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An asymptotic finite plane deformation analysis of the elastostatic fields at a notch vertex of an incompressible hyperelastic material

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Abstract

This Note is devoted to the theoretical study of the elastostatic fields at a vertex notch under general far-field loading conditions. The analysis is based on the finite plane deformation hyperelasticity theory for an incompressible Mooney–Rivlin material. We approach the solution, near the singularity, by a mixed asymptotic development. We show that the shape of the solution depends on the opening angle of the notch and that there is singularity if the notch is concave. Furthermore, we show that a pure loading mode II gives rise to the opening of the notch vertex in contrast to the linear elasticity. ***To cite this article: M. Arfaoui et al., C. R. Mecanique 336 (2008).***

Résumé

Analyse asymptotique en déformation finie plane des champs élastostatiques au coin d'un secteur pour un matériau hyperélastique incompressible. Cette Note est consacrée à l'étude théorique des champs élastostatiques au coin d'un secteur soumis à des chargements généraux à l'infini. L'analyse est basée sur la théorie de l'hyperélasticité en déformation plane pour un matériau incompressible de Mooney–Rivlin. Nous approchons la solution en déplacement, près de la singularité, par un développement asymptotique mixte. La forme de la solution dépend de l'angle d'ouverture du coin et nous montrons qu'il y a singularité si le secteur est concave. Par ailleurs, nous montrons qu'un chargement en mode II pur donne lieu à l'ouverture du coin contrairement à l'élasticité linéaire. ***Pour citer cet article : M. Arfaoui et al., C. R. Mecanique 336 (2008).***

Keywords: Notch vertex; Hyperelasticity; Singularity; Asymptotic analysis; Mixed mode loading; Stress

Mots-clés : Coin d'un secteur ; Hyperélasticité ; Singularité ; Analyse asymptotique ; Chargement en mode mixte ; Contrainte

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1. Introduction

In solid mechanics, the determination of deformation and stress fields in proximity of cracks, corners, voids of inclusions, and other material or geometrical imperfections, is of paramount interest. In these situations, the associated elastic problem may admit a singular solution. In the linear elastic theory, several methods of singularity analysis are used to calculate the singular stresses in the vicinity of the vertex of a two-dimensional notch [1,2]. Nevertheless, these studies often yield strain fields which are locally unbounded and therefore in conflict with the underlying assumption that justifies the kinematic linearization.

Without any doubt, the most important papers, within the framework of fully nonlinear finite elasticity, are those written by Knowles and Stenberg [3,4]. They gave the earliest systematic analyses of the local fields for a symmetrically loaded traction-free crack in a homogeneous slab. Another fundamental paper is that of Stephenson [5]. He investigated the general plane strain problems (mixed-mode loading). This analyse showed that the global nonlinear Mode II crack problem cannot admit a solution that is antisymmetric about the crack axis for a class of an incompressible Mooney–Rivlin materials.

With reference to notch problems, some analyses have been performed to compute the singular elastostatic fields near the notch vertex by removing the requirement of infinitesimal deformations. Within the framework of fully nonlinear elastostatics, early investigations seem to be due to Tarantino [6,7], who carried out a global plane-stress analysis for an infinite compressible and incompressible Mooney–Rivlin sheet. The interesting work of Ru [8] aims to give a finite plane-strain analysis of the deformations and stresses near the vertex of a compressible elastic bi-material notch. Each of two edge-bonded dissimilar wedges is hyperelastic with the harmonic-type strain energy density which facilitate the analysis. We note also the work of Gao [9] who used an alternative approach by dividing the singular field into shrinking and expanding sectors for which the asymptotic equations are derived separately.

The purpose of the present work is closely related to that of crack analyses developed in [3–5]. The notch problem is formulated and solved for an incompressible hyperelastic material under plane deformation condition in a fully nonlinear finite elasticity context. In order to calculate the deformation and stress singular fields near the notch vertex an asymptotic analysis is carried out. Finally, the structure of the singular deformation field is examined in detail. Emphasis is placed on describing the notch-profile after deformation, proving Stephenson's conjecture [5] in our context and evaluating the asymptotic order of pressure and stress singularities. The most important differences with respect to the predictions of the linear theory are evidenced and discussed.

