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Nonlinear double porosity models with non-standard growth

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Abstract

We study the solutions to quasilinear elliptic equations with high contrast coefficients. The energy formulation leads to work with variable exponent Lebesgue spaces $L^{p_\varepsilon(\cdot)}$ in domains with a complex microstructure scaled by a small parameter ε. We derive rigorously the corresponding homogenized problem. It is completely described in terms of local energy characteristics of the original domain. ***To cite this article: C. Choquet, L. Pankratov, C. R. Mecanique 337 (2009).***

Résumé

Modèles non linéaires de type double-porosité à croissance non standart. Nous étudions les solutions d'équations quasilinéaires elliptiques à coefficients fortement contrastés. La formulation variationnelle associée conduit à travailler dans des espaces de Lebesgue à exposant variable $L^{p_\varepsilon(\cdot)}$ dans des domaines à la microstructure complexe caractérisée par un petit paramètre ε. Nous obtenons rigoureusement le problème homogénéisé correspondant. Il est déterminé par les caractéristiques variationnelles locales de la microstructure. ***Pour citer cet article : C. Choquet, L. Pankratov, C. R. Mecanique 337 (2009).***

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Nous considérons le problème variationnel (2), où K_ε est une fonction qui dégénère sur une partie asymptotiquement dense du domaine (voir (K.1)–(K.2)). Le domaine est un milieu dispersé vérifiant (C.1)–(C.2). En contrôlant les caractéristiques locales (7)–(8) du domaine, nous obtenons rigoureusement le problème homogénéisé correspondant à (2). Il est décrit dans le théorème 2.1 : la solution u^ε de (2) converge dans $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ vers la solution u de $J_{\mathrm{hom}}[u] \to \min$, $u \in W_0^{1,p_0(\cdot)}(\Omega)$, avec $J_{\mathrm{hom}}[u] = \int_\Omega (\mathsf{A}(x, \nabla u) + \frac{\rho(x)}{\sigma(x)}|u|^{\sigma(x)} + \mathsf{b}(x,u) - g(x)\rho(x)u)\,\mathrm{d}x$. De

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plus, dans la partie matricielle du domaine, $(\frac{1}{p_\varepsilon(x)}|u^\varepsilon|^{\sigma(x)-2}(u(x)u^\varepsilon - |u^\varepsilon|^2) + \frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)})$ converge faiblement vers $\mathsf{b}(x,u)$. Après avoir construit (§3) des approximations *ad hoc* de u^ε, nous prouvons le théorème 2.1 en quatre étapes : 0. il existe une fonction u telle que $u^\varepsilon \to u$ dans $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$; 1. on a $\overline{\lim}\, J^\varepsilon[u^\varepsilon] \leqslant J_{\mathrm{hom}}[w]$ pour toute fonction $w \in W^{1,p_0(\cdot)}(\Omega)$; 2. on a $\underline{\lim}\, J^\varepsilon[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u]$ (ainsi u minimise J_{hom} dans $W_0^{1,p_0(\cdot)}(\Omega)$) ; 3. on prouve le résultat de convergence dans la partie matricielle. Finalement, nous illustrons notre résultat dans le cadre d'un exemple périodique, retrouvant ainsi une formulation plus usuelle du problème homogénéisé.

1. Introduction

We study rigorously the asymptotic behavior as $\varepsilon \to 0$ of solutions to quasilinear equations of the form

$$-\operatorname{div}\big(K_\varepsilon(x)\big|\nabla u^\varepsilon\big|^{p_\varepsilon(x)-2}\nabla u^\varepsilon\big) + \big|u^\varepsilon\big|^{\sigma(x)-2}u^\varepsilon = g^\varepsilon(x), \quad x \in \Omega \tag{1}$$

with a high contrast coefficient $K_\varepsilon(x)$. This equation is naturally associated with variable exponent Lebesgue and Sobolev spaces, $L^{\sigma}(\cdot)$ and $W^{1,p_\varepsilon(\cdot)}$. These types of functional space with non-standard growth are nowadays mainly used for the modeling of electrorheological fluids (see, e.g., [1,2]) and for image restoration [3]. Eqs. (1) arise, e.g., from compressible flows in porous media, and non-Newtonian flow through thin fissures.

Our homogenization problem is closely related to the so-called double-porosity models widely discussed in the literature (see [4,5]). A linear double-porosity model was first mathematically derived in [6]. Nonlinear models, a general non-periodic model and a random model were handled in [7,8], [9] and [10], respectively. Here, instead of using the above geometrical assumptions, following the approach introduced in [11], we only impose conditions on the *local energetic characteristics* of the problem. These characteristics contain the useful informations on the microstructure and are adapted to the equation through penalty terms. This method covers a variety of concrete behaviors including periodicity and almost periodicity.

Notational conventions. In what follows C, C_1, etc. are generic constants independent of ε. The space $L^{p(\cdot)}(\Omega)$ of measurable functions ϕ in Ω such that $\Upsilon_{p(\cdot),\Omega}(\phi) = \int_\Omega |\phi(x)|^{p(x)}\,\mathrm{d}x < +\infty$ endowed with the norm $\|\phi\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0\colon\ \Upsilon_{p(\cdot),\Omega}(\frac{\phi}{\lambda}) \leqslant 1\}$ is a Banach space. Following [12], the Sobolev space with variable exponent p is defined by $W^{1,p(\cdot)}(\Omega) = \{\phi \in L^{p(\cdot)}(\Omega)\colon\ |\nabla\phi| \in L^{p(\cdot)}(\Omega)\}$. The scalar product in $\mathbb{R}^n$ is denoted by $(\cdot,\cdot)$. We denote by $\mathbf{1}_k^\varepsilon$ the characteristic function of the set Ω_k^ε, $k = f, m$. Finally, for any $0 < \varepsilon \ll h \ll 1$, K_h^z is an open cube centered at $z \in \Omega$ with lengths equal to h.

