A new formulation of immiscible compressible two-phase flow in porous media

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Abstract

A new formulation is proposed to describe immiscible compressible two-phase flow in porous media. The main feature of this formulation is the introduction of a global pressure. The resulting equations are written in a fractional flow formulation and lead to a coupled system which consists of a nonlinear parabolic (the global pressure equation) and a nonlinear diffusion–convection one (the saturation equation) which can be efficiently solved numerically. ***To cite this article: B. Amaziane, M. Jurak, C. R. Mecanique 336 (2008).***
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Résumé

Une nouvelle formulation des écoulements diphasiques immiscibles compressibles en milieux poreux. Une nouvelle formulation est proposée pour décrire les écoulements diphasiques immiscibles compressibles en milieux poreux. Ce modèle est basé sur la notion de pression globale. Les équations du problème sont écrites sous forme de flux fractionnel et introduisent un système couplé faisant intervenir une équation parabolique nonlinéaire (équation de la pression globale) et une équation de diffusion–convection nonlinéaire (équation de la saturation). Cette formulation permet la mise en oeuvre d'une méthode de calcul numérique performante pour le système diphasique complet. ***Pour citer cet article : B. Amaziane, M. Jurak, C. R. Mecanique 336 (2008).***
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1. Introduction

Motivation for the following problem arises in the area of modeling flow and transport of two-phase problems related to the environment and the energy. Many difficult problems arise in the numerical simulation of complex fluid processes in reservoir simulation, subsurface contaminant transport and remediation, gas migration through

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engineered and geological barriers for a deep repository for radioactive waste, sequestration of CO_2 and other applications.

Historically, there have been two main approaches to modeling multiphase flow in porous media. The first is based on individual balance equations for each of the fluids, while the second involves manipulation and combination of those balance equations into modified forms, with concomitant introduction of ancillary functions that we will refer to as the fractional flow or global pressure saturation formulation. The notion of global pressure was first introduced by [1,2] and was then revisited by other authors, see for instance [3]. It has been since used in a wide range of engineering specialities related to numerical simulation in hydrology and petroleum reservoir engineering, see for instance [4] and references therein. It has been proven that this fractional flow approach is far more efficient than the original two-pressure approach from the computational point of view [4].

In this Note, we focus our attention on the study of immiscible compressible two-phase flow in porous media. The fractional flow formulation employs the saturation of one of the phases and a pressure as independent variables. The fractional flow approach treats the two-phase flow problem as a total fluid flow of a single mixed fluid, and then describes the individual phases as fractions of the total flow. This approach leads to two coupled equations: the global pressure equation; and the saturation equation.

Numerical methods are very sensitive to the choice of form of the governing equation. In the light of the new and continuing developments in numerical methods for the solution of the multiphase flow equations, it is worthwhile revisiting the question of the form of the governing equations and exploring the implications of this equation form for a numerical method based on it.

The aim of this Note is to derive a new fractional flow formulation for immiscible compressible two-phase flow in porous media which can be efficiently solved numerically. We will restrict our attention to water (incompressible) and gas such as hydrogen (compressible), however, the methodology and the analysis can be extended to problems where both fluids are assumed to be compressible.

The rest of the Note is organized as follows. In the next section, we review the differential problem describing immiscible compressible two-phase flow in porous media. In Section 3 we briefly recall a simplified fractional flow formulation described in [1]. Then, in Section 4 we derive a new more general fractional flow formulation. This formulation leads to a coupled system which consists of a nonlinear parabolic (the global pressure equation) and a nonlinear diffusion-convection one (the saturation equation). Finally, numerical comparison of coefficients that enter into the simplified and the new formulations for water–gas flow is presented and some conclusions are drawn.

