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Boundary conditions for the high order homogenized equation: laminated rods, plates and composites

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Abstract

The high order homogenization technique generates the so called infinite order homogenized equation. Its coefficients were widely discussed in composite mechanics literature because they are closely related to the so called high order strain gradients theories. However, it was not clear what is the correct mathematical setting for this equation and what are the asymptotically exact boundary conditions. In the present Note we give a variational formulation for the high order homogenized equation by the projection of the initial problem on the "ansatz subspace". This formulation generates the appropriate boundary conditions for the high order homogenized equation. The error estimates for the solution of the original problem and the homogenized one are obtained. ***To cite this article: G. Panasenko, C. R. Mecanique 337 (2009).***

Résumé

Conditions aux limites pour l'équation homogénéisée d'ordre élévé : barres, plaques et composites stratifiés. La technique d'homogénéisation d'ordre élévé mène à l'équation homogénéisée d'ordre élévé. Ses coefficients ont été largement discutés dans la literature de la mécanique des composites parce qu'ils sont liés aux théories des gradients des déformations d'ordre élévé. Neanmoins, la nature mathématique de cette équation n'a pas été complètement clarifiée et les conditions aux limites asymptotiquement exactes n'ont pas été définies. Dans cette Note nous donnons la formulation variationnelle de l'équation homogénéisée d'ordre élévé. Cette formulation est dérivée par la projection du problème initial sur l'espace du dévéloppement asymptotique. Elle engendre les conditions aux limites appropriées pour l'équation homogénéisée d'ordre élévé. L'estimation de la difference entre la solution exacte et la solution approchée est obtenue. ***Pour citer cet article : G. Panasenko, C. R. Mecanique 337 (2009).***

Keywords: Homogenization; High order homogenized equation; High order boundary conditions; Strain gradient theories

Mots-clés : Homogénéisation ; Equation homogénéisée d'ordre élévé ; Conditions aux limites d'ordre élévé ; Théories des gradients des déformations

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1. Introduction

The high order homogenized equations are well known in the structural mechanics and mechanics of composites in the form of theories involving higher derivatives (gradients) of the unknown function, for example, displacement (see [1–3]). These theories are expected to be more precise than the classical ones; they may describe some special properties of the continuum such as, the dispersion or the dissipation. These models can be constructed phenomenologically; however the derivation of these theories at the macroscopic scale from the standard microscopic models is still a very delicate question. On the other hand, in the plate theory there were proposed some special boundary conditions called "artificial" which better take into account the higher asymptotic approximations of the three-dimensional solution (see [4,5] where the second order accuracy has been obtained). These two questions can be related, because the arbitrary high order macroscopic approximation for a microstructured composite material is exactly the high order homogenized equation proposed by N. Bakhvalov in [6], see also [7]. This approach has been then extended for the thin structures in [8,9] and in [10]. It leads to the infinite order homogenized equation which formally is a pseudo-differential equation with small higher order coefficients. This equation has not been interpreted mathematically although there were a lot of mechanical interpretations of the infinite order equation and its coefficients. In particular, one can find the variational properties of the infinite order formally homogenized equations [10], the numerical study of the sign of the coefficients, [11,12], the discussion how to truncate this equation [13]. The problem is that the sign of the coefficient of the elder derivative after truncation may be "wrong" from the point of view of the well-posedness of the truncated equation. A part of this problem there is a question about the appropriate boundary conditions associated to the high order differential equation. For the equation set in the whole space a well-posed truncation procedure was proposed in [14]; it corresponds to some projection of the variational formulation on the "ansatz space". This approach automatically adds to the truncation some small stabilizing terms. However, to our knowledge, the problem is still open in the case of the boundary value problems: what should be the appropriate boundary conditions to the high order homogenized equation? Can they be local or they are always non-local? We will answer these questions by constructing a special representation for the boundary layer ansatz. We emphasize that the well posed setting of the boundary value problem for the high order homogenized equation is closely related to similar questions in high order strain gradient theories [15] where they are still open.