2. Formulation of the global notch problem

Let Ω be the domain of the (x_1, x_2) plane characterizing the cross section of an infinite cylindrical body in its undeformed configuration. We assume that the cylindrical body is subjected to a plane strain (deformation) so that the position of material point (x_1, x_2) after deformation is (y_1, y_2). The plane domain Ω is described by:

$$\Omega = \{(r, \theta) \mid 0 < r < +\infty, -\omega \leqslant \theta \leqslant \omega\} \tag{1}$$

where (r, θ) are the material polar coordinates. In this model, the possibility of the notch faces coalescing until they form an interface is admitted. The deformation is represented by a vectorial transformation $\underline{y}$:

$$y_\alpha = y_\alpha(x_1, x_2) \quad (\alpha = 1, 2) \quad \text{for all } (x_1, x_2) \in \Omega \text{ with } x_\alpha(r, \theta) \tag{2}$$

which maps Ω on to a domain Ω^* of the same plane. The geometry transformation subjected to the incompressibility constraint is described by the second order tensor of a gradient transformation $\underline{\underline{F}}$:

$$[F_{\alpha\beta}] = \left[\frac{\partial y_\alpha}{\partial x_\beta}\right] (\alpha, \beta = 1, 2) \quad \text{on } \Omega \quad \text{and} \quad J = \det \underline{\underline{F}} = 1 \tag{3}$$

We introduce a class of polyconvex hyperelastic incompressible potential W per unit undeformed volume governed by the Moony–Rivlin potential which satisfy the strong ellipticity condition:

$$W(I) = \frac{\mu(I-2)}{2} \quad \text{on } \Omega,\ \mu > 0,\ I = \operatorname{tr}(\underline{\underline{F}}^T \underline{\underline{F}}) \tag{4}$$

For hyperelastic potential (4), the first Piola–Kirchhoff and Cauchy stresses tensors $\underline{\underline{\tau}}$ and $\underline{\underline{\sigma}}$ are written, respectively:

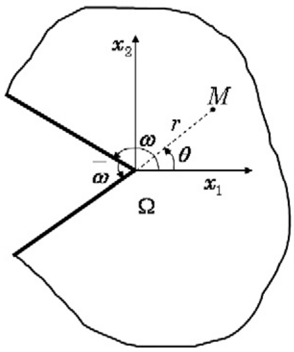

Fig. 1. The notch problem: the notch angle ω and the opening angle $\bar{\omega}$.

$$\underline{\underline{\tau}} = \frac{\partial W}{\partial \underline{\underline{F}}} - p\underline{\underline{F}}^{-T} = \mu \underline{\underline{F}} - p\underline{\underline{F}}^{-T} \quad \text{on } \Omega \text{ and}$$

$$\underline{\underline{\sigma}} = \underline{\underline{\tau}}\,\underline{\underline{F}}^T = \left(\frac{\partial W}{\partial \underline{\underline{F}}}\right)\underline{\underline{F}}^T - p\underline{\underline{1}} = \mu \underline{\underline{F}}\,\underline{\underline{F}}^T - p\underline{\underline{1}} \quad \text{on } \Omega^* \tag{5}$$

while p is an arbitrary hydrostatic pressure field arising from the constraint of incompressibility. In the absence of body forces, equilibrium in the undeformed configuration then demands that:

$$\operatorname{Div}\underline{\underline{\tau}} = \underline{0} \text{ on } \Omega \quad \Leftrightarrow \quad \frac{\partial p}{\partial r} = \mu \frac{\partial y_\alpha}{\partial r}\Delta y_\alpha \text{ and } \frac{\partial p}{\partial \theta} = \mu \frac{\partial y_\alpha}{\partial \theta}\Delta y_\alpha \text{ on } \Omega \text{ (sum on } \alpha = 1, 2) \tag{6}$$

To satisfy traction-free boundaries conditions at the notch faces, we impose the following conditions:

$$\underline{\underline{\tau}}\,\underline{n} = \underline{0} \text{ at } \theta = \pm\omega \quad \Leftrightarrow \quad p = \frac{\mu}{r^2}\frac{\partial y_\alpha}{\partial \theta}\frac{\partial y_\alpha}{\partial \theta} \text{ and } \frac{\partial y_\alpha}{\partial r}\frac{\partial y_\alpha}{\partial \theta} = 0 \text{ at } \theta = \pm\omega \text{ (sum on } \alpha) \tag{7}$$

where $\underline{n}$ is the external normal to the notch face. At infinity, the transformation $\underline{y}$ must be compatible with the kinematic mixed-mode loading conditions.

