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Data completion for the Stokes system

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Abstract

This Note is concerned with the severely ill-posed Cauchy–Stokes problem. This inverse problem is rephrased into an optimization one: An energy-like error functional is introduced. We prove that the optimality condition of the first order is equivalent to solving an interfacial equation which turns out to be a Cauchy–Steklov–Poincaré operator. Numerical trials highlight the efficiency of the present method. ***To cite this article: A. Ben Abda et al., C. R. Mecanique 337 (2009).***

Résumé

Complétion des données pour le système de Stokes. Cette Note concerne le problème mal-posé de Cauchy–Stokes. On réécrit ce problème en terme d'optimisation d'une fonctionnelle d'erreur de type écart à la loi de comportement. La condition d'optimalité du premier ordre conduit à une équation d'interface Cauchy–Steklov–Poincaré sur la partie à compléter. Des essais numériques illustrent l'efficacité de cette méthode. ***Pour citer cet article : A. Ben Abda et al., C. R. Mecanique 337 (2009).***

Keywords: Computational fluid mechanics; Cauchy–Stokes problem; Data completion; Inverse problem; Steklov–Poincaré operator

Mots-clés : Mécanique des fluides numérique ; Complétion de données ; Opérateur de Steklov–Poincaré ; Problème de Cauchy–Stokes ; Problème inverse

Version française abrégée

On propose dans cette Note une méthode de résolution du problème de Cauchy–Stokes. Ce problème est connu pour être mal posé au sens d'Hadamard. On se donne une vitesse U et une force F sur une partie de la frontière, noté Γ_d, d'un domaine Ω, on veut compléter les données sur l'autre partie de la frontière inconnue Γ_u. Une fonctionnelle d'erreur de type énergie est construite dans ce contexte, elle a été introduite dans [1] et [2] dans le cadre de l'équation de Laplace. Le problème inverse est résolu via la minimisation de cette fonctionnelle. Nous prouvons que la condition d'optimalité du premier ordre est équivalente à résoudre une équation d'interface qui s'interprète en terme d'opérateur de Cauchy–Steklov–Poincaré sur la partie à compléter.

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La méthode est testée sur un exemple où les conditions aux limites à reconstruire sont des fonctions polynomiales. Un cas test qui présente une singularité au voisinage de la frontière inconnue Γ_u atteste de l'efficacité de ce procédé de reconstruction.

1. Introduction

Let $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. We assume that Γ is partitioned into two parts Γ_d and Γ_u having both non-vanishing measure.

In this work we are interested in a data completion problem related to the Stokes system. It consists in recovering the data on the incomplete (inaccessible, buried, ...) boundary Γ_u from the over-specified data on the accessible boundary Γ_d.

Assume a given velocity U and a force F on Γ_d, the data completion problem for the Stokes operator can be formulated as a Cauchy problem: *find the velocity u and the pressure p solution to*

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega \\ \nabla.u = 0 & \text{in } \Omega \\ u = U & \text{on } \Gamma_d \\ \sigma(u).n = F & \text{on } \Gamma_d \end{cases} \tag{1}$$

where ν is the fluid kinematic viscosity and σ is the stress tensor $\sigma(u) = 2\nu D(u) - pI$ with $D(u)$ is the deformation tensor $D(u) = 1/2(\nabla u + \nabla u^T)$, I is the $d \times d$ identity matrix and n is the unit outward normal vector.

This problem is known since Hadamard to be ill-posed in the sense that the dependence of (u, p) on the data (U, F) is not continuous. In order to reconstruct the unknown boundary data $u_{|\Gamma_u}$ and $\sigma(u).n_{|\Gamma_u}$ on Γ_u we will use the Steklov–Poincaré operator (see [3,4] or [1,5,6] for the Laplace equation). The inverse problem is formulated as an optimization one.