2. Model setting

Consider the model equation set in the half-strip $G_{\varepsilon+} = \{x \in \mathbb{R}^2,\ x_1 > 0,\ x_2 \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\}$

$$-\frac{\partial}{\partial x_k}\left(A_{kj}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right) = f, \quad x \in G_{\varepsilon+} \tag{1}$$

where $x = (x_1, x_2) \in G_{\varepsilon+}$ and $f = f(x_1)$ is $C^\infty-$smooth and it satisfies the inequality

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}x_1^l} f(x_1)\right| \leqslant c_1 \mathrm{e}^{-c_2 x_1}, \quad c_1, c_2 > 0,\ l = 0, 1, \ldots, K$$

ε is a small positive parameter. The coefficients $A_{kj}(\xi_2)$ are piecewise-smooth in the interval $[-\frac{1}{2}, \frac{1}{2}]$ having only the discontinuities of the first kind; for all $\xi_2 \in [-\frac{1}{2}, \frac{1}{2}]$ and for all $i, j \in \{1, 2\}$, $A_{kj}(\xi_2) = A_{jk}(\xi_2)$; we assume that there exists a positive constant $\kappa > 0$ such that, for any $(\eta_1, \eta_2) \in \mathbb{R}^2$, for all $\xi_2 \in [-\frac{1}{2}, \frac{1}{2}]$,

$$A_{kj}(\xi_2)\eta_j\eta_k \geqslant \kappa\eta_j\eta_j$$

We admit here the repeating indices convention, i.e. the summation is taken from 1 to 2 over the repeating indices. We set the boundary conditions

$$A_{2j}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j} = 0, \quad x_2 = \pm\frac{\varepsilon}{2},\ x_1 > 0 \tag{2}$$

$$u_\varepsilon = 0, \quad x_1 = 0 \tag{3}$$

Let us define the space $H^1_{0,\infty} = \{w \in H^1_{\mathrm{loc}},\ w|_{x_1=0} = 0,\ \nabla w \in (L^2(G_{\varepsilon+})^2\}$ and consider problem (1)–(3) with a more general right hand side $f = f_{0,\varepsilon} - \frac{\partial f_{j,\varepsilon}}{\partial x_j}$ defined in $G_{\varepsilon+}$ and satisfying

$$\left|f_{i,\varepsilon}(x_1,x_2)\right| \leqslant c_1 \mathrm{e}^{-c_2 x_1}, \quad c_1, c_2 > 0,\ i = 0,1,2$$

where $f_{i,\varepsilon}$ are piecewise-continuous functions.

Let us give the variational formulation, that is, find $u_\varepsilon \in H^1_{0,\infty}$ such that for any $w \in H^1_{0,\infty}$

$$\int_{G_{\varepsilon+}} A_{kj}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial w}{\partial x_k}\,\mathrm{d}x = \int_{G_{\varepsilon+}} \left(f_{0,\varepsilon}(x)w(x) + f_{j,\varepsilon}(x)\frac{\partial w}{\partial x_j}\right)\mathrm{d}x \tag{4}$$

Theorem 2.1. *There exists a unique solution of problem* (4)*. It satisfies the estimate* $\|\nabla u_\varepsilon\|_{L^2(G_{\varepsilon+})} \leqslant C(\|\int_{x_1}^{+\infty} f_{0,\varepsilon}(t)\,\mathrm{d}t\|_{L^2(G_{\varepsilon+})} + \sum_{j=1}^2 \|f_{j,\varepsilon}\|_{L^2(G_{\varepsilon+})})$*, where constant C is independent of* ε*.*

The proof of this theorem is analogous to [16]. The solution stabilizes at infinity to a constant [22].