2. Governing equations

The usual equations describing immiscible compressible two-phase isothermal flow in a porous medium are given by the mass balance equation and Darcy's law for each of the fluid phases (see, e.g., [5]):

$$\Phi \frac{\partial}{\partial t}(\rho_\alpha S_\alpha) + \operatorname{div}(\rho_\alpha \mathbf{q}_\alpha) = \mathcal{F}_\alpha \quad \text{and} \quad \mathbf{q}_\alpha = -\mathbb{K}\frac{kr_\alpha(S_\alpha)}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha \mathbf{g}) \tag{1}$$

where Φ and $\mathbb{K}$ are the porosity and absolute permeability of the porous medium; $\alpha = w$ denotes the wetting phase (e.g. water), $\alpha = g$ indicates the nonwetting phase (e.g. gas), ρ_α, S_α, p_α, $\mathbf{q}_\alpha$ and μ_α are, respectively, the density, (reduced) saturation, pressure, volumetric velocity, and viscosity of the α-phase, $\mathcal{F}_\alpha$ is the source/sink term, kr_α is the relative permeability of the α-phase, and $\mathbf{g}$ is the gravitational, downward-pointing, constant vector. In addition to (1), we also have the customary property for saturations and the capillary pressure function:

$$S_w + S_g = 1, \quad \text{and} \quad p_c(S_w) = p_g - p_w \tag{2}$$

The primary variables are S_α, p_α, and $\mathbf{q}_\alpha$. Here we assume that the porosity Φ and the absolute permeability $\mathbb{K}$ are functions of space. The water density ρ_w and viscosities μ_w, μ_g are constant; the gas density satisfies the ideal gas law: $\rho_g(p_g) = c_g p_g$, where c_g is a constant. Finally, we assume that the capillary pressure and relative permeabilities depend upon the water saturation solely. For notational simplicity, we neglect their dependence on space. For expositional convenience, we introduce the phase mobility functions: $\lambda_\alpha(S_w) = kr_\alpha(S_w)/\mu_\alpha$, $\alpha = w, g$ and the total mobility $\lambda(S_w, p_g) = \rho_w \lambda_w(S_w) + \rho_g(p_g)\lambda_g(S_w)$. Finally, we define the fractional flow functions: $f_\alpha(S_w, p_g) = \rho_\alpha \lambda_\alpha(S_w)/\lambda(S_w, p_g)$, $\alpha = w, g$, and the following nonlinear functions:

$$\bar{\rho}(S_w, p_g) = \left(\lambda_w(S_w)\rho_w^2 + \lambda_g(S_w)\rho_g^2(p_g)\right)/\lambda(S_w, p_g)$$

$$b_g(S_w, p_g) = \rho_w\rho_g(p_g)\lambda_w(S_w)\lambda_g(S_w)\left(\rho_w - \rho_g(p_g)\right)/\lambda(S_w, p_g)$$

$$a(S_w, p_g) = -\rho_w\rho_g(p_g)\lambda_w(S_w)\lambda_g(S_w)p_c'(S_w)/\lambda(S_w, p_g)$$

where $p_c'(S_w) = \mathrm{d}p_c/\mathrm{d}S_w$.

The governing equations (1), (2) are a set of coupled, nonlinear partial differential equations (PDEs). The basic equations can be mathematically manipulated into several alternate forms with various choices of primary dependent variables. The choice of equation form and primary solution defined by variables have considerable implications for the mathematical analysis and the numerical method used to solve these equations.

Here we rewrite the equations in the gas pressure and the water saturation formulation with a total flux, i.e. p_g (gas pressure), $\mathbf{Q}_t = \rho_w\mathbf{q}_w + \rho_g(p_g)\mathbf{q}_g$ (total flux), and S_w (water saturation) are chosen as primary variables. In this case, one checks easily, that the set of equations can be written as a system of two coupled evolution equations given by

$$\Phi\frac{\partial}{\partial t}\left(S_w\rho_w + (1-S_w)\rho_g(p_g)\right) - \mathrm{div}\left(\lambda(S_w, p_g)\mathbb{K}\left[\nabla p_g - f_w(S_w, p_g)\nabla p_c(S_w) - \bar{\rho}(S_w, p_g)\mathbf{g}\right]\right)$$
$$= \mathcal{F}_w + \mathcal{F}_g \tag{3}$$

$$\mathbf{Q}_t = -\lambda(S_w, p_g)\mathbb{K}\left(\nabla p_g - f_w(S_w, p_g)\nabla p_c(S_w) - \bar{\rho}(S_w, p_g)\mathbf{g}\right) \tag{4}$$

$$\Phi\rho_w\frac{\partial S_w}{\partial t} + \mathrm{div}\left(f_w(S_w, p_g)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g(S_w, p_g)\right) - \mathrm{div}\left(\mathbb{K}a(S_w, p_g)\nabla S_w\right) = \mathcal{F}_w \tag{5}$$