2. Statement of the problem and the main result

Let $\Omega = \Omega_f^\varepsilon \cup \overline{\Omega_m^\varepsilon}$ be a bounded domain of $\mathbb{R}^n$ ($n \geqslant 2$) with Lipschitz boundary $\partial\Omega$, $\{\Omega_m^\varepsilon\}_{(\varepsilon>0)}$ being a family of open subsets in Ω. We assume that the set Ω_m^ε is distributed in an asymptotically regular way in Ω. It may consist of N_ε ($N_\varepsilon \to +\infty$) small isolated components or it may be defined as fibers becoming more and more dense.

Our aim is to study the asymptotic behavior, as $\varepsilon \to 0$, of the solution u^ε of the following variational problem:

$$J^\varepsilon\big[u^\varepsilon\big] \to \min, \quad u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega) \tag{2}$$

$$J^\varepsilon[u] \overset{\mathrm{def}}{=} \begin{cases} \int_\Omega (\varkappa_\varepsilon(x)\,|\nabla u|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}|u|^{\sigma(x)} - \mathbf{1}_f^\varepsilon(x)g(x)u)\,\mathrm{d}x & \text{if } u \in W^{1,p_\varepsilon(\cdot)}(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{3}$$

with $\varkappa_\varepsilon(x) = K_\varepsilon(x)/p_\varepsilon(x)$. The growth functions p_ε and σ possess the sufficient properties to ensure the existence of a unique solution $u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ of the variational problem (2) (see [12]):

(A.1) for any $\varepsilon > 0$, p_ε satisfies the following log-Hölder continuity property

$$\big|p_\varepsilon(x) - p_\varepsilon(y)\big| \leqslant \omega_{p_\varepsilon}\big(|x-y|\big) \quad \forall x, y \in \Omega, \text{ with } \overline{\lim_{\tau\to 0}}\, \omega_{p_\varepsilon}(\tau)\ln(1/\tau) \leqslant C \tag{4}$$

and there exist two real numbers p^- and p^+ such that

$$1 < \mathsf{p}^- \leqslant p_\varepsilon^- \equiv \min_{x\in\overline{\Omega}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\overline{\Omega}} p_\varepsilon(x) \equiv p_\varepsilon^+ \leqslant \mathsf{p}^+ < +\infty \quad \text{in } \overline{\Omega} \tag{5}$$

(A.2) the function σ satisfies a log-Hölder continuity property and there are real numbers σ^- and σ^+ such that:

$$0 < \sigma^- \equiv \min_{x\in\overline{\Omega}} \sigma(x) \leqslant \sigma(x) \leqslant \max_{x\in\overline{\Omega}} \sigma(x) \equiv \sigma^+ < \min_{x\in\overline{\Omega}} \frac{p_0(x)n}{n-p_0(x)} \quad \text{in } \overline{\Omega} \tag{6}$$

We specify the asymptotic behavior of the growth functions $\{p_\varepsilon\}$:

(A.3) the sequence $\{p_\varepsilon\}$ converges uniformly in Ω to a function p_0 satisfying a log-Hölder continuity property.

Function K_ε describes the high-contrasted medium. It is a measurable function in Ω such that:

(K.1) there exists a real number k_0 such that $0 < k_0 \leqslant K_\varepsilon(x) \leqslant k_0^{-1}$ in Ω_f^ε;
(K.2) for any $\varepsilon > 0$, there exists a real number k_ε such that $\sup_{x\in\Omega_m^\varepsilon} K_\varepsilon(x) = \mathrm{k}_\varepsilon > 0$ and $\mathrm{k}_\varepsilon \to 0$ as $\varepsilon \to 0$.

We now specify the microstructure of the domain. We make the following assumptions:

(C.1) the local concentration of the set Ω_f^ε has a positive continuous limit: there exists a continuous positive function ρ such that, for any open cube K_h^x centered at $x \in \Omega$ with lengths equal to $h > 0$

$$\lim_{h\to 0} \lim_{\varepsilon\to 0} h^{-n} \operatorname{meas}\left(K_h^x \cap \Omega_f^\varepsilon\right) = \rho(x)$$

(C.2) for any $\varepsilon > 0$, there is a constant $\mathrm{C}_{p_\varepsilon} \geqslant 0$ such that, if the function $p_\varepsilon^\star$ is defined by $p_\varepsilon^\star = p_\varepsilon - \mathrm{C}_{p_\varepsilon}$ in Ω, then:
  (i) $\lim_{\varepsilon\to 0} \mathrm{C}_{p_\varepsilon} = 0$;
  (ii) there exists a family of extension operators $\mathbf{P}^\varepsilon : W^{1,p_\varepsilon^\star(\cdot)}(\Omega_f^\varepsilon) \to W^{1,p_\varepsilon^\star(\cdot)}(\Omega)$ such that, for any $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)$,

$$\mathbf{P}^\varepsilon v^\varepsilon = v^\varepsilon \quad \text{in } \Omega_f^\varepsilon \quad \text{and} \quad \left\|\mathbf{P}^\varepsilon v^\varepsilon\right\|_{W^{1,p_\varepsilon^\star(\cdot)}(\Omega)} \leqslant \Phi\left(\left\|v^\varepsilon\right\|_{W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)}\right)$$

  where Φ is a strictly monotone continuous function in $\mathbb{R}^+$ such that $\Phi(0) = 0$ and $\Phi(t) \to +\infty$ as $t \to +\infty$.