This Note is outlined as follows: In the next section, an energy-like error functional is introduced in the context of the ill-posed problem of recovering boundary data. The data completion problem is formulated as an optimization one. The first-order optimality condition is rephrased in terms of an interfacial problem using the Steklov–Poincaré operator. The numerical procedure for solving the Cauchy–Stokes problem is described in Section 3. In Section 4 we present some numerical illustrations. Some comments are given in the closing section.

2. Formulation of the problem

The considered data completion problem (1) can be stated as: for a given compatible data $(U, F) \in H^{1/2}(\Gamma_d)^d \times H^{-1/2}(\Gamma_d)^d$, for which existence and uniqueness of the solution are guaranteed, find the unknown boundary data $(V, G) \in H^{1/2}(\Gamma_u)^d \times H^{-1/2}(\Gamma_u)^d$ such that

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega \\ \nabla.u = 0 & \text{in } \Omega \\ u = U,\ \sigma(u).n = F & \text{on } \Gamma_d \\ u = V,\ \sigma(u).n = G & \text{on } \Gamma_u \end{cases} \tag{2}$$

To solve this problem, (V, G) will be characterized as the minimum of an energy-like functional [1,2,7].

2.1. Minimization problem

First, for each given $(v, g) \in H^{1/2}(\Gamma_u)^d \times H^{-1/2}(\Gamma_u)^d$, we consider the two mixed well-posed problems:

$$(P_D)\quad \begin{cases} -\nu\Delta u_D^g + \nabla p_D^g = 0 & \text{in } \Omega \\ \nabla.u_D^g = 0 & \text{in } \Omega \\ u_D^g = U & \text{on } \Gamma_d \\ \sigma(u_D^g).n = g & \text{on } \Gamma_u \end{cases} \qquad (P_N)\quad \begin{cases} -\nu\Delta u_N^v + \nabla p_N^v = 0 & \text{in } \Omega \\ \nabla.u_N^v = 0 & \text{in } \Omega \\ u_N^v = v & \text{on } \Gamma_u \\ \sigma(u_N^v).n = F & \text{on } \Gamma_d \end{cases}$$

In the second step we build the following energy-like error functional:

$$E(v,g) = 1/2 \int_{\Omega} \sigma\big(u_D^g - u_N^v\big) : \nabla\big(u_D^g - u_N^v\big)\,\mathrm{d}x$$

Then, the unknown data (V, G) can be characterized as the solution of the minimization problem:

$$(V,G) = \arg\min_{v,g} E(v,g)$$

Remark 1. The energy-type error functional has been already introduced for data completion in the framework of Laplace equation in [1,2]. The approach followed here solves, as shown in the next section, the interfacial equation rather than the optimization problem.

Let us observe that, for compatible data, the following proposition holds:

Proposition 1. *When (U, F) is a compatible pair, the minimum of E is reached when*:

$$\begin{cases} u_D^g = u_N^v + \text{Const.} & \text{on } \Gamma_u \\ (D(u_N^v) - p_N^v I).n = (D(u_D^g) - p_D^g I).n & \text{on } \Gamma_u \end{cases} \tag{3}$$

Proof. We have

$$\frac{\partial E(v,g)}{\partial v}(w) = \int_{\Omega} 2\nu D\big(u_N^v - u_D^g\big) : \nabla r_N^w\,\mathrm{d}x = \int_{\Gamma_u} \sigma\big(r_N^w\big).n\big(u_N^v - u_D^g\big)\,\mathrm{d}s \quad \forall w \in H^{1/2}(\Gamma_u)^d \tag{4}$$

$$\frac{\partial E(v,g)}{\partial g}(h) = \int_{\Omega} 2\nu D\big(u_D^g - u_N^v\big) : \nabla r_D^h\,\mathrm{d}x = \int_{\Gamma_u} \sigma\big(u_D^g - u_N^v\big).n r_D^h\,\mathrm{d}s \quad \forall h \in H^{-1/2}(\Gamma_u)^d \tag{5}$$