The governing equation for pressure (3) is a nonlinear parabolic PDE and the equation for saturation (5) is a nonlinear convection–diffusion PDE.

3. Simplified global formulation

In this section, we give a short description of a simplified fractional flow formulation of the system (3)–(5), i.e., in terms of a global pressure p and saturation under the assumption that we can ignore the error caused by calculating the gas density at p instead of p_g. This kind of approximation is widely used in petroleum engineering for production of hydrocarbons from petroleum reservoirs (oil and gas flow in reservoirs) and in hydrology (water and air in dams or soil). However, this condition is not satisfied for all the existing immiscible compressible two-phase flow.

Eqs. (3)–(5) are strongly coupled, as noted. To reduce the coupling, we now write them in a different formulation, where a global pressure is used. To this end, define the global pressure p as the solution of the following nonlinear equation [1]:

$$\Pi(S_w, p) = p_g, \quad \text{where } \Pi(S_w, p) = p + \gamma(S_w, p), \text{ with } \gamma(S_w, p) = \int_1^{S_w} f_w(s, p)p_c'(s)\,\mathrm{d}s \tag{6}$$

where we have assumed, for simplicity, that $p_c(1) = 0$. Then it follows from (3)–(5) that the system can be written as follows

$$\Phi\frac{\partial}{\partial t}\left(S_w\rho_w + (1-S_w)\rho_g(p)\right) - \mathrm{div}\left(\lambda(S_w, p)\mathbb{K}\left[\omega(S_w, p)\nabla p - \bar{\rho}(S_w, p)\mathbf{g}\right]\right) = \mathcal{F}_w + \mathcal{F}_g \tag{7}$$

$$\mathbf{Q}_t = -\lambda(S_w, p)\mathbb{K}\left(\omega(S_w, p)\nabla p - \bar{\rho}(S_w, p)\mathbf{g}\right) \tag{8}$$

$$\Phi\rho_w\frac{\partial S_w}{\partial t} + \mathrm{div}\left(f_w(S_w, p)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g(S_w, p)\right) - \mathrm{div}\left(\mathbb{K}a(S_w, p)\nabla S_w\right) = \mathcal{F}_w \tag{9}$$

where $\omega(S_w, p) = 1 + \frac{\partial}{\partial p}\gamma(S_w, p)$. Eqs. (7)–(9) are the global pressure and saturation equations, respectively.

We end this section with three remarks. First, instead of the total flux, we could just use the total velocity $\mathbf{q} = \mathbf{q}_w + \mathbf{q}_g$, and derive a similar set of PDEs. However, it follows from the expression of the total flux that it behaves more smoothly. It is easy to see that $\lambda(S_w, p)\omega(S_w, p)\nabla p = \rho_w\lambda_w(S_w)\nabla p_w + \rho_g(p_g)\lambda_g(S_w)\nabla p_g$. This implies that the global pressure is the pressure that would produce a flow of fluid (with mobility $\lambda\omega$) equal to the sum of the flows of fluids w and g. Third, the phase pressure–saturation form (3)–(5) is much more complicated than the global pressure–saturation form (7)–(9). In particular, the coupling between the pressure and saturation equations in (3)–(5) is stronger then in (7)–(9), and thus these equations are more expensive to solve, for more details see [4].