Condition (C.2) extends the classical result of [13] to variable exponent Sobolev spaces.
The local characteristic of the sets Ω_f^ε and Ω_m^ε associated to the functional (3) are described by:

- the functional $c_{p_\varepsilon(\cdot)}^{\varepsilon,h}$ associated to the energy in Ω_f^ε, defined for $z \in \Omega$, $\mathbf{a} \in \mathbb{R}^n$ by

$$c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z;\mathbf{a}) \stackrel{\text{def}}{=} \inf_{v^\varepsilon} \int_{K_h^z \cap \Omega_f^\varepsilon} \left(\varkappa_\varepsilon(x)\left|\nabla v^\varepsilon(x)\right|^{p_\varepsilon(x)} + h^{-p_\varepsilon(x)-\gamma}\left|v^\varepsilon(x) - (x-z,\mathbf{a})\right|^{p_\varepsilon(x)}\right) \mathrm{d}x \tag{7}$$

  where $\gamma \in \mathbb{R}^+$, and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z \cap \Omega_f^\varepsilon)$;
- the functional $b_{p_\varepsilon(\cdot)}^{\varepsilon,h}$ associated to the energy exchange between Ω_f^ε and Ω_m^ε, defined for $z \in \Omega$, $\beta \in \mathbb{R}$ by

$$b_{p_\varepsilon(\cdot)}^{\varepsilon,h}(z;\beta) \stackrel{\text{def}}{=} \inf_{w^\varepsilon} \int_{K_h^z} \left(\varkappa_\varepsilon(x)\left|\nabla w^\varepsilon\right|^{p_\varepsilon(x)} + \frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)}\left|w^\varepsilon\right|^{\sigma(x)} + h^{-p_\varepsilon(x)-\gamma}\mathbf{1}_f^\varepsilon(x)\left|w^\varepsilon - \beta\right|^{p_\varepsilon(x)}\right) \mathrm{d}x \tag{8}$$

  the infimum being taken over $w^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(K_h^z)$.

We assume that the local characteristics of Ω are such that, for any sequence $\{p_\varepsilon\}_{(\varepsilon>0)}$ satisfying (A.1)–(A.3):

(C.3) for any $x \in \Omega$, $\mathbf{a} \in \mathbb{R}^n$, there is a continuous function $\mathsf{A}(x,\mathbf{a})$ and a real number γ_0, $0 < \gamma_0 < \mathsf{p}^-$, such that

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}}\, h^{-n} c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(x,\mathbf{a}) = \lim_{h\to 0} \underline{\lim_{\varepsilon\to 0}}\, h^{-n} c_{p_\varepsilon(\cdot)}^{\varepsilon,h}(x,\mathbf{a}) = \mathsf{A}(x,\mathbf{a}) \tag{9}$$

(C.4) for any $x \in \Omega$, $\beta \in \mathbb{R}$, there is a continuous function $\mathsf{b}(x,\beta)$ and a real number γ_1, $0 < \gamma_1 < \mathsf{p}^-$, such that

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\, h^{-n} b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(x,\beta) = \lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}}\, h^{-n} b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(x,\beta) = \mathsf{b}(x,\beta) \tag{10}$$

Contrary to the standard growth setting as considered in [7], the local characteristic $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z;\beta)$ is not homogeneous with respect to the parameter β. This induces the appearance of a nonlinear function $\mathsf{b}(x,u)$ in the homogenized functional which is described in the following theorem.

Theorem 2.1. *Let (u^ε) be a sequence satisfying* (2). *Under the standing assumptions, the solution u^ε to* (2) *converges strongly in $L^{p_0(\cdot)}(\Omega^\varepsilon_f)$ to u, solution to the following variational problem*:

$$J_{\text{hom}}[u] \to \min, \quad u \in W^{1,p_0(\cdot)}_0(\Omega) \tag{11}$$

$$J_{\text{hom}}[u] \stackrel{\text{def}}{=} \begin{cases} \int_\Omega (\mathsf{A}(x,\nabla u) + \frac{\rho(x)}{\sigma(x)}|u|^{\sigma(x)} + \mathsf{b}(x,u) - g(x)\rho(x)u)\,\mathrm{d}x & \text{if } u \in W^{1,p_0(\cdot)}_0(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{12}$$

Moreover, for any smooth function ζ in Ω, we have:

$$\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon_m}\left(\frac{1}{p_\varepsilon(x)}\left|u^\varepsilon\right|^{\sigma(x)-2}\left(u(x)u^\varepsilon - \left|u^\varepsilon\right|^2\right) + \frac{1}{\sigma(x)}\left|u^\varepsilon\right|^{\sigma(x)}\right)\zeta(x)\,\mathrm{d}x = \int_\Omega \mathsf{b}(x,u)\zeta(x)\,\mathrm{d}x \tag{13}$$

Note that the existence of the unique solution $u \in W^{1,p_0(\cdot)}(\Omega)$ to (11) follows from the previous theorem. Indeed, using (C.3)–(C.4), one states that the functions $\mathsf{A}(x,\mathbf{a})$ and $\mathsf{b}(x,\beta)$ are convex and locally Lipschitz with respect to their second variable, and $|\mathsf{A}(x,\mathbf{a})| \leqslant C\ |\mathbf{a}|^{p_0(x)}$, $|\mathsf{b}(x,\beta)| \leqslant C\ |\beta|^{\sigma(x)}$ $\forall x \in \Omega$. The continuity of J_{hom} in the space $W^{1,p_0(\cdot)}(\Omega)$ follows from the latter results. The uniqueness of the solution to (11) immediately follows from the strict convexity of J_{hom}. Finally, a minimizer of (12) is constructed in the proof of Theorem 2.1.