where (r_N^w, s_N^w) and (r_D^h, s_D^h) are respectively the solution to

$$\begin{cases} -\nu\Delta r_N^w + \nabla s_N^w = 0 & \text{in } \Omega \\ \nabla.r_N^w = 0 & \text{in } \Omega \\ \sigma(r_N^w).n = 0 & \text{on } \Gamma_d \\ r_N^w = w & \text{on } \Gamma_u \end{cases} \qquad \begin{cases} -\nu\Delta r_D^h + \nabla s_D^h = 0 & \text{in } \Omega \\ \nabla.r_D^h = 0 & \text{in } \Omega \\ r_D^h = 0 & \text{on } \Gamma_d \\ \sigma(r_D^h).n = h & \text{on } \Gamma_u \end{cases}$$

Consider the Steklov–Poincaré operator

$$\begin{aligned} &S_N : H^{1/2}(\Gamma_u)^d \to H^{-1/2}(\Gamma_u)^d \\ &w \to \sigma\big(r_N^w\big).n \end{aligned} \tag{6}$$

One can observe that the kernel $N(S_N)$ and the range $R(S_N)$ of the operator S_N are defined by

$$N(S_N) = \mathbb{R} \quad \text{and} \quad R(S_N) = H^{-1/2}(\Gamma_u)^d$$

Then, it follows that $S_N : H^{1/2}(\Gamma_u)^d / N(S_N) \to H^{-1/2}(\Gamma_u)^d$ is an isomorphism. Consequently, Eq. (4) implies the first condition of the proposition: $u_N - u_D = \text{Const.}$ on Γ_u.

For the second condition, we introduce the inverse of Steklov–Poincaré operator:

$$\begin{aligned} &S_D^{-1} : H^{-1/2}(\Gamma_u)^d \to H^{1/2}(\Gamma_u)^d \\ &h \to r_D^h \end{aligned} \tag{7}$$

From the fact that (U, F) is a compatible pair one can deduce that S_D^{-1} is an isomorphism. Then the equality

$$\big(D(u_D) - p_D I\big).n = \big(D(u_N) - p_N I\big).n \quad \text{on } \Gamma_u$$

follows immediately from Eq. (5). □

2.2. The interfacial operators

The solutions (u_D^g, p_D^g) and (u_N^v, p_N^v) can be decomposed as

$$\left(u_D^g, p_D^g\right) = \left(u_D^0, p_D^0\right) + \left(r_D^g, s_D^g\right) \quad \text{and} \quad \left(u_N^v, p_N^v\right) = \left(u_N^0, p_N^0\right) + \left(r_N^v, s_N^v\right)$$

Then, the equalities (3) can be rewritten as

$$\begin{cases} r_N^v - r_D^g = u_D^0 - u_N^0 & \text{on } \Gamma_u \\ \sigma(r_N^v).n - \sigma(r_D^g).n = \sigma(u_D^0).n - \sigma(u_N^0).n & \text{on } \Gamma_u \end{cases}$$

Using the definitions of the fields r_N^v and r_D^g, we deduce the interfacial system satisfied by (v, g)

$$\begin{cases} v - S_D^{-1}(g) = u_D^0 - u_N^0 & \text{on } \Gamma_u \\ -S_N(v) + g = \sigma(u_N^0).n - \sigma(u_D^0).n & \text{on } \Gamma_u \end{cases}$$

which can be written as:

$$\begin{pmatrix} I & -S_D^{-1} \\ -S_N & I \end{pmatrix} \begin{pmatrix} v \\ g \end{pmatrix} = T, \quad \text{where } T = \begin{pmatrix} u_D^0 - u_N^0 \\ \sigma(u_N^0).n - \sigma(u_D^0).n \end{pmatrix} \text{ depends only on the data } (U, F)$$