We now turn to the determination of the local structure of the pressure field from (6):

$$\mu \Delta y_1 = \frac{1}{r}\left(\frac{\partial p}{\partial r}\frac{\partial y_2}{\partial \theta} - \frac{\partial p}{\partial \theta}\frac{\partial y_2}{\partial r}\right) \tag{8}$$

Solving the global notch problem, at least in closed form, is a very complex task. Thus, asymptotic analysis are commonly performed. In this case, solutions, which hold exclusively for points close to the vertex of the notch, are sought. The local notch problem can be stated by requiring that the transformation satisfies the field equations and the condition that the surfaces of the notch must be traction-free. The far field loading conditions in the formulation of the local notch problem are ignored. We note that without corresponding requirements on the transformation field at infinity this formulation is not a complete statement. However, the significance of the fields resulted by the local formulation lies in their characterisation of the singular elastostatic field behaviour in the vertex region, namely, as $r \to 0$.

Let $\Im$ be the class of all $\{\underline{y}, \underline{\underline{\sigma}}, p\}$ that satisfy the boundary value problems. Then it is easy to prove that:

$$\{\underline{y}, \underline{\underline{\sigma}}, p\} \subset \Im \Leftrightarrow \{\underline{\underline{Q}}\,\underline{y}, \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,Q^T, p\} \subset \Im \quad \forall \underline{\underline{Q}} \quad \text{Orthogonal second order tensor} \tag{9}$$

This is assured by the objectivity of the constitutive equation (5) and by virtue of the form of the boundary conditions (7). This property will be used later to better understand the nature of the local deformation field [5].

3. Asymptotic analysis of the elastostatic field near the notch vertex

3.1. First order

On supposing that the global notch problem admits a solution, in order to investigate the singularity induced by the vertex presence, it is assumed that such solution has the following form [3]:

$$y_\alpha(r,\theta) = r^{m_1} U_\alpha(\theta) + o\big(r^{m_1}\big), \quad U_\alpha(\theta) \in C^\infty\big([-\omega,\omega]\big) \text{ and } U_\alpha(\theta) \neq 0 \tag{10}$$

m_1 must be real constant to avoid the appearance of oscillations arising in the linearized local solution and satisfies the inequality $0 < m_1 < 1$ in order to ensure bounded displacement but admit unbounded gradients at the notch vertex. As suggested by [5], the exponent m_1 in (10) replaces, without loss of generality, an eventual pair of exponents $m_1^{(\alpha)}$. This is justified by the objectivity of the constitutive equations. Finally, we suppose that the pressure field associated with the global solution satisfies:

$$p(r,\theta) = r^{l_1} P_1(\theta) + o\big(r^{l_1}\big) \quad \text{as } r \to 0,\ P_1(\theta) \in C^1\big([-\omega,\omega]\big),\ l_1 \in \mathrm{R}_+^* \tag{11}$$

We now seek to determine the smallest exponent $m_1 \in]0,1[$ and the function U_α appearing in (10) consistent with the incompressibility constraint (3), the governing field equations (6) and boundary conditions (7). The incompressibility constraint (3), together with (10), give:

$$J = m_1 r^{2(m_1-1)}\big[U_1(\theta)\dot{U}_2(\theta) - U_2(\theta)\dot{U}_1(\theta)\big] + o\big(r^{2(m_1-1)}\big) = 1$$
$$\Rightarrow \quad U_\alpha = a_\alpha U \ (\alpha = 1,2) \quad \text{on } [-\omega,\omega],\ a_\alpha \in \mathrm{R}_+^* \text{ with } a = \sqrt{a_1^2 + a_2^2} \tag{12}$$

Governing field equations (6) and the boundary conditions (7), with (10), lead to:

$$m_1^2 U + \ddot{U} = 0 \quad \text{on } [-\omega,\omega] \text{ and } \dot{U}(\pm\omega) = 0 \text{ with } \dot{U} = \frac{\partial U}{\partial \theta} \tag{13}$$

Therefore, the first asymptotic solution is:

$$y_\alpha(\theta) = a_\alpha r^{m_1} \sin(m_1\theta) + o\big(r^{m_1}\big) \quad \text{as } r \to 0 \text{ with } m_1 = \frac{\pi}{2\omega} \tag{14}$$

However, the inequality $m_1 < 1$ implies that $\omega > \pi/2$. Namely, the result (14) holds for concave notch problems only. Consequently, the problem in the case of re-entrant notches (wedge, for $\omega \leqslant \pi/2$) does not admit singular solutions. Nevertheless, such a solution provides the following estimate:

$$J = o\big(r^{\pi/\omega - 2}\big) \to \infty \quad \text{and} \quad p(r,\theta) = o\big(r^{\pi/\omega-2}\big) \to \infty \quad \text{as } r \to 0 \tag{15}$$

and is therefore inadequate which reflects the degenerate character of the asymptotic approximation established so far.