3. Auxiliary results

In this section, under the standing assumptions, we construct a convenient approximation for the solution to (3).

Lemma 3.1. *For each $h > 0$, there exist a set $\mathcal{B}^{\varepsilon,h} \subset \Omega^\varepsilon_f$ and a function $Y^{\varepsilon,h} \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ such that*:

(i) $0 \leqslant Y^{\varepsilon,h}(x) \leqslant 1$ *in* Ω *and* $Y^{\varepsilon,h}(x) = 1$ *in* $\Omega^\varepsilon_f \setminus \mathcal{B}^{\varepsilon,h}$;

(ii) $\overline{\lim}_{\varepsilon\to 0}\operatorname{meas}\mathcal{B}^{\varepsilon,h} = O(h^{\frac{\gamma}{\mathsf{p}^+ + 1}})$ *as* $h \to 0$;

(iii) *for any function $w \in C^1_0(\Omega)$, we have, as $h \to 0$,*

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega\left(\varkappa_\varepsilon(x)\left|w\nabla Y^{\varepsilon,h}\right|^{p_\varepsilon(x)} + \frac{1}{\sigma(x)}\left|wY^{\varepsilon,h}\right|^{\sigma(x)}\right)\mathrm{d}x \leqslant \int_\Omega\left(\mathsf{b}(x,w) + \frac{\rho(x)}{\sigma(x)}|w|^{\sigma(x)}\right)\mathrm{d}x + o(1) \tag{14}$$

Lemma 3.2. *Let $\mathcal{B}^{\varepsilon,h}$ be the set defined in Lemma* 3.1. *Let $w \in C^1_0(\Omega)$. Then there are a set $\mathcal{D}^{\varepsilon,h} \subset \Omega$ and a function $V^{\varepsilon,h} = V^{\varepsilon,h}(\cdot,w) \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ such that*

(i) $\mathcal{B}^{\varepsilon,h} \subset \mathcal{D}^{\varepsilon,h}$ *and* $\overline{\lim}_{\varepsilon\to 0}\operatorname{meas}\mathcal{D}^{\varepsilon,h} = o(1)$ *as* $h \to 0$;

(ii) $\max_{x\in\Omega}|V^{\varepsilon,h}(x) - w(x)| \leqslant Ch$;

(iii) *the following relations hold true as $h \to 0$:*

$$\overline{\lim_{\varepsilon\to 0}}\int_{\mathcal{D}^{\varepsilon,h}\cup\Omega^\varepsilon_m}\varkappa_\varepsilon(x)\left|\nabla V^{\varepsilon,h}\right|^{p_\varepsilon(x)} = o(1), \qquad \overline{\lim_{\varepsilon\to 0}}\int_{\Omega^\varepsilon_f}\varkappa_\varepsilon(x)\left|\nabla V^{\varepsilon,h}\right|^{p_\varepsilon(x)} \leqslant \int_\Omega A(x,\nabla w) + o(1) \tag{15}$$

Lemma 3.3. *Let $\{p_\varepsilon^\star\}_{(\varepsilon>0)}$ be the sequence of functions defined in condition* (C.2) *and let the sequence $\{\pi_\varepsilon^\star\}_{(\varepsilon>0)}$ defined in Ω by $\pi_\varepsilon^\star=\min\{p_\varepsilon^\star, p_0\}$. Assume that a sequence $\{u^\varepsilon\}_{(\varepsilon>0)}\subset W_0^{1,p_\varepsilon^\star(\cdot)}(\Omega)$ converges to a function $u\in C_0^1(\Omega)$ in $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ and, moreover, $\int_\Omega(\mathbf{1}_f^\varepsilon(x)\varkappa_\varepsilon(x)|\nabla u^\varepsilon|^{p_\varepsilon(x)}+\frac{1}{\sigma(x)}|u^\varepsilon|^{\sigma(x)})\,\mathrm{d}x\leqslant C$. Then there are a set $\mathcal{G}^\varepsilon\subset\Omega$ with $\Omega_m^\varepsilon\subset\mathcal{G}^\varepsilon$, a function $\hat u^\varepsilon$ and a subsequence $\varepsilon_k\to 0$ (still denoted by ε for convenience) such that*

(i) $\lim_{\varepsilon\to 0}\operatorname{meas}\mathcal{G}_f^\varepsilon=0$, *where* $\mathcal{G}_f^\varepsilon=\mathcal{G}^\varepsilon\cap\Omega_f^\varepsilon$;
(ii) $\hat u^\varepsilon=u^\varepsilon$ *in* $\Omega_f^\varepsilon\setminus\mathcal{G}_f^\varepsilon$ *and, moreover,* $\lim_{\varepsilon\to 0}\|\hat u^\varepsilon\|_{W^{1,\pi_\varepsilon^\star(\cdot)}(\mathcal{G}_f^\varepsilon)}=0$;
(iii) *the following inequality holds true*:

$$\lim_{\varepsilon\to 0}\int_{\mathcal{G}^\varepsilon}\left(\varkappa_\varepsilon(x)\left|\nabla u^\varepsilon\right|^{\pi_\varepsilon^\star(x)}+\frac{\mathbf{1}_m^\varepsilon(x)}{\sigma(x)}\left|u^\varepsilon\right|^{\sigma(x)}\right)\mathrm{d}x\geqslant\int_\Omega \mathsf{b}(x,u)\,\mathrm{d}x \tag{16}$$