3. Asymptotic expansion of the solution

Consider now problem (1)–(3) with $f = f(x_1)$. An asymptotic expansion of the solution can be constructed by the method of boundary layer in homogenization [17,18,9] (another ansatz has been proposed by E. Sanchez-Palencia in [19]and J.L. Lions in [20]):

$$u_\varepsilon^{a\,(K)} = \sum_{l=0}^{K+1} \varepsilon^l N_l\left(\frac{x_2}{\varepsilon}\right) D_1^l v_\varepsilon^{(K)}(x_1) + \sum_{l=1}^{K+1} \varepsilon^l N_l^{\mathrm{BL}}\left(\frac{x}{\varepsilon}\right) D_1^l v_\varepsilon^{(K)}(x_1) \tag{5}$$

where $N_l(\xi_2)$ are solutions of the sequence of cell problems:

$$\frac{\partial}{\partial \xi_2}\left(A_{22}\frac{\partial N_l}{\partial \xi_2}\right) = -\frac{\partial}{\partial \xi_2}(A_{21}N_{l-1}) - A_{12}\frac{\partial N_{l-1}}{\partial \xi_2} - A_{11}N_{l-2} + h_l,\ \xi_2 \in \left(-\frac{1}{2},\frac{1}{2}\right) \tag{6_1}$$

$$A_{22}\frac{\partial N_l}{\partial \xi_2} + A_{21}N_{l-1} = 0, \quad \xi_2 = \pm\frac{1}{2} \tag{6_2}$$

$$\langle N_l\rangle = 0 \quad \text{for } l > 0, \quad N_0 = 1 \tag{6_3}$$

$$h_l = \left\langle A_{12}\frac{\partial N_{l-1}}{\partial \xi_2} + A_{11}N_{l-2}\right\rangle \tag{7}$$

where $\langle\,\rangle = \int_{-1/2}^{1/2} \mathrm{d}\xi_2$, and by convention, N_l with negative subscripts l vanishes. Functions N_l^{BL} exponentially decay and tend to zero as $\xi_1 \to +\infty$ and they are described in [9,7], $h_0 = h_1 = 0,\ h_2 > 0$:

$$h_2 = \left\langle A_{12}\frac{\partial N_1}{\partial \xi_2} + A_{11}\right\rangle = \langle A_{12}A_{22}^{-1}\rangle\langle A_{22}^{-1}\rangle^{-1}\langle A_{22}^{-1}A_{21}\rangle + \langle A_{11} - A_{12}A_{22}^{-1}A_{21}\rangle \tag{8}$$

$\langle N_l\rangle = 0$ for $l > 0$, $N_0 = 1$, $D_1 = \frac{\partial}{\partial x_1}$.

However, for our objective we will construct below **another ansatz** modifying the boundary layer part as follows:

$$u_\varepsilon^{a\,(K)} = \sum_{l=0}^{K+1} \varepsilon^l N_l\left(\frac{x_2}{\varepsilon}\right) D_1^l v_\varepsilon^{(K)}(x_1) + \sum_{l=1}^{K+1} \varepsilon^l N_l^{\mathrm{BL0}}\left(\frac{x}{\varepsilon}\right) D_1^l v_\varepsilon^{(K)}\big|_{x_1=0} \tag{9}$$

$$v_\varepsilon^{(K)}(x_1) = \sum_{j=0}^{K+1} \varepsilon^j v_j(x_1) \tag{10}$$

where N_l^{BL0} are solutions of the following boundary value problems set in the half-strip $\Pi_+ = \{\xi \in \mathbb{R}^2 \mid \xi_1 \in (0,+\infty),\ \xi_2 \in (-\frac{1}{2},\frac{1}{2})\}$:

$$L_{\xi\xi} N_l^{\mathrm{BL0}} = 0, \quad \xi \in \Pi_+ \tag{11}$$

$$A_{2j} \frac{\partial N_l^{\mathrm{BL0}}}{\partial \xi_j} = 0, \quad \xi_2 = \pm \frac{1}{2} \tag{12}$$

$$N_l^{\mathrm{BL0}}(0, \xi_2) = -N_l(\xi_2) + h_l^{\mathrm{BL0}} \tag{13}$$

where the constants h_l^{BL0} such that,

$$N_l^{\mathrm{BL0}}(\xi) \to 0, \quad \xi_1 \to +\infty \tag{14}$$

$L_{\xi\xi} = \frac{\partial}{\partial \xi_k}(A_{kj}(\xi_2)\frac{\partial}{\partial \xi_j})$ and v_j are solutions of the chain of problems

