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Numerical analysis of a quasistatic piezoelectric problem with damage ☆

José R. Fernández [a,*], Rebeca Martínez [a], Georgios E. Stavroulakis [b]

[a] *Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, Escola Politécnica Superior, Campus Universitario, 27002 Lugo, Spain*

[b] *Department of Production Engineering and Management, Technical University of Crete, GR-73100 Chania, Greece*

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Abstract

The quasistatic evolution of the mechanical state of a piezoelectric body with damage is numerically studied in this paper. Both damage and piezoelectric effects are included into the model. The variational formulation leads to a coupled system composed of two linear variational equations for the displacement field and the electric potential, and a nonlinear parabolic variational equation for the damage field. The existence of a unique weak solution is stated. Then, a fully discrete scheme is introduced by using a finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivatives. Error estimates are derived on the approximate solutions, from which the linear convergence of the algorithm is deduced under suitable regularity conditions. Finally, a two-dimensional example is presented to demonstrate the behaviour of the solution. ***To cite this article: J.R. Fernández et al., C. R. Mecanique 336 (2008).***

Résumé

Analyse numérique d'un problème quasi statique piézoélectrique avec endommagement. On considère l'analyse numérique d'un problème quasi statique en piézoélectricité. L'endommagement est aussi inclu dans le modèle. Le problème variationnel est formulé comme deux équations variationnelles linéaires pour les déplacements et le potentiel électrique et une équation variationnelle non-linéaire parabolique pour l'endommagement. L'existence et l'unicité de solution faible pour ce problème sont établies. On étudie l'approche numérique du problème, avec une méthode d'éléments finis pour l'approximation en la variable espace et un schéma d'Euler pour la discrétisation temporelle. Alors, on démontre des résultats d'estimation de l'erreur et de convergence linéaire de l'algorithme sous des hypothèses de régularité additionnelles. Finalement, on présente des résultats numériques en dimension deux pour démontrer le comportement de la solution. ***Pour citer cet article : J.R. Fernández et al., C. R. Mecanique 336 (2008).***

Keywords: Computational solid mechanics; Damage; Piezoelectricity; Error estimates; Numerical simulations

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* Corresponding author.
E-mail addresses: jramon@usc.es (J.R. Fernández), rebeca.martinez2@rai.usc.es (R. Martínez), gestavr@dpem.tuc.gr (G.E. Stavroulakis).

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Soit un corps piézoélectrique occupant une ouverture $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, de frontière Γ, suffisamment regulière et divisée en deux parties disjointes et mesurables Γ_D et Γ_N, et en Γ_A et Γ_S tels que $\mathrm{mes}(\Gamma_D) > 0$ et $\mathrm{mes}(\Gamma_A) > 0$. Soit $\boldsymbol{\nu}$ le vecteur unitaire de la normale sortante à Γ et soit $[0, T]$, $T > 0$, un intervalle de temps. Le corps est supposé fixe sur la partie Γ_D alors que des forces mécaniques volumiques et surfaciques de densités $\boldsymbol{f}_B$ et $\boldsymbol{f}_N$ agissent respectivement dans Ω et sur Γ_N. En plus, on suppose que le potentiel électrique est fixe sur la partie Γ_A et que des charges électriques volumiques et surfaciques de densités q_B et q_S sont présentes respectivement dans Ω et sur Γ_S. On néglige les termes d'inertie dans l'équation du mouvement ce qui revient à considérer une approche quasi statique du problème.

La formulation variationnelle du problème s'écrit alors de la forme suivante (voir la Section 1 pour les notations et les fonctionnelles) :

Problème PV. Trouver le champs de déplacements $\boldsymbol{u} : [0, T] \to V$, le potentiel électrique $\varphi : [0, T] \to W_A$ et l'endommagement $\zeta : [0, T] \to H^1(\Omega)$ tels que $\zeta(0) = \zeta_0$ et p.p. $t \in (0, T)$:

$$\big(\eta_*\big(\zeta(t)\big)\big[\mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \mathcal{E}^*\nabla\varphi(t)\big], \boldsymbol{\varepsilon}(\boldsymbol{w})\big)_Q = \big(\boldsymbol{f}(t), \boldsymbol{w}\big)_V \quad \forall \boldsymbol{w} \in V,$$
$$\big(\zeta'(t), \xi\big)_Y + a\big(\zeta(t), \xi\big) = \big(\phi\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \zeta(t), \nabla\varphi(t)\big), \xi\big)_Y \quad \forall \xi \in H^1(\Omega),$$
$$\big(\eta_*\big(\zeta(t)\big)\big[\beta\,\nabla\varphi(t) - \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big], \nabla\psi\big)_H = \big(q(t), \psi\big)_W \quad \forall \psi \in W.$$

Sous les hypothèses (i)–(vi) on peut démontrer, Théorème 1.1, que le Problème PV a une solution unique.