4. Proof of Theorem 2.1

The minimizer u^ε of problem (2) is such that $\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega_f^\varepsilon)}\leqslant C$. It follows from (C.2) that there is a function $\mathsf{u}^\varepsilon=\mathbf{P}^\varepsilon u^\varepsilon$ such that $\mathsf{u}^\varepsilon=u^\varepsilon$ in Ω_f^ε and $\|\mathsf{u}^\varepsilon\|_{W^{1,p_\varepsilon^\star(\cdot)}(\Omega)}\leqslant C$. Since p_ε converges uniformly to p_0, then there exists a parameter ς that does not depend on ε such that $\|\mathsf{u}^\varepsilon\|_{W^{1,p_0(\cdot)-\varsigma}(\Omega)}\leqslant C$ and the family $\{\mathsf{u}^\varepsilon\}_{(\varepsilon>0)}$ is a compact set in the space $L^{p_0(\cdot)}(\Omega)$. Then one can extract a subsequence (still denoted by $\{\mathsf{u}^\varepsilon\}$) which converges to a function $u\in L^{p_0(\cdot)}(\Omega)$. In particular,

$$u^\varepsilon\to u\quad\text{in } L^{p_0(\cdot)}\big(\Omega_f^\varepsilon\big) \tag{17}$$

It remains to show that u is the solution to the homogenized problem (11). The proof will be done in three steps. In step 1, we prove that $\overline{\lim}\,J^\varepsilon[u^\varepsilon]\leqslant J_{\mathrm{hom}}[w]$ for any $w\in W^{1,p_0(\cdot)}(\Omega)$. Step 2 is devoted to the proof of the inequality $\underline{\lim}\,J^\varepsilon[u^\varepsilon]\geqslant J_{\mathrm{hom}}[u]$. Thus u is the minimizer of functional J_{hom} in $W_0^{1,p_0(\cdot)}(\Omega)$. Finally, we prove in step 3 the weak convergence of $\mathbf{1}_m^\varepsilon(\frac{1}{p_\varepsilon}|u^\varepsilon|^{\sigma(\cdot)-2}(uu^\varepsilon-|u^\varepsilon|^2)+\frac{1}{\sigma}|u^\varepsilon|^{\sigma(\cdot)})$ to $\mathsf{b}(\cdot,u)$.

Step 1. *Upper bound.* Thanks to density arguments it is sufficient to state the result for an arbitrary function $w\in C_0^1(\Omega)$. Let $Y^{\varepsilon,h}$, $V^{\varepsilon,h}$, $\mathcal{D}^{\varepsilon,h}$ be the corresponding functions and set defined in Lemmas 3.1 and 3.2. We define the function $T^{\varepsilon,h}\in W^{1,p_\varepsilon(\cdot)}(\Omega)$ by

$$T^{\varepsilon,h}(x)\stackrel{\mathrm{def}}{=}Y^{\varepsilon,h}(x)V^{\varepsilon,h}(x),\quad x\in\Omega \tag{18}$$

Since u^ε minimizes the functional J^ε, $J^\varepsilon[u^\varepsilon]\leqslant J^\varepsilon[T^{\varepsilon,h}]$ and it is sufficient to prove that

$$\overline{\lim_{h\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\ J^\varepsilon\big[T^{\varepsilon,h}\big]\leqslant J_{\mathrm{hom}}[w] \tag{19}$$

Let us enumerate the basic ingredients leading to (19). First, it follows from condition (C.1), assertions (i)–(ii) of Lemma 3.1, and assertion (ii) of Lemma 3.2 that

$$\lim_{h\to 0}\ \overline{\lim_{\varepsilon\to 0}}\int_\Omega g^\varepsilon(x)T^{\varepsilon,h}(x)\,\mathrm{d}x=\int_\Omega g(x)\rho(x)\,w(x)\,\mathrm{d}x \tag{20}$$

It follows from assertion (i) of Lemma 3.1 and assertion (i) of Lemma 3.2 that

$$\overline{\lim_{h\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\int_\Omega\frac{1}{\sigma(x)}\left|Y^{\varepsilon,h}\right|^{\sigma(x)}\left|\left|V^{\varepsilon,h}\right|^{\sigma(x)}-|w|^{\sigma(x)}\right|\mathrm{d}x=0 \tag{21}$$

We note that $\nabla T^{\varepsilon,h}=\nabla V^{\varepsilon,h}$ in $\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h}$. Thus we have

$$\overline{\lim_{h\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\int_{\Omega_f^\varepsilon\setminus\mathcal{D}^{\varepsilon,h}}\varkappa_\varepsilon(x)\left|\nabla T^{\varepsilon,h}\right|^{p_\varepsilon(x)}\mathrm{d}x\leqslant\int_\Omega \mathsf{A}(x,\nabla w)\,\mathrm{d}x \tag{22}$$