3. The numerical procedure

The Stokes–Cauchy problem is solved iteratively using the following preconditioned gradient algorithm:

$$\begin{pmatrix} v_{k+1} \\ g_{k+1} \end{pmatrix} = \begin{pmatrix} v_k \\ g_k \end{pmatrix} - \rho M \left[\begin{pmatrix} I & -S_D^{-1} \\ -S_N & I \end{pmatrix} \begin{pmatrix} v_k \\ g_k \end{pmatrix} - T \right]$$

where $M = \begin{pmatrix} I & 0 \\ 0 & S_D^{-1} \end{pmatrix}$ is a preconditioning operator and ρ is a relaxation parameter.

The algorithm:

- Initialization: set $k = 0$ and chosen v_0 and g_0.
- Repeat until $E(v_k, g_k) \leqslant \varepsilon$, where ε is a given tolerance level:
  - solve the problems (P_D) and (P_N) (using $v = v_k$ and $g = g_k$),
  - computation of the gradient: compute w_D^k and w_N^k solutions to the problems (A_D) and (A_N),
  - set $v_{k+1} = v_k - \rho w_N^k$ and $g_{k+1} = g_k - \rho w_D^k$,
  - $k \to k + 1$,

  where (A_D) and (A_N) are two auxiliary problems defined by

$$(A_N) \quad \begin{cases} -\Delta w_N^k + \nabla q_N^k = 0 & \text{in } \Omega \\ \nabla . w_N^k = 0 & \text{in } \Omega \\ w_N^k = u_D - u_N & \text{on } \Gamma_u \\ \sigma(w_N^k).n = 0 & \text{on } \Gamma_d \end{cases} \qquad (A_D) \quad \begin{cases} -\Delta w_D^k + \nabla q_D^k = 0 & \text{in } \Omega \\ \nabla . w_D^k = 0 & \text{in } \Omega \\ \sigma(w_D^k).n = \sigma(u_N).n - \sigma(u_D).n & \text{on } \Gamma_u \\ w_D^k = 0 & \text{on } \Gamma_d \end{cases}$$

4. Numerical illustration

Let us consider a viscous incompressible fluid that is confined between two concentric circular cylinders of infinite length. We assume that the velocity field and the pressure do not depend on the longitudinal coordinate. We deal therefore with a two-dimensional problem defined in a cross section Ω.

In order to validate the proposed approach, we consider here the identification of the velocity field and the stress force on the inner circle Γ_u form an over-specified data (U, F) on the outer circle Γ_d. The data (U, F) is generated from the following two analytic Stokes solutions

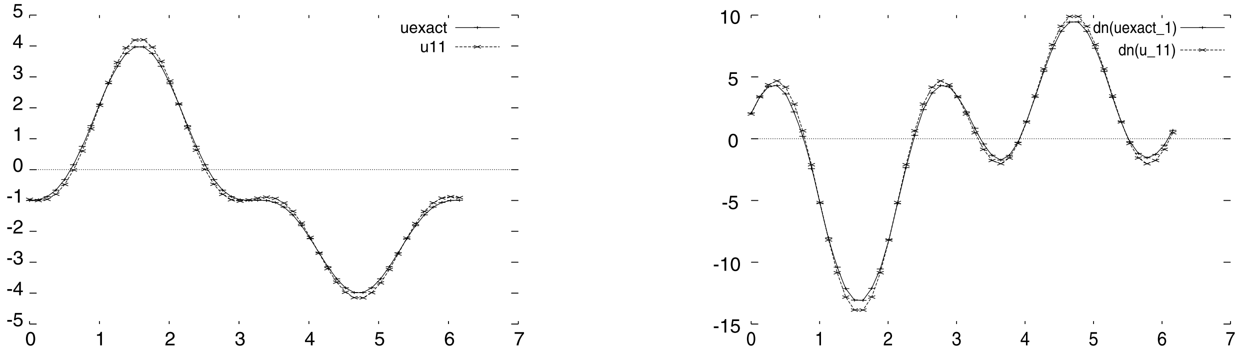

Fig. 1. Smooth data: reconstructed velocity and the stress tensor on Γ_u for polynomial case.