Ensuite, dans la Section 2 on présente l'approche numérique de ce problème variationnel, en utilisant une méthode d'éléments finis pour approcher la variable espace et le schéma d'Euler pour discrétiser les derivées temporelles (voir le Problème VP^{hk}). Alors, on démontre des résultats d'estimation de l'erreur, Proposition 2.1, et de convergence linéaire de l'algorithme sous des hypothèses de régularité additionnelles (voir le Théorème 2.1).

Afin de vérifier la précision et l'efficacité de la méthode numérique décrite ci-dessus, on a mis en oeuvre quelques experiences numériques en dimension 2 que nous résumons dans la Section 3. La résolution numérique du problème discrétisé se réalise en trois étapes. D'abord on calcule les déplacements et le potentiel électrique au temps initial comme la solution d'un système linéaire non symétrique qui est résolu avec la méthode LU. Après, on obtient l'endommagement au temps t_n en utilisant la méthode de Cholesky pour résoudre le système linéaire symétrique (4). Les déplacements et le potentiel électrique sont alors obtenus comme la solution d'un système linéaire non symétrique résolu, une nouvelle fois, par la méthode LU. Finalement, on présente un exemple bidimensionnel avec une différence de potentiel électrique sur la frontière. Les résultats obtenus sont montrés dans les Figs. 1 et 2. Comme l'on peut voir, il y a une déformation dans le corps due au potentiel électrique, qui donne lieu à l'endommagement du matériel.

1. Mechanical problem and variational formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a domain occupied by an piezoelectric body with boundary $\partial\Omega = \Gamma$, assumed to be sufficiently smooth and decomposed into two disjoint measurable parts Γ_D and Γ_N, on one hand, and into two disjoint measurable parts Γ_A and Γ_S, on the other hand, such that meas $(\Gamma_D) > 0$, meas $(\Gamma_A) > 0$. For each $\boldsymbol{x} \in \Gamma$, let $\boldsymbol{\nu}(\boldsymbol{x})$ be the unit normal outward vector to Γ. Let us denote by $[0, T]$, $T > 0$, the time interval of interest. Volume forces of density $\boldsymbol{f}_B$ are applied in $\Omega \times (0, T)$, volume electric charges of density q_B are present in $\Omega \times (0, T)$, traction forces of density $\boldsymbol{f}_N$ act on $\Gamma_N \times (0, T)$ and surface electric charges of density q_S are found on $\Gamma_S \times (0, T)$.

We denote the displacement field, the stress tensor, the linearized strain tensor and the electric potential by $\boldsymbol{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ and φ, respectively. We let ζ denote the damage field, which is defined in $\Omega \times (0, T)$ and measures the fractional decrease in the strength of the material.

The material is assumed piezoelectric with constitutive law (see [1]),

$$\boldsymbol{\sigma} = \zeta\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \zeta\mathcal{E}^*\mathbf{E}(\varphi),$$

where $\mathcal{A} = (a_{ijkl})_{i,j,k,l=1}^d$ is the elasticity tensor, $\mathbf{E}(\varphi) = (E_i(\varphi))_{i=1}^d$ is the electric field defined by $E_i(\varphi) = -\frac{\partial\varphi}{\partial x_i}$, $i = 1, \dots, d$, and $\mathcal{E}^* = (e^*_{ijk})_{i,j,k=1}^d$ denotes the transpose of the third-order piezoelectric tensor $\mathcal{E}$. Moreover, according to [2,1], the constitutive law employed for the electric potential is

$$\mathbf{D} = \zeta\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \zeta\beta\mathbf{E}(\varphi),$$

where $\mathbf{D}$ denotes the electric displacement field and β the dielectric tensor.