We decompose the remaining term in $J^\varepsilon[T^{\varepsilon,h}]$ as follows:

$$\int_{\Omega_m^\varepsilon} \varkappa_\varepsilon(x)\big|\nabla T^{\varepsilon,h}\big|^{p_\varepsilon(x)}\,\mathrm{d}x = \int_{\Omega_m^\varepsilon\cup\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)\big|\nabla Y^{\varepsilon,h}\big|^{p_\varepsilon(x)}\big(\big|V^{\varepsilon,h}\big|^{p_\varepsilon(x)} - \big|w(x)\big|^{p_\varepsilon(x)}\big)\,\mathrm{d}x + \int_{\Omega_m^\varepsilon\cup\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)\big(\big|\nabla T^{\varepsilon,h}\big|^{p_\varepsilon(x)} - \big|V^{\varepsilon,h}\nabla Y^{\varepsilon,h}\big|^{p_\varepsilon(x)}\big)\,\mathrm{d}x + \int_{\Omega_m^\varepsilon\cup\mathcal{D}^{\varepsilon,h}} \varkappa_\varepsilon(x)\big|w(x)\nabla Y^{\varepsilon,h}\big|^{p_\varepsilon(x)}\,\mathrm{d}x \tag{23}$$

Finally, it follows from (22)–(23), assertion (iii) of Lemma 3.1, assertion (ii) of Lemma 3.2 and Hölder's inequality that

$$\int_{\Omega} \varkappa_\varepsilon(x)\big|\nabla T^{\varepsilon,h}\big|^{p_\varepsilon(x)}\,\mathrm{d}x \leqslant \int_{\Omega} \mathsf{A}(x,\nabla w)\,\mathrm{d}x + \int_{\Omega} \varkappa_\varepsilon(x)\big|w(x)\nabla Y^{\varepsilon,h}\big|^{p_\varepsilon(x)}\,\mathrm{d}x + \mathbf{j}_1^{\varepsilon,h} \tag{24}$$

where $\overline{\lim}_{h\to 0}\,\overline{\lim}_{\varepsilon\to 0}\,|\mathbf{j}_1^{\varepsilon,h}| = 0$. Now inequality (19) immediately follows from (20), (21), (24) and assertion (iii) of Lemma 3.1.

Step 2. *Lower bound.* In view of Lemma 3.3, we aim to reduce the problem to the case where $u \in C_0^1(\Omega)$. But the functional J^ε is not continuous in the $W^{1,p_0(\cdot)}$ topology unless we restrict ourself to the case when $p_\varepsilon \leqslant p_0$. So, we note that

$$\varliminf_{\varepsilon\to 0} J^\varepsilon\big[u^\varepsilon\big] \geqslant \varliminf_{\varepsilon\to 0}\int_{\Omega} \varkappa_\varepsilon(x)\big|\nabla u^\varepsilon\big|^{\pi_\varepsilon^\star(x)}\big(\big|\nabla u^\varepsilon\big|^{p_\varepsilon(x)-\pi_\varepsilon^\star(x)} - 1\big)\,\mathrm{d}x + \varliminf_{\varepsilon\to 0} J^{\pi_\varepsilon^\star}\big[u^\varepsilon\big]$$

where

$$J^{\pi_\varepsilon^\star}[u] \stackrel{\text{def}}{=} \begin{cases} \int_\Omega \big(\varkappa_\varepsilon(x)|\nabla u|^{\pi_\varepsilon^\star(x)} + \frac{1}{\sigma(x)}|u|^{\sigma(x)} - g^\varepsilon(x)\,u\big)\,\mathrm{d}x \stackrel{\text{def}}{=} \int_\Omega \mathsf{F}_{\pi_\varepsilon^\star}(x,u^\varepsilon,\nabla u^\varepsilon)\,\mathrm{d}x & \text{if } u\in W^{1,\pi_\varepsilon^\star(\cdot)}(\Omega)\\ +\infty & \text{otherwise}\end{cases}$$

Since $\max_{0<B<1}(-B^{\pi_\varepsilon^\star(x)}(B^{p_\varepsilon(x)-\pi_\varepsilon^\star(x)}-1)) \geqslant C(\varepsilon)$, for any $x\in\Omega$, with $C(\varepsilon)\to 0$ as $\varepsilon\to 0$, it is now sufficient to prove that $\varliminf_{\varepsilon\to 0} J^{\pi_\varepsilon^\star}[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u]$.

First let u be an arbitrary $C_0^1(\Omega)$ function and $\{u^\varepsilon\}_{(\varepsilon>0)}$ be a sequence which converges to the function u strongly in $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ and such that $J^{\pi_\varepsilon^\star}[u^\varepsilon] \leqslant C$. We will show that

$$\varliminf_{\varepsilon\to 0} J^{\pi_\varepsilon^\star}\big[u^\varepsilon\big] \geqslant J_{\mathrm{hom}}[u] \tag{25}$$

Let $\{x^\alpha\}$ be a set of points in Ω forming an h-periodic space lattice. We cover the domain Ω by cubes $K_h^{x^\alpha}$ with non-intersecting interiors. We now apply Lemma 3.3 to the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ and the function u. Using the regularity of u and the strong convergence of the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ to u in $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$, we check that

$$\varliminf_{\varepsilon\to 0}\sum_\alpha \int_{K_h^\alpha\cap(\Omega_f^\varepsilon\setminus\mathcal{G}_f^\varepsilon)} \varkappa_\varepsilon(x)\big|\nabla\hat{u}^\varepsilon\big|^{\pi_\varepsilon^\star(x)}\,\mathrm{d}x \geqslant \varliminf_{\varepsilon\to 0}\sum_\alpha c^{\varepsilon,h}_{\pi_\varepsilon^\star(\cdot)}\big(x^\alpha,\nabla u\big(x^\alpha\big)\big) - \varliminf_{\varepsilon\to 0} k_0^{-1}\int_{\mathcal{G}_f^\varepsilon}\big|\nabla\hat{u}^\varepsilon\big|^{\pi_\varepsilon^\star(x)}\,\mathrm{d}x + o(1)\quad \text{as } h\to 0$$