$$h_2 D_1^2 v_j = f_j(x_1), \quad x_1 \in (0, +\infty); \qquad v_j(0) = g_j^0, \quad v_j(x_1) \to const, \quad x_1 \to \infty \tag{15}$$

where

$$f_j(x_1) = -\sum_{p=0}^{j-1} h_{j-p+2} D_1^{j-p+2} v_p(x_1) + \delta_{j0} f(x_1), \qquad g_j^0 = -\sum_{p=0}^{j-1} h_{j-p}^{\mathrm{BL0}} D_1^{j-p} v_p\big|_{x_1=0}$$

These relations as well as problems for N_l and N_l^{BL0} are obtained by the standard substitution of the ansatz (9), (10) into equation and the boundary conditions. The technique is similar to that of [9], Chapter 2, Section 2.2. Together with the assertion of Theorem 1.1 it justifies the following theorem:

Theorem 3.1. *The estimate holds* $\|\nabla(u_\varepsilon - u_\varepsilon^{a\,(K)})\|_{L^2(G_{\varepsilon+})} = \mathrm{O}(\varepsilon^K \sqrt{\varepsilon})$.

4. Projection of the problem on the "ansatz subspace"

Consider now the following subspace $H^1_{(0,\infty),K+1}$ of the space $H^1_{0,\infty}$:

$$\Bigg\{\varphi \in H^1_{0,\infty} \,\Bigg|\, \varphi(x) = \sum_{l=0}^{K+1} \varepsilon^l \left(N_l\left(\frac{x_2}{\varepsilon}\right) D_1^l w(x_1) + N_l^{\mathrm{BL0}}\left(\frac{x}{\varepsilon}\right) D_1^l w\big|_{x_1=0}\right), \ w \in H^{K+2}_{\mathrm{loc}};$$
$$D_1 w \in H^{K+1}\big((0,+\infty)\big); \ w(0) + \sum_{l=1}^{K+1} \varepsilon^l h_l^{\mathrm{BL0}} D_1^l w\big|_{x_1=0} = 0\Bigg\} \tag{16}$$

where $N_0^{\mathrm{BL0}} = 0$. The last relation of (16) is a direct consequence of the relation $\varphi|_{x_1=0} = 0$ and relation (13).

Let us consider the projection of problem (4) on the subspace $H^1_{(0,\infty),K+1}$ as in the method of asymptotic partial decomposition of domain [9], Chapter 6, Section 6.2, [21]:

– find $\bar{u}_\varepsilon \in H^1_{(0,\infty),K+1}$ such that for any $\varphi \in H^1_{(0,\infty),K+1}$

$$\int_{G_{\varepsilon+}} A_{kj}\left(\frac{x_2}{\varepsilon}\right) \frac{\partial \bar{u}_\varepsilon}{\partial x_j} \frac{\partial \varphi}{\partial x_k} \,\mathrm{d}x = \int_{G_{\varepsilon+}} f(x_1)\varphi(x)\,\mathrm{d}x \tag{17}$$

Let E_{K+2} be the space $\{w \in H^{K+2}_{\mathrm{loc}};\ D_1 w \in H^{K+1}((0,+\infty));\ w(0) + \sum_{l=1}^{K+1} \varepsilon^l h_l^{\mathrm{BL0}} D_1^l w|_{x_1=0} = 0\}$. Then taking into account (11), (12) we get that (17) is equivalent to the following problem:

– find $\bar{v}_\varepsilon \in E_{K+2}$ such that, for any $w \in E_{K+2}$,

$$\int_0^{+\infty} \sum_{l,m=1}^{K+2} \varepsilon^{l+m-1} \tilde{h}_{lm} D_1^l v D_1^m w \,\mathrm{d}x_1 - \sum_{m=1}^{K+1} \varepsilon^m \tilde{h}_{m0}^{ib} v\big|_{x_1=0} D_1^m w\big|_{x_1=0} - \sum_{l=1}^{K+1} \varepsilon^l \tilde{h}_{0l}^{ib} D_1^l v\big|_{x_1=0} w\big|_{x_1=0}$$
$$- \sum_{l,m=1}^{K+1} \varepsilon^{l+m} \big(\tilde{h}_{lm}^{ib} + \tilde{h}_{ml}^{ib}\big) D_1^l v\big|_{x_1=0} D_1^m w\big|_{x_1=0} + \sum_{l,m=1}^{K+1} \varepsilon^{l+m} \tilde{h}_{lm}^{\mathrm{BL0}} D_1^l v\big|_{x_1=0} D_1^m w\big|_{x_1=0}$$