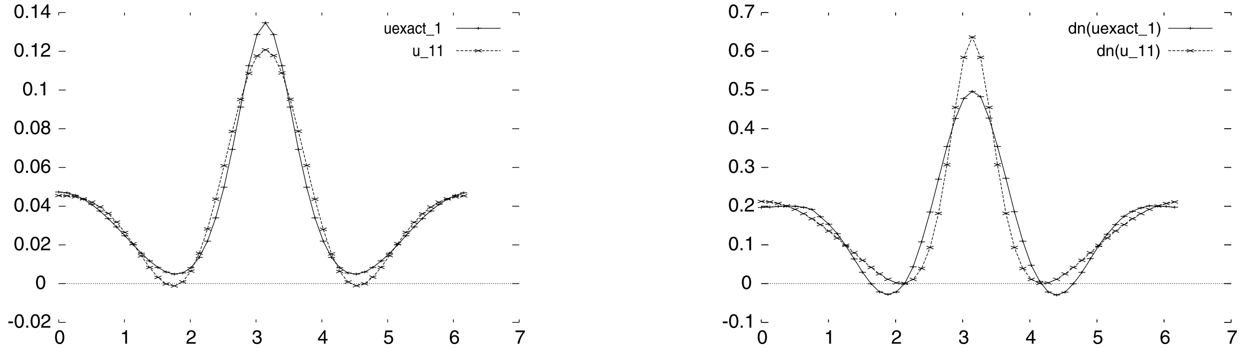

Fig. 2. Singular data: reconstructed velocity and the stress tensor on Γ_u with $a = 0.5$.

(i) $u = (4y^3 - x^2, 4x^3 + 2xy - 1)$, $p = 24xy - 2x$, $\forall (x, y) \in \Omega$,
(ii) $u = \frac{1}{4\pi}(\log \frac{1}{\sqrt{(x-a)^2+y^2}} + \frac{(x-a)^2}{(x-a)^2+y^2}, \frac{y(x-a)}{(x-a)^2+y^2})$, $p = \frac{1}{2\pi} \frac{x-a}{(x-a)^2+y^2}$, $\forall (x, y) \in \Omega$.

As one can remark, the second analytic solution involves a singularity in the vicinity of the inner boundary. The numerical experiments are performed on a thick annular domain with radii $R_1 = 2$ and $R_2 = 1$.

The numerical simulation is run under the *Freefem++* software environment [8], it is a free software based on the Finite Element Method. The domain Ω is discretized using a uniform mesh with 50 nodes on Γ_u and 100 nodes on Γ_d. The reconstructed data are computed using the iterative algorithm described in Section 3 with $\varepsilon = 10^{-5}$ and $\rho = 0.1$.

The numerical results are given in Figs. 1 and 2. Two test cases are considered using respectively a regular and a singular boundary data (U, F). We present in Fig. 1 the reconstructed velocity v and stress tensor g on Γ_u using polynomial boundary data (U, F). The reconstructed solution of singular data (using $a = 0.5$) is shown in Fig. 2. We only present the first component of the reconstructed velocity v or stress tensor g. The first component of the exact and the computed velocity are respectively denoted by $uexact_1$ and u_{11}. The exact and the computed stress tensor are respectively denoted by $dn(uexact_1)$ and $dn(u_{11})$.

5. Conclusion

In this work, we have investigated the Cauchy problem for the viscous stationary Stokes system. An energy-like error functional is introduced. The minimisation process is achieved through the resolution of the first optimality condition which relies on solving an interfacial equation. This method is, up to the authors knowledge, implemented for the first time for a viscous and incompressible fluid. The preliminary numerical experiments are very encouraging.

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