It follows from the latter relation and Lemma 3.3(ii) that

$$\varliminf_{\varepsilon\to 0} J^{\pi_\varepsilon^\star}\big[u^\varepsilon\big] \geqslant \varliminf_{\varepsilon\to 0}\bigg(\sum_\alpha c^{\varepsilon,h}_{\pi_\varepsilon^\star(\cdot)}\big(x^\alpha,\nabla u\big(x^\alpha\big)\big) + \sum_\alpha \int_{K_h^\alpha\cap\Omega_f^\varepsilon}\bigg(\frac{1}{\sigma(x)}\big|u^\varepsilon\big|^{\sigma(x)} - g^\varepsilon(x)u^\varepsilon\bigg)\,\mathrm{d}x + \int_{\mathcal{G}^\varepsilon}\mathsf{F}_{\pi_\varepsilon^\star}\big(x,u^\varepsilon,\nabla u^\varepsilon\big)\,\mathrm{d}x\bigg) + o(1)\quad \text{as } h\to 0 \tag{26}$$

We pass to the limit in the inequality (26) first as $\varepsilon \to 0$ and then as $h \to 0$. Taking into account the strong convergence of the sequence $\{u^\varepsilon\}_{(\varepsilon>0)}$ to u in the space $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$, the properties of the function p_ε, conditions (C.1), (C.3), and Lemma 3.3 we obtain (25).

This result in $C_0^1(\Omega)$ remains true in $W^{1,p_0(\cdot)}(\Omega)$ because the family $\{J^{\pi_\varepsilon^\star}\}$ is uniformly in ε continuous in the $W^{1,p_0(\cdot)}(\Omega)$ topology. In addition, as emphasized after the main theorem, the functional J_{hom} is continuous in the $W^{1,p_0(\cdot)}(\Omega)$ topology. This completes the proof of the "lim inf"-inequality for the functional $J^{\pi_\varepsilon^\star}$, and thus for J^ε.

Step 3. *Convergence result* (13) *in the matrix part.* Suppose that the solution u of the homogenized problem is a sufficiently smooth function (if not we use smooth approximations of u to construct $\tilde u^\varepsilon$). Let $\tilde u^\varepsilon$ be the function defined in (18) with $w = u$. Since u^ε is the solution to the variational problem (2), then

$$\int_\Omega \left(-\operatorname{div}\left(K_\varepsilon(x)\left|\nabla u^\varepsilon\right|^{p_\varepsilon(x)-2}\nabla u^\varepsilon\right) + \left|u^\varepsilon\right|^{\sigma(x)-2}u^\varepsilon\right)\left(\tilde u^\varepsilon - u^\varepsilon\right)\mathrm{d}x = \int_\Omega g^\varepsilon(x)\left(\tilde u^\varepsilon - u^\varepsilon\right)\mathrm{d}x \tag{27}$$

It also follows from steps 1–2 that $\lim_{\varepsilon\to 0} J^\varepsilon[\tilde u^\varepsilon] - \lim_{\varepsilon\to 0} J^\varepsilon[u^\varepsilon] = 0$. This relation together with (27) implies:

$$\lim_{\varepsilon\to 0}\left\|\tilde u^\varepsilon - u^\varepsilon\right\|_{L^{\sigma(\cdot)}(\Omega)} = 0 \tag{28}$$

Consider now the functional $b^{\varepsilon,h}_{p_\varepsilon(\cdot)}(z;\beta)$ defined in (8). It is clear that the minimizer $w_z^{\varepsilon,h}$ of the functional (8) satisfies the Neumann boundary value problem for the following equation:

$$\begin{aligned}&-\operatorname{div}\left(K_\varepsilon(x)\nabla w_z^{\varepsilon,h}\left|\nabla w_z^{\varepsilon,h}\right|^{p_\varepsilon(x)-2}\right) + \mathbf{1}_m^\varepsilon(x) w_z^{\varepsilon,h}\left|w_z^{\varepsilon,h}\right|^{\sigma(x)-2}\\ &\quad + p_\varepsilon(x)h^{-p_\varepsilon(x)-\gamma}\mathbf{1}_f^\varepsilon(x)\left(w^\varepsilon-\beta\right)\left|w^\varepsilon-\beta\right|^{p_\varepsilon(x)-2} = 0 \quad \text{in } K_h^z\end{aligned} \tag{29}$$

Using (29) and condition (C.4), we prove that

$$\mathsf{b}(x,\beta) = \lim_{\varepsilon\to 0} h^{-n}(\varepsilon)\left(\int_{K_h^z}\frac{1}{p_\varepsilon(x)}\left|w_z^{\varepsilon,h}\right|^{\sigma(x)-2}\left(\beta w_z^{\varepsilon,h} - \left|w_z^{\varepsilon,h}\right|^2\right)\mathbf{1}_m^\varepsilon(x) + \int_{K_h^z}\frac{1}{\sigma(x)}\left|w_z^{\varepsilon,h}\right|^{\sigma(x)}\mathbf{1}_m^\varepsilon(x)\right)$$

and then

$$\lim_{\varepsilon\to 0}\int_{\Omega_m^\varepsilon}\left(\frac{1}{p_\varepsilon(x)}\left|\tilde u^\varepsilon\right|^{\sigma(x)-2}\left(u\tilde u^\varepsilon - \left|\tilde u^\varepsilon\right|^2\right) + \frac{1}{\sigma(x)}\left|\tilde u^\varepsilon\right|^{\sigma(x)}\right)\zeta(x)\,\mathrm{d}x = \int_\Omega \mathsf{b}(x,u)\zeta(x)\,\mathrm{d}x \tag{30}$$

for any smooth function ζ. Relation (13) follows from (28) and (30). This completes the proof of Theorem 2.1.