$$= \varepsilon \int_0^{+\infty} f(x_1) w(x_1)\,\mathrm{d}x_1 + \varepsilon^2 \sum_{m=1}^{K+1} \varepsilon^m \tilde{g}_{m\varepsilon}^{\mathrm{BL0}} D_1^m w\big|_{x_1=0} \tag{18}$$

where

$$\tilde{h}_{lm} = \left\langle A_{kj}(\xi_2)\left(\frac{\partial N_l}{\partial \xi_j} + \delta_{j1} N_{l-1}\right)\left(\frac{\partial N_m}{\partial \xi_k} + \delta_{k1} N_{m-1}\right)\right\rangle$$

the derivative $\frac{\partial N_l}{\partial \xi_j}$ is dropped if $l = K+2$, and $\frac{\partial N_m}{\partial \xi_k}$ is dropped if $m = K+2$; these coefficients were introduced in [14] (up to some space averaging over the translation parameter) and they are the same as in the method of partial homogenization [9], Chapter 6, Section 6.4, [21];

$$\tilde{h}_{lm}^{ib} = \left\langle A_{1k}(\xi_2) \frac{\partial N_l^{\mathrm{BL0}}}{\partial \xi_k}\bigg|_{\xi_1=0} N_m(\xi_2)\right\rangle, \qquad \tilde{h}_{lm}^{\mathrm{BL0}} = \left\langle A_{kj}(\xi_2) \frac{\partial N_l^{\mathrm{BL0}}}{\partial \xi_j} \frac{\partial N_m^{\mathrm{BL0}}}{\partial \xi_k}\right\rangle_{\Pi_+}$$
$$\tilde{g}_{m\varepsilon}^{\mathrm{BL0}} = \left\langle f(\varepsilon\xi_1) N_m^{\mathrm{BL0}}(\xi)\right\rangle_{\Pi_+}$$

In what follows we assume that there exists a solution of this problem which belongs to the space $H^1_{(0,\infty),K+1}$. In particular, a sufficient condition is the uniform (with respect to ε) equivalence of the norm

$$\left(\int_{G_{\varepsilon+}} A_{kj}\left(\frac{x_2}{\varepsilon}\right) \frac{\partial \Phi(w_\varepsilon)}{\partial x_j} \frac{\partial \Phi(w_\varepsilon)}{\partial x_k}\,\mathrm{d}x\right)^{1/2},$$

$$\Phi(w_\varepsilon) = \sum_{l=0}^{K+1} \varepsilon^l \left(N_l\left(\frac{x_2}{\varepsilon}\right) D_1^l w(x_1) + N_l^{\mathrm{BL0}}\left(\frac{x}{\varepsilon}\right) D_1^l w\big|_{x_1=0}\right) \tag{19}$$

and the norm $\|w\|_{K+2,\varepsilon} = (\sum_{l=0}^{K+1} \varepsilon^{2l} \|D_1^{l+1} w\|^2_{L^2(0,+\infty)})^{1/2}$ for $w \in H^1_{(0,\infty),K+1}$.