4. A new global formulation

In this section, we extend the global pressure–saturation formulation with the total flux in the general case where the assumption done in the last section on the gas density may not be satisfied. The purpose of the following computations is to define a global pressure p such that Eqs. (1)–(2) are exactly equivalent to a set of two coupled equations in p and S_w only. For this, let us set $p_g = \Pi^n(S_w, p) = p + \gamma^n(S_w, p)$ where γ^n is a nonlinear function which will be defined later (the superscript n stands for new). Then we want to obtain $\nabla p_g = \omega^n(S_w, p)\nabla p + f_w(S_w, \Pi^n(S_w, p))p_c'(S_w)\nabla S_w$, which can be written in the following form:

$$\frac{\partial \Pi^n}{\partial S_w}(S_w, p)\nabla S_w + \frac{\partial \Pi^n}{\partial p}(S_w, p)\nabla p = \omega^n(S_w, p)\nabla p + f_w\big(S_w, \Pi^n(S_w, p)\big)p_c'(S_w)\nabla S_w$$

Since p and S_w are independent variables we have

$$\frac{\partial \Pi^n}{\partial S_w}(S_w, p) = f_w\big(S_w, \Pi^n(S_w, p)\big)p_c'(S_w) \tag{10}$$

$$\frac{\partial \Pi^n}{\partial p}(S_w, p) = \omega^n(S_w, p) \tag{11}$$

We integrate (10) to obtain Π^n then ω^n is defined by (11). Setting $\Pi^n(1, p) = p$ we get

$$\Pi^n(S_w, p) = p + \gamma^n(S_w, p) \quad \text{where } \gamma^n(S_w, p) = \int\limits_1^{S_w} f_w\big(s, \Pi^n(s, p)\big)p_c'(s)\,\mathrm{d}s \tag{12}$$

then it is easy to see that we have $p_w = \Pi^n(S_w, p) - p_c(S_w) \leqslant p \leqslant \Pi^n(S_w, p) = p_g$.

The difference between (12) and (6) is that in (12) we have a family of Cauchy problems to solve. By considering the global pressure p as a parameter, we get

$$\frac{\mathrm{d}\Pi^n(S, p)}{\mathrm{d}S} = \frac{\rho_w \lambda_w(S)p_c'(S)}{\rho_w \lambda_w(S) + c_g \lambda_g(S)\Pi^n(S, p)}, \quad S \in \,]0, 1[, \; \Pi^n(1, p) = p \tag{13}$$

We have to show that this problem has a global solution in $]0, 1]$. The main difficulty is that the right-hand side function in (13) may be infinite at $S = 1$. Note that $u = p_c(S_w)$ can easily be inverted and we can write $S_w = S_w(u)$. Then, for any function of water saturation $f(S_w)$ we write $\hat{f}(u) = f(S_w(u))$. Let us introduce the function $\hat{\Pi}^n(u, p)$ as a solution of the following Cauchy problem (p being a parameter),

$$\frac{\mathrm{d}\hat{\Pi}^n(u, p)}{\mathrm{d}u} = \frac{\rho_w \hat{\lambda}_w(u)}{\rho_w \hat{\lambda}_w(u) + c_g \hat{\lambda}_g(u)\hat{\Pi}^n(u, p)}, \quad u > 0, \; \hat{\Pi}(0, p) = p \tag{14}$$

Now it is easy to see that problem (14) has a global solution, and that $\Pi^n(S_w, p) = \hat{\Pi}^n(p_c(S_w), p)$. Thus the function Π^n is well defined. Then we have from (11)

$$\omega^n(S_w, p) = 1 + \int\limits_1^{S_w} \frac{\partial}{\partial p} f_w\big(s, \Pi^n(s, p)\big)p_c'(s)\,\mathrm{d}s \tag{15}$$

Again, for each p, we have to solve the following linear Cauchy problem:

$$\frac{\mathrm{d}\omega^n(S, p)}{\mathrm{d}S} = -\frac{c_g \rho_w \lambda_w(S)\lambda_g(S)p_c'(S)}{(\rho_w \lambda_w(S) + c_g \lambda_g(S)\Pi^n(S, p))^2}\omega^n(S, p), \quad S \in \,]0, 1[, \; \omega^n(1, p) = 1$$

Now it is easy to obtain an analytic solution of this problem, and $\omega^n(S, p)$ is well determined and strictly positive.