5. Periodic example

It is important to show that the "intersection" of the conditions (A.1)–(A.3), (K.1)–(K.2), (C.1)–(C.4) is not empty. We thus illustrate our result with a periodical example. We assume that, in the standard periodic cell $\mathcal{Y} = (0,1)^n$, there is an obstacle $\mathcal{M} \subset \mathcal{Y}$ with Lipschitz boundary $\partial\mathcal{M}$. We assume that this geometry is repeated periodically in the whole $\mathbb{R}^n$. The geometric structure within the domain Ω is then obtained by intersecting the ε-multiple of this geometry with Ω. Let p_0 be a *log-Hölder continuous* function such that $2 < \mathsf{p}^- \equiv \min_{x\in\overline{\Omega}} p_0(x) \leqslant p_0(x) \leqslant \max_{x\in\overline{\Omega}} p_0(x) \equiv \mathsf{p}^+ < +\infty$ in $\overline{\Omega}$. Let $\{p_\varepsilon\}_{(\varepsilon>0)}$ be a sequence satisfying (A.1)–(A.3), defined by

$$p_\varepsilon(x) \stackrel{\text{def}}{=} p_0(x) + \mathbf{d}_\varepsilon(x), \qquad \mathbf{d}_\varepsilon = o(1) \quad \text{as } \varepsilon \to 0, \qquad \lim_{\varepsilon\to 0}\varepsilon^{-\mathbf{d}_\varepsilon(\cdot)} = \mathbf{d}(\cdot) \tag{31}$$

We denote by $\mathsf{u}^a = \mathsf{u}^a(x,y)$ and $\mathsf{w}^\beta = \mathsf{w}^\beta(x,y)$ the unique solutions in respectively $W_\#^{1,p_0(\cdot)}(\mathcal{F})$ and $W_\#^{1,p_0(\cdot)}(\mathcal{M})$ of the following cell problems:

$$\begin{aligned}&\operatorname{div}_y\left(k_f\left|\nabla_y \mathsf{u}^a\right|^{p_0(x)-2}\nabla_y\mathsf{u}^a\right) = 0 \quad \text{in } \mathcal{F}\\ &\left(k_f\left|\nabla_y\mathsf{u}^a\right|^{p_0(x)-2}\nabla_y\mathsf{u}^a - \mathbf{a}, \boldsymbol{\nu}_\mathcal{M}\right) = 0 \quad \text{on } \partial\mathcal{M}; \qquad y \to \mathsf{u}^a(y) \quad \mathcal{Y}\text{-periodic}\end{aligned} \tag{32}$$

$$\begin{aligned}&-\operatorname{div}_y\left(k_m\mathbf{d}(x)\left|\nabla_y\mathsf{w}^\beta\right|^{p_0(x)-2}\nabla_y\mathsf{w}^\beta\right) + \left|\mathsf{w}^\beta\right|^{p_0(x)-2}\mathsf{w}^\beta = 0 \quad \text{in } \mathcal{M}\\ &\mathsf{w}^\beta(y) = \beta \quad \text{on } \partial\mathcal{M}; \qquad y \to \mathsf{w}^\beta(y) \quad \mathcal{Y}\text{-periodic}\end{aligned} \tag{33}$$

where $\mathcal{F} = \mathcal{Y} \setminus \overline{\mathcal{M}}$, $\boldsymbol{\nu}_{\mathcal{M}}$ is the outward normal vector to $\partial\mathcal{M}$, $\mathbf{a} \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$. In the cell problems (32) and (33) x is a parameter. Regularity results for u^a and w^β are thus easily deduced from [14]. The following result holds.

Theorem 5.1. *Under the aforementioned assumptions* (*see the beginning of the present section and especially* (31))*, the solution u^ε of* (2) *converges strongly in $L^{p_0(\cdot)}(\Omega_f^\varepsilon)$ to u the solution to the variational problem*:

$$J_{\mathrm{hom}}[u] = \int_\Omega \left(\mathsf{A}(x, \nabla u) + \frac{\rho^\star}{\sigma(x)} |u|^{\sigma(x)} + \mathsf{b}(x, u) - g(x)\rho^\star u \right) \mathrm{d}x \to \min, \quad u \in W_0^{1,p_0(\cdot)}(\Omega)$$

where $\rho^\star = \operatorname{meas} \mathcal{F}$, $\mathsf{A}(x, \mathbf{a}) = \frac{1}{p_0(x)} \int_{\mathcal{F}} |\nabla_y \mathsf{u}^a(x, y) - \mathbf{a}|^{p_0(x)}\, \mathrm{d}y$, $\mathsf{b}(x, \beta) = \int_{\mathcal{M}} (\frac{1}{p_0(x)} (\beta \mathsf{w}^\beta |\mathsf{w}^\beta|^{\sigma(x)-2} - |\mathsf{w}^\beta|^{\sigma(x)}) + \frac{1}{\sigma(x)} |\mathsf{w}^\beta|^{\sigma(x)})\, \mathrm{d}y$.

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