is the linear algebraic system of equations of flux conservation at each truncation section $S_{i,j}$; the unknowns are coefficients $c_{i,j}$ and its dimension is equal to the global number of segments e_{ij} in the graph. The same approach was applied for the Laplace equation in [5].

2. For non-linear problem, such as Eq. (4), the superposition principle is not true. However, in some special cases the complete decomposition (parallelization) is possible.

2.1. In the case of one bundle structure with given velocities $g_t(x) = G(\frac{x-x_{b_t}}{\varepsilon})$ on Γ_t, such that vertex x_{b_t} belongs to the boundary of cylinder B_j^ε, the flux passing over every section of this cylinder (and in particular, over the truncation section S_{ij}) has the rate equal to

$$D_j = -\int_{\Gamma_t} g_t(s) \cdot n \, \mathrm{d}s \tag{5}$$

where n is an outer normal. It is a consequence of the incompressibility condition $\operatorname{div} u = 0$. Therefore, the constants of the Poiseuille flow c_{ij} can be calculated explicitly for all cylinders. So, the Navier–Stokes equations in all $B_i^{\varepsilon,\delta}$ have boundary condition with known right-hand side (known Poiseuille function on every truncation) and these problems are completely decoupled: we get $n+1$ independent problems on the truncated parts $B_i^{\varepsilon,\delta}$ with given Poiseuille flow boundary conditions at all truncated sections of the cylinders.

2.2. Similar result holds in the case of a multi-bundle structure, such that all flux rates D_j can be found from the linear system of equations

$$\sum_{j:\ O_i \in \bar{e}_j} D_j = 0, \quad i = 1, \ldots, N \tag{6}$$

for all vertices O_i of the graph.

2.3. In the general case, when number N of vertices O_j may be less than the number of segments e_j, the system of Eqs. (6) doesn't determine all the fluxes D_j. In this case the problems on $B_i^{\varepsilon,\delta}$ are coupled. However the asymptotic analysis of the pressure shows that in the first approximation it is continuous on the graph $B = \bigcup_j e_j$, linear for every segment e_j of the graph function, satisfying relation

$$D_j = -K_j \frac{\partial p}{\partial x_1^{e_j}} \tag{7}$$

for all e_j. Here K_j stands for the permeability coefficient, i.e. the flux of the Poiseuille flow corresponding to a unitary pressure drop. For example, in the 2D case for the channel of the thickness $\frac{\varepsilon\alpha}{2}$ and $\nu = 1$, the Poiseuille function is equal to

$$\left(\frac{1}{2}\left(x_2^2 - \left(\frac{\varepsilon\alpha}{2}\right)^2\right), 0\right)^T$$

where x_2 is the transversal variable in the local system, and

$$K_j = -\int_{-\frac{\varepsilon\alpha}{2}}^{\frac{\varepsilon\alpha}{2}} \frac{1}{2}\left(x_2^2 - \left(\frac{\varepsilon\alpha}{2}\right)^2\right) \mathrm{d}x_2 = \frac{1}{12}(\varepsilon\alpha)^3$$

Finally for the macroscopic pressure we get the differential equation set on the graph:

$$D_j' = 0 \tag{8}$$

where D_j is related to the pressure by Eq. (7), and at the vertices we have the junction conditions (6) and the continuity of the pressure. At the single vertices we have condition (5). Problem on the graph (5)–(8) may be formulated in variational sens and its existence and uniqueness of solution follow immediately from the Riesz theorem or Lax–Milgram lemma. Solving this problem, we find the macroscopic pressure on the graph B, then we can find the fluxes D_j on all cylinders B_j^ε with a relative error of order ε, and so decouple (with accuracy of order ε) the sub-problems on all $B_i^{\varepsilon,\delta}$.

4. MAPDD for non-Newtonian flows

The blood motion is described in a more adequate way by non-Newtonian flow laws (see [6–10]), for example, by equations

$$-\operatorname{div}\big(\nu(Du)Du\big) + \nabla p = 0 \tag{9}$$

$$\operatorname{div} u = 0 \tag{10}$$

where $D = \frac{1}{2}(\nabla + \nabla^T)$, and $\nu(y)$ is some given relation between the symmetric matrix y and a scalar viscosity ν. The results on existence of solutions of such equations in some cases were obtained in [6–9], however, an asymptotic expansion of solution in a tube structure is not yet constructed because of absence of results on stability. Although the MAPDD is not justified in this case, its algorithm can be formulated as above for Newtonian flows with the only one difference: the Poiseuille flow should be replaced by the quasi-Poiseuille function u_P. As in the case of Newtonian flows, in the local coordinates $\tilde{x}^e$ its normal components are equal to zero, while the longitudinal one $\tilde{u}_{P,\gamma}$ is a function of the transversal variables $\tilde{x}' = (\tilde{x}_2^e, \ldots, \tilde{x}_2^e)$ and it is an exact solution of Eqs. (9), (10) in an infinite cylinder $B_j^\infty = \beta_j \times l_j$, with the boundary condition of vanishing of the velocity $\tilde{u}_{P,\gamma}$ on the whole lateral boundary of the cylinder ($\tilde{u}_{P,\gamma} = 0$) with $p = \gamma \tilde{x}_1^e$; here l_j is a straight line, orthogonal to β_j,γ is a real parameter and $\tilde{x}_1^e$ is the longitudinal local variable. Let us define the flux

$$D_j(\gamma) = \int \tilde{u}_{P,\gamma}\, \mathrm{d}\tilde{x}'$$

where the integration holds in the cross-section of the cylinder. The quasi-Poiseuille in the thin cylinder B_j^ε is calculated as $\varepsilon \tilde{u}_{P,\gamma}(\frac{\tilde{x}'}{\varepsilon})$, and it corresponds to $p = \varepsilon^{-1}\gamma \tilde{x}_1^e$. Assume that such a quasi-Poiseuille function exists for any real γ. Then we can do the same parallelization for the MAPDD algorithm (replacing the Poiseuille flow by the quasi-Poiseuille flow). In particular, Eq. (7) is replaced by

$$D_j = D_j\left(\frac{\partial p}{\partial \tilde{x}_1^e}\right)$$

with $\gamma = \frac{\partial p}{\partial \tilde{x}_1^e}$.

The above MAPDD algorithm for the Carreau law in the 2D case with

$$\nu(y) = M\big(1 + (\lambda y_{12})^2\big)^{\frac{n-1}{2}}$$

$n = 0.7$, $M = 7$, $\lambda = 0.11$ was tested in numerical COMSOL simulations developed in collaboration with A. Nachit and A.M. Zine. These numerical experiments, corresponding to the blood flow data [10], confirmed a high accuracy of the method even in the case when $\delta = 1.5\varepsilon$, i.e. for δ much less than $\delta = \mathrm{const}\,\varepsilon|\ln\varepsilon|$ in the theoretical predictions.

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