3.2. Second order

The first order approximation to the local deformation in the vicinity of the notch vertex does not constitute an invertible mapping. Consequently, we should refine (10) by seeking at least a two term approximation:

$$y_\alpha = a_\alpha r^{m_1} U(\theta) + r^{m_2} V_\alpha(\theta) + o\big(r^{m_2}\big)\ (\alpha = 1,2), \quad V_\alpha(\theta) \in C^\infty\big([-\omega,\omega]\big),\ m_1 < m_2 \in \mathrm{R}_+^* \tag{16}$$

m_1 and U are now given by (14). Again, by virtue of the discussion that led to the adoption of (10), no generality is lost in assuming equal exponents in the second term of (16). From the incompressibility constraint (3) with (16) one can write:

$$m_1 U \dot{\Psi}_2 - m_2 \dot{U} \Psi_2 = 0 \quad \text{on } [-\omega,\omega] \quad \text{if } m_1 < m_2 < 2 - m_1,\ \Psi_2 = a_1 V_2 - a_2 V_1 \tag{17}$$
$$m_1 U \dot{\Psi}_2 - m_2 \dot{U} \Psi_2 = 1 \quad \text{on } [-\omega,\omega] \quad \text{if } m_1 < m_2 = 2 - m_1 \tag{18}$$

The boundary conditions can be deduced from (17) and (18):

$$\dot{\Psi}_2(\pm\omega) = 0 \quad \text{if } m_1 < m_2 < 2 - m_1 \tag{19}$$
$$\dot{\Psi}_2(\pm\omega) = 1/m_1 U(\pm\omega) \quad \text{if } m_2 = 2 - m_1 \tag{20}$$

By inserting (16) into the field equations (6) and the boundary conditions (7), and recalling that satisfies (13), we can prove that χ_2 must verify the following eigenvalue equation:

$$\begin{cases} \ddot{\chi}_2 + m_2^2 \chi_2 = 0 & \text{on } [-\omega, \omega] \\ \dot{\chi}_2(\pm\omega) = 0 \end{cases} \quad \text{if } m_1 < m_2 \leqslant 2 - m_1, \; \chi_2 = a_1 V_1 + a_2 V_2 \tag{21}$$

Solving the differential equations (17) and (21) with the boundary conditions (19) and (21) gives:

$$m_2 = 2m_1 = \pi/\omega, \quad \Psi_2 = b_2 \sin^2(m_1\theta), \quad \chi_2(\theta) = c_2 \cos(m_2\theta), \quad b_2, c_2 \in \mathrm{R} \tag{22}$$

In addition m_2 defined by (22) conforms to (17) which lead to the condition on ω: $3\pi/4 < \omega \leqslant \pi$.

We now turn to the determination of the local structure of the pressure field, which has been assumed to admit the representation (11). From (7), (8), (11), (16) and (22) follows:

$$p(r,\theta) = r^{l_1} P_1(\theta) + \mathrm{o}\big(r^{l_1}\big) \quad \text{with } l_1 = m_1 = \pi/2\omega \text{ and } P_1(\theta) = -\mu/\big(a^2\big) b_2 \cos(l_1\theta) \tag{23}$$

We note that the dominant term in (16) has a degenerate Jacobian determinant $J = \mathrm{o}(r^{3\pi/(2\omega)-2})$. Thus a higher order asymptotic analysis is needed. The results of the second order asymptotic analysis are valid for notches obtained as $3\pi/4 < \omega \leqslant \pi$. When this condition is violated, namely when $\pi/2 < \omega \leqslant 3\pi/4$, we must relax the constraint $m_2 < 2 - m_1$ and take $m_2 = 2 - m_1 > m_1$. This possibility will be discussed briefly in Section 3.4 (for more details see [10]).