The rest of the computations to obtain the fractional flow formulation is the same as in Section 3 with the difference that p should be replaced by $\Pi^n(S_w, p)$ in all coefficients. Therefore we define:

$$\lambda^n(S_w, p) = \rho_w \lambda_w(S_w) + c_g \lambda_g(S_w)\Pi^n(S_w, p), \qquad f_w^n(S_w, p) = \rho_w \lambda_w(S_w)/\lambda^n(S_w, p)$$

$$\bar{\rho}^n(S_w, p) = \bar{\rho}\big(S_w, \Pi^n(S_w, p)\big), \quad a^n(S_w, p) = a\big(S_w, \Pi^n(S_w, p)\big), \quad \text{and} \quad b_g^n(S_w, p) = b_g\big(S_w, \Pi^n(S_w, p)\big)$$

Then we obtain a new global pressure–saturation formulation of the problem given by:

$$\Phi \frac{\partial}{\partial t}\big(S_w \rho_w + c_g(1 - S_w)\Pi^n(S_w, p)\big) - \mathrm{div}\big(\lambda^n(S_w, p)\mathbb{K}\big(\omega^n(S_w, p)\nabla p - \bar{\rho}^n(S_w, p)\mathbf{g}\big)\big) = \mathcal{F}_w + \mathcal{F}_g \qquad (16)$$

$$\mathbf{Q}_t = -\lambda^n(S_w, p)\mathbb{K}\big(\omega^n(S_w, p)\nabla p - \bar{\rho}^n(S_w, p)\mathbf{g}\big) \qquad (17)$$

$$\Phi \rho_w \frac{\partial S_w}{\partial t} + \mathrm{div}\big(f_w^n(S_w, p)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g^n(S_w, p)\big) - \mathrm{div}\big(\mathbb{K}a^n(S_w, p)\nabla S_w\big) = \mathcal{F}_w \qquad (18)$$

Note that the expression of ω in the simplified formulation (7)–(9) and ω^n in the new formulation (16)–(18) are not simply related. It should be noted that this new formulation requires to solve a family of ordinary differential equations which could numerically be done by using standard libraries existing in the literature.

5. Numerical comparison

In this section we compare numerically coefficients in simplified and new global pressure formulations.

Data are chosen from the benchmark problem Couplex-Gaz [6] proposed by the ANDRA (the French National Radioactive Waste Management Agency), we consider the van Genuchten functions with the parameter $n = 1.54$.