We now describe the damage process. As a result of the tensile or compressive stresses in the body, micro-cracks and micro-cavities open and grow and this causes the load bearing capacity of the material to decrease. Following [3], the evolution of the microscopic cracks and cavities responsible for the damage is described by the parabolic partial differential equation

$$\zeta' - \kappa\Delta\zeta = \phi\big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta, \nabla\varphi\big),$$

where ∇ denotes the gradient operator, Δ is the Laplace operator, the prime denotes the time derivative, $\kappa > 0$ is a diffusion constant and ϕ represents the damage source function.

We assume that there is no damage influx throughout the boundary Γ and, therefore, $\partial\zeta/\partial\boldsymbol{\nu} = 0$ on Γ.

Finally, we describe the boundary conditions for the displacement and electric potential fields.

On the boundary part Γ_D we assume that the body is clamped; that is, $\boldsymbol{u} = \mathbf{0}$ on $\Gamma_D \times (0, T)$. A density of traction forces, denoted by $\boldsymbol{f}_N$, acts on the boundary part Γ_N and so, $\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N$ on $\Gamma_N \times (0, T)$. Moreover, we assume that Ω is subjected to a time-independent prescribed electric potential φ_A on Γ_A and to a density of surface electric charges q_S on Γ_S; that is, $\varphi = \varphi_A$ on $\Gamma_A \times (0, T)$ and $\mathbf{D}\cdot\boldsymbol{\nu} = q_S$ on $\Gamma_S \times (0, T)$.

For technical reasons associated with the loss of coercivity in the elasticity equation, and possible singularities in ϕ as $\zeta \to 0$, we introduce the truncation operator η_* defined as $\eta_*(r) = 1$ if $r > 1$, $\eta_*(r) = r$ if $1 > r > \zeta_*$ and $\eta_*(r) = \zeta_*$ if $r < \zeta_*$. This lower limit for the damage, ζ_*, is postulated because when the damage is substantial, the material is likely to develop a crack and to model the material as elastic ceases to make sense.

Let $Y = L^2(\Omega)$, $H = [L^2(\Omega)]^d$, and denote the space of second order symmetric tensors by $Q = \{\boldsymbol{\tau} \in [L^2(\Omega)]^{d\times d};\ \tau_{ij} = \tau_{ji},\ i, j = 1, \dots, d\}$.

Let V and W be the spaces defined by

$$V = \big\{\boldsymbol{w} \in \big[H^1(\Omega)\big]^d;\ \boldsymbol{w} = \mathbf{0} \text{ on } \Gamma_D\big\}, \qquad W = \big\{\psi \in H^1(\Omega);\ \psi = 0 \text{ on } \Gamma_A\big\},$$

and let $W_A = \{\psi \in H^1(\Omega);\ \psi = \varphi_A \text{ on } \Gamma_A\}$.

The following assumptions are made on the given data.

(i) The elasticity tensor $\mathcal{A}$ is assumed to be bounded, symmetric and V-elliptic.
(ii) The piezoelectric tensor $\mathcal{E} = (e_{ijk})_{i,j,k=1}^d$ is bounded and it satisfies $e_{ijk} = e_{ikj}$ for $i, j, k = 1, \dots, d$.
(iii) The dielectric tensor β is assumed to be bounded, symmetric and W-elliptic.
(iv) The damage source function ϕ is a Lipschitz and bounded function which satisfies $\phi(\boldsymbol{\varepsilon}, \zeta, \nabla\varphi) \leqslant 0$ if $\zeta \geqslant 1$, and $\phi(\boldsymbol{\varepsilon}, \zeta, \nabla\varphi) \geqslant 0$ if $\zeta \leqslant \zeta_*$.
(v) The density of volume forces, tractions, volume electric charges and surface electric charges have the regularity $\boldsymbol{f}_B \in C([0, T]; H)$, $\boldsymbol{f}_N \in C([0, T]; [L^2(\Gamma_N)]^d)$, $q_B \in C([0, T]; Y)$ and $q_S \in C([0, T]; L^2(\Gamma_S))$.
(vi) The initial condition ζ_0 satisfies $\zeta_0 \in H^1(\Omega)$, $\zeta_0(\boldsymbol{x}) \in (\zeta_*, 1]$ a.e. $\boldsymbol{x} \in \Omega$.