3.3. *Third order for* $3\pi/4 < \omega \leqslant \pi$

With a view to refining these estimates, when $m_2 < 2 - m_1$, we first replace (16) by:

$$y_\alpha = a_\alpha r^{m_1} U(\theta) + r^{m_2} V_\alpha(\theta) + r^{m_3} R_\alpha(\theta) + \mathrm{o}\big(r^{m_3}\big)$$
$$R_\alpha(\theta) \in C^\infty\big([-\omega,\omega]\big), \quad m_1 < m_2 < m_3 \in \mathrm{R}_+^* \tag{24}$$

The functions U and V_α are already known. Combining (24) with (6) and invoking the boundary conditions (7), one finds after considerable computations that:

$$R_1 = \frac{1}{a^2}[a_1\chi_3 - a_2\Psi_3], \quad R_2 = \frac{1}{a^2}[a_2\chi_3 + a_1\Psi_3], \quad m_3 = 2 - m_1 = 2 - \frac{\pi}{2\omega} \tag{25}$$

$$\Psi_3(\theta, m_1) = -\frac{1}{m_1 m_3} F\left(\frac{1}{2} - \frac{1}{m_1}, \frac{1}{2}; \frac{3}{2} - \frac{1}{m_1}; \sin^2(m_1\theta)\right) \quad \text{and} \quad \chi_3 = 0 \text{ on } [-\omega,\omega] \tag{26}$$

F stands for the hypergeometric function [5]. We note that the solution (24), (25) and (26) is available for $3\pi/4 < \omega < \pi$. In the case of $\omega = \pi$, the solution was done in [5].

At this stage and in order to deduce the intrinsic or canonical simple representation of y_α, we apply the objectivity property (9) by using a particular form of $\underline{\underline{Q}}$:

$$[Q_{\alpha\beta}] = \begin{bmatrix} a_2/a & -a_1/a \\ a_1/a & a_2/a \end{bmatrix} \tag{27}$$

to (24). One arrives at:

$$y_1(r,\theta) = -\frac{1}{a} r^{m_2} \Psi_2(\theta) - \frac{1}{a} r^{m_3} \Psi_3(\theta) + \mathrm{o}\big(r^{m_3}\big)$$
$$y_2(r,\theta) = a r^{m_1} U(\theta) + \frac{1}{a} r^{m_2} \chi_2(\theta) + \mathrm{o}\big(r^{m_2}\big) \tag{28}$$

The particular form of $\underline{\underline{Q}}$ (27) corresponds to a rotation angle given by the mixed-mode loading of the first asymptotic term, $\tan(\phi) = -a_1/a_2$. We shall call the particular field deduced from (9) and (27) a canonical field, because it is the standard representative element of the set $\Im$ of local singular fields. To specify all the other elements of $\Im$ we simply apply the reverse formula of (9) and (27). We will see later the consequence of the canonical field. In order to calculate an additional term for the pressure field, we now suppose that the pressure field conforms to:

$$p(r,\theta) = r^{l_1} P_1(\theta) + r^{l_2} P_2(\theta) + \mathrm{o}\big(r^{l_2}\big), \quad l_2 > l_1 \tag{29}$$

with l_1, P_1 supplied by (23) and l_2, P_2, unknown. Eqs. (8), (28), (29) and the boundary conditions (7) lead to $l_2 = m_3 - m_1 = 2 - \frac{\pi}{\omega}$ with P_2 is a functional of Ψ_3 [10].

3.4. Higher order for $\pi/2 < \omega \leqslant 3\pi/4$

These results are valid for notches obtained as $3\pi/4 < \omega \leqslant \pi$. When this condition is violated, namely when $\pi/2 < \omega \leqslant 3\pi/4$, we must relax the constraint $m_2 < 2 - m_1$ and take $m_2 = 2 - m_1 > m_1$. We show (see [10] for more details) that when $\pi/2 < \omega < 3\pi/4$, the solution is achieved by an asymptotic development to the second order which has the same form as the third order term of the solution for $3\pi/4 < \omega < \pi$ (25) and (26). In the case of $\omega = 3\pi/4$, the solution involve a logarithm term (ε small parameter and $\tilde{\Psi}$ is a known function):

$$y_1(r,\theta) = \frac{27}{8a} r^{\frac{4}{3}} \operatorname{Ln}(r) \sin^2\left(\frac{2}{3}\theta\right) + r^{\frac{4}{3}} \tilde{\Psi}_2(\theta) + o\left(r^{\frac{4}{3}}\right)$$
$$y_2(r,\theta) = a r^{\frac{2}{3}} \sin\left(\frac{2}{3}\theta\right) + o\left(r^{\frac{2}{3}+\varepsilon}\right) \tag{30}$$