5. Derivation of the high order homogenized equation and the high order boundary conditions

Varying $w \in H^1_{(0,\infty),K+1}$ and integrating by parts in (18) we derive:

- the high order homogenized equation of order $2K+4$, the same as in [14], and [21]

$$\sum_{l,m=1}^{K+2} \varepsilon^{l+m-2} (-1)^m \tilde{h}_{lm} D_1^{l+m} v(x_1) - f(x_1) = 0, \quad x_1 \in (0, +\infty) \tag{20}$$

- and $K+2$ boundary conditions at $x_1 = 0$ (here r stands for the order of the derivative of w in the trace terms containing $D_1^r w|_{x_1=0}$):

$$\begin{aligned}&\sum_{l=1}^{K+2} \sum_{m=r+1}^{K+2} \varepsilon^{l+m-1} (-1)^{m-r} \tilde{h}_{lm} D_1^{l+m-r} v\big|_{x_1=0} - \sum_{l=1}^{K+2} \sum_{m=1}^{K+2} \varepsilon^{r+l+m-1} (-1)^m \tilde{h}_{lm} h_r^{\mathrm{BL0}} D_1^{l+m} v\big|_{x_1=0}\\ &\quad - \sum_{l=1}^{K+1} \varepsilon^{r+l} \left(\tilde{h}_{lr}^{ib} + \tilde{h}_{rl}^{ib}\right) D_1^l v\big|_{x_1=0} + \sum_{l=1}^{K+1} \varepsilon^{r+l} \tilde{h}_{r0}^{ib} h_l^{\mathrm{BL0}} D_1^l v\big|_{x_1=0} + \sum_{l=1}^{K+1} \varepsilon^{r+l} \tilde{h}_{0l}^{ib} h_r^{\mathrm{BL0}} D_1^l v\big|_{x_1=0}\\ &\quad + \sum_{l=1}^{K+1} \varepsilon^{r+l} \tilde{h}_{lr}^{\mathrm{BL0}} D_1^l v\big|_{x_1=0} = \varepsilon^{r+2} \tilde{g}_{r\varepsilon}^{\mathrm{BL0}}, \quad r = 1, \ldots, K+1\end{aligned} \tag{21}$$

and

$$v|_{x_1=0} + \sum_{l=1}^{K+1} \varepsilon^l h_l^{\mathrm{BL0}} D_1^l v\big|_{x_1=0} = 0 \tag{22}$$

In order to get (21), after integrating by parts in (18) we take into account that

$$w(0) = -\sum_{l=1}^{K+1} \varepsilon^l h_l^{\mathrm{BL0}} D_1^l w\big|_{x_1=0}$$

we replace as well $v(0)$ by the analogous sum from (22). That is why (21) contains neither value $v(0)$ nor terms with $r = 0$.

We seek for a bounded at infinity solution having the derivative from $H^{K+1}((0, +\infty))$.

Theorem 5.1. *Let $v = \bar{v}_\varepsilon$ be a solution of problem* (18). *The estimate holds*

$$\left\| \nabla \left(u_\varepsilon - \sum_{l=0}^{K+1} \varepsilon^l N_l \left(\frac{x_2}{\varepsilon} \right) D_1^l \bar{v}_\varepsilon + \sum_{l=1}^{K+1} \varepsilon^l N_l^{\mathrm{BL0}} \left(\frac{x}{\varepsilon} \right) D_1^l \bar{v}_\varepsilon \big|_{x_1=0} \right) \right\|_{L^2(G_{\varepsilon+})} = \mathrm{O}\big(\varepsilon^K \sqrt{\varepsilon}\big)$$

It should be noted that the constructed boundary conditions as well as the equation are “local”, i.e. without non-local terms. The analogous boundary conditions could be written for the high order homogenized equation in the case of the Neumann condition at $x_1 = 0$ instead of the Dirichlet one, for the problem set in a thin rectangle $(0, 1) \times (-\varepsilon/2, \varepsilon/2)$ (then the boundary conditions similar to (21), (22) appear at $x_1 = 1$), for the three-dimensional laminated rod with layers parallel to the axis of the rod, for a thin laminated plate, for the elasticity equation instead of Eq. (1), for Eq. (1) set in the layer $(0, 1) \times \mathbb{R}^2$ with coefficients depending on x_2, x_3 only. The same procedure of projection on the “ansatz subspace” generated by the “standard” ansatz (5) [17] also leads to some high order homogenized model but with coefficients depending on the distance from the boundary, which is not natural for a macroscopically homogeneous medium.

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