From the Riesz' Theorem, we define the linear mappings $\boldsymbol{f} : [0, T] \to V$ and $q : [0, T] \to W$ as $(\boldsymbol{f}(t), \boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_B(t)\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{f}_N(t)\cdot\boldsymbol{w}\,\mathrm{d}\Gamma$ for all $\boldsymbol{w} \in V$ and $(q(t), \psi)_W = \int_\Omega q_B(t)\psi\,\mathrm{d}\boldsymbol{x} - \int_{\Gamma_S} q_S(t)\psi\,\mathrm{d}\Gamma$ for all $\psi \in W$, respectively, and we construct the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by $a(\xi, \eta) = \kappa\int_\Omega \nabla\xi\cdot\nabla\eta\,\mathrm{d}\boldsymbol{x}$ for all $\xi, \eta \in H^1(\Omega)$.

Using Green's formula and the previous boundary conditions, the variational formulation is as follows:

Problem VP. Find a displacement field $\boldsymbol{u} : [0, T] \to V$, an electric potential field $\varphi : [0, T] \to W_A$ and a damage field $\zeta : [0, T] \to H^1(\Omega)$ such that $\zeta(0) = \zeta_0$ and for a.e. $t \in (0, T)$,

$$\big(\eta_*\big(\zeta(t)\big)\big[\mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \mathcal{E}^*\nabla\varphi(t)\big], \boldsymbol{\varepsilon}(\boldsymbol{w})\big)_Q = \big(\boldsymbol{f}(t), \boldsymbol{w}\big)_V \quad \forall\boldsymbol{w} \in V, \tag{1}$$

$$\big(\zeta'(t),\xi\big)_Y + a\big(\zeta(t),\xi\big) = \big(\phi\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big),\eta_*\big(\zeta(t)\big),\nabla\varphi(t)\big),\xi\big)_Y \quad \forall\xi\in H^1(\Omega), \tag{2}$$

$$\big(\eta_*\big(\zeta(t)\big)\big[\beta\,\nabla\varphi(t) - \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big],\nabla\psi\big)_H = \big(q(t),\psi\big)_W \quad \forall\psi\in W. \tag{3}$$

The existence of a unique weak solution to Problem VP is stated in the following:

Theorem 1.1. *Let assumptions* (i)–(vi) *hold. Then, there exists a unique solution to Problem* VP *such that,* $\boldsymbol{u}\in C([0,T];V)$, $\varphi\in C([0,T];W_A)$, $\zeta\in H^1(0,T;Y)\cap L^2(0,T;H^2(\Omega))\cap C([0,T];H^r(\Omega))$ *for some* $0<r<1$ *and* $\zeta(t)\in[\zeta_*,1]$ *a.e. in* Ω.

The proof of the above theorem is done by using the theory of maximal monotone operators, the Schauder fixed-point theorem and a comparison result (see the recent paper [4] for details).

2. Numerical analysis of a fully discrete scheme

The discretization of Problem VP will be done in two steps. First, we consider three finite dimensional spaces $V^h\subset V$, $W^h\subset W$ and $E^h\subset H^1(\Omega)$ which approximate the spaces V, W and $H^1(\Omega)$, respectively. Here, $h>0$ denotes the spatial discretization parameter and, for the sake of simplicity, we assume that $\varphi_A=0$ and then $W_A=W$. It is straightforward to extend the results presented below to more general situations. Secondly, the time derivatives are discretized by using a uniform partition of the time interval $[0,T]$, denoted by $0=t_0<t_1<\cdots<t_N=T$ and let k be the time step size, $k=T/N$. For a continuous function $f(t)$, let $f_n=f(t_n)$ and for a sequence $\{w_n\}_{n=0}^N$ we let $\delta w_n=(w_n-w_{n-1})/k$ be its corresponding divided differences. Moreover, c is a positive constant which depends on the problem data but it is independent of the discretization parameters k and h.

The fully discrete approximation of Problem VP, based on a hybrid combination of the forward and the backward Euler schemes, is as follows.