4. Discussion of the deformation and stresses near the notch vertex and closure

Of particular concern is the deformation image of the notch vertex faces $\theta = \pm\omega$. Thus, to dominant order, the two notch vertex faces at $\theta = \pm\omega$ are transformed into the curves represented by:

$$y_2 = \pm a^{m_1/m_2+1}(-y_1)^{m_1/m_2} \quad \text{if } \omega \in]\pi/2, 3\pi/4[\cup]3\pi/4, \pi]$$
$$y_1 = (81/8a^3) y_2^2 \operatorname{Ln}(\pm y_2/a) \quad \text{if } \omega = 3\pi/4 \tag{31}$$

From (31), it follows that points near the notch vertex in the undeformed body lie to the right of this curve after deformation. We note on the basis of (31) that the notch vertex is bound to open, regardless of the magnitude and nature of the particular loading at infinity. This conclusion is in marked contrast to the predictions of the linearized theory for a mode II loading [10]. From (28) and (30) we get the important result that, for arbitrarily given loading conditions at the infinite edge, the asymptotic deformation field is obtained by a mere rigid rotation of the canonical symmetric deformation field, or, likewise, of the solution of the mode I notch problem. More specifically, (28) and (30) imply, in contrast to predictions of the linear elastostatic theory, that, even if the applied loading is antisymmetric about the plane of the notch (mode II), the notch faces should open symmetrically at the notch vertex. Consequently, the general nonlinear global notch problem, cannot admit an antisymmetric solution. The above result confirms how this important property, which was demonstrated by [5] for a crack problem concerning an incompressible Mooney–Rivlin material under plane strain, also holds in our context. At this stage we calculate the dominant terms of the associated local true-stress field for. On account of (5) one has:

$$\begin{cases} \sigma_{11} = \dfrac{\mu}{a^2} r^{4m_1-2}\left[(1-m_1)^2\left(\Psi_2(\theta)\right)^2 + \left(\dot{\Psi}_2(\theta)\right)^2\right] - r^{2-2m_1} P_1(\theta) + o\left(r^m\right) \\ \sigma_{22} = \mu a^2 r^{2m_1-2}\left[(1-m_1)^2 U^2(\theta) + \dot{U}^2(\theta)\right] + o\left(r^{2m_1-2}\right) \\ \sigma_{12} = \sigma_{21} = -r^{m_1+m_2-2}\left\{m_1(1-m_1)\Psi_2 U + \dot{\Psi}_2 \dot{U}\right\} + o\left(r^{m_1+m_2-2}\right) \end{cases}$$
$$\text{for } \omega \in \left]\frac{\pi}{2}, \frac{3\pi}{4}\right[\cup \left]\frac{3\pi}{4}, \pi\right] \quad \text{with } m = \max(4m_1 - 2, 2 - 2m_1) \tag{32}$$

The most singular of the stress components in (32) is σ_{22}, which becomes infinite at the notch vertex like r^{2m_1-2}. In the transition case $\omega = 3\pi/4$, Eqs. (5) and (30) may be used to deduce the corresponding stress field. We now determine the dominant character of the true stresses when the latter are referred to the spatial coordinates y_α. We introduce the special coordinate $y = \sqrt{y_1^2 + y_2^2}$, evaluated along the line $\theta = 0$ and one draws from (5), (28) and (30) that:

$$\begin{cases} y \sim r^{2m_1} \\ \sigma_{22} \sim y^{1-1/m_1} \end{cases} \text{if } \omega \in \left]\frac{3\pi}{4}, \pi\right], \qquad \begin{cases} y \sim r^{2-m_1} \\ \sigma_{22} \sim y^{(2m_1-2)/(2-m_1)} \end{cases} \text{if } \omega \in \left]\frac{\pi}{2}, \frac{3\pi}{4}\right[$$
$$\begin{cases} y \sim r^{4/3} \\ \sigma_{22} \sim y^{-1/2} \end{cases} \text{if } \omega = \frac{3\pi}{4} \tag{33}$$

For the crack problem, $\omega = \pi$, it follows from both (32) and (33) that the most singular component of the Cauchy stress tensor has the asymptotic behaviour y^{-1}, which is stronger than the inverse square root singularity predicted by linear fracture mechanics [2]. Eqs. (32) and (33) show how the order of the stress singularities depends on the local geometry of the notch: as the opening angle $\overline{\omega}$ increases, the stress singularities decrease. In particular, for $\bar{\omega} = \frac{\pi}{2}$, the asymptotic behaviour of the component σ_{22} reduces to $(y^{-1/2})$. On the other hand, the order of stress singularities as the opening angle varies does not depend on the type of far-field loading conditions, namely, it is the same for a mode I and mode II problems. This is still in contrast with the predictions of the linear theory [2].

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