Differences between corresponding coefficients in PDEs (7)–(9) and (16)–(18) behave in a consistent way when varying the global pressure p, so that it is sufficient to represent one of them. We choose to present the diffusion coefficient $a(S_w, p)$ from (9), and the corresponding diffusion coefficient $a^n(S_w, p) = a(S_w, \Pi^n(S_w, p))$ from (18).
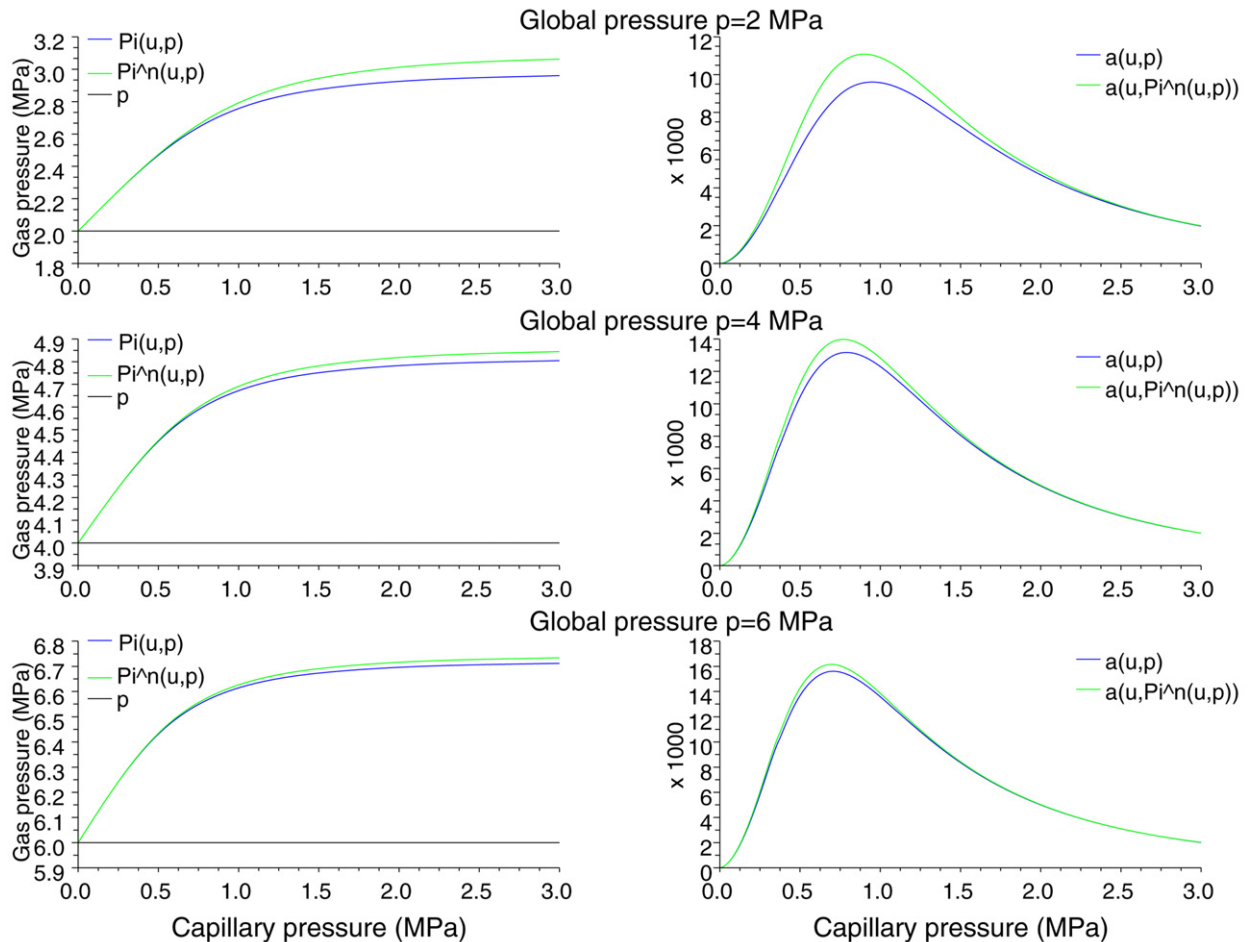

Fig. 1. Comparison of coefficients between simplified and new models.

Furthermore, to see better graphics of coefficients we have presented them as functions of the capillary pressure u instead of the water saturation S_w. The two diffusion coefficients are presented in the right column of Fig. 1. In the left column we plot gas pressure $\hat{\Pi}^n(u, p)$ (see (14)) from the new global formulation and the gas pressure $\hat{\Pi}(u, p)$ given from the simplified global formulation (see (6)), where hat denotes again that the water saturation is replaced by the capillary pressure. We present functions for three different fixed values of the global pressure p, namely, 2, 4 and 6 MPa.

From the behavior of coefficients we can draw several conclusions. If we look only at the error committed by calculating the coefficients in global pressure instead of gas pressure then we see that this error can be significant only when typical capillary pressure in the system is comparable to global or gas pressure. That may be the case for small operating pressures, for example in hydro-geological applications of water–air system. In the other hand that difference can be safely ignored in typical oil field conditions. On the other hand, the error committed in calculating gas density in (7) by replacing the gas pressure by the global pressure stays significant and leads to unacceptable loss of mass balance which is an important property of the physical solution. This can clearly be seen from the left column of Fig. 1.

Acknowledgements

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