Problem VP^{hk}. Find a discrete displacement field $\boldsymbol{u}^{hk}=\{\boldsymbol{u}_n^{hk}\}_{n=0}^N\subset V^h$, a discrete electric potential field $\varphi^{hk}=\{\varphi_n^{hk}\}_{n=0}^N\subset W^h$ and a discrete damage field $\zeta^{hk}=\{\zeta_n^{hk}\}_{n=0}^N\subset E^h$ such that $\zeta_0^{hk}=\zeta_0^h$ and for $n=1,\dots,N$,

$$\big(\delta\zeta_n^{hk},\xi^h\big)_Y + a\big(\zeta_n^{hk},\xi^h\big) = \big(\phi\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}_{n-1}^{hk}\big),\zeta_{n-1}^{hk},\nabla\varphi_{n-1}^{hk}\big),\xi^h\big)_Y \quad \forall\xi^h\in E^h, \tag{4}$$

$$\big(\eta_*\big(\zeta_n^{hk}\big)\big[\mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_n^{hk}\big)+\mathcal{E}^*\nabla\varphi_n^{hk}\big],\boldsymbol{\varepsilon}\big(\boldsymbol{w}^h\big)\big)_Q = \big(\boldsymbol{f}_n,\boldsymbol{w}^h\big)_V \quad \forall\boldsymbol{w}^h\in V^h, \tag{5}$$

$$\big(\eta_*\big(\zeta_n^{hk}\big)\big[\beta\,\nabla\varphi_n^{hk} - \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_n^{hk}\big)\big],\nabla\psi^h\big)_H = \big(q_n,\psi^h\big)_W \quad \forall\psi^h\in W^h, \tag{6}$$

where ζ_0^h is an appropriate approximation of the initial condition ζ_0, and $\boldsymbol{u}_0^{hk}\in V^h$ and $\varphi_0^{hk}\in W^h$ are the solutions to the following problems:

$$\big(\eta_*\big(\zeta_0^h\big)\big[\mathcal{A}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_0^{hk}\big)+\mathcal{E}^*\nabla\varphi_0^{hk}\big],\boldsymbol{\varepsilon}\big(\boldsymbol{w}^h\big)\big)_Q = \big(\boldsymbol{f}_0,\boldsymbol{w}^h\big)_V \quad \forall\boldsymbol{w}^h\in V^h, \tag{7}$$

$$\big(\eta_*\big(\zeta_0^h\big)\big[\beta\,\nabla\varphi_0^{hk} - \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_0^{hk}\big)\big],\nabla\psi^h\big)_H = \big(q_0,\psi^h\big)_W \quad \forall\psi^h\in W^h. \tag{8}$$

Using standard arguments for variational equations, we deduce the existence and uniqueness of the solution to Problem VP^{hk}.

In this section, our interest is focused on the estimation of the numerical errors $\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V$, $\|\varphi_n-\varphi_n^{hk}\|_W$ and $\|\zeta_n-\zeta_n^{hk}\|_Y$. We have the following:

Proposition 2.1. *Let the assumptions* (i)–(vi) *hold. Let* $\{\boldsymbol{u},\varphi,\zeta\}$ *and* $\{\boldsymbol{u}^{hk},\varphi^{hk},\zeta^{hk}\}$ *denote the solutions to problems VP and VP^{hk}, respectively. Let us assume the following regularity conditions on the continuous solution:* $\boldsymbol{u}\in C([0,T];[W^{1,\infty}(\overline{\Omega})]^d)$, $\varphi\in C([0,T];W^{1,\infty}(\overline{\Omega}))\cap C^1([0,T];W)$ *and* $\zeta\in C^1([0,T];Y)\cap C([0,T];H^2(\Omega))$. *Therefore, we have the following error estimates for all* $\boldsymbol{w}^h=\{\boldsymbol{w}_n^h\}_{n=1}^N\subset V^h$, $\psi^h=\{\psi_n^h\}_{n=1}^N\subset W^h$ *and* $\xi^h=\{\xi_n^h\}_{n=1}^N\subset E^h$,

$$
\begin{aligned}
&\max_{0\leqslant n\leqslant N}\left\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_W^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2\right\}+k\sum_{n=1}^{N}\|\zeta_n-\zeta_n^{hk}\|_{H^1(\Omega)}^2\\
&\quad\leqslant c\Bigg(k\sum_{j=1}^{N}\left[\|\zeta_j'-\delta\zeta_j\|_Y^2+\|\boldsymbol{u}_j-\boldsymbol{u}_{j-1}\|_V^2+\|\zeta_j-\xi_j^h\|_{H^1(\Omega)}^2\right]+\|\zeta_0-\zeta_0^h\|_Y^2\\
&\qquad+k^2+\frac{1}{k}\sum_{j=1}^{N-1}\|\zeta_j-\xi_j^h-(\zeta_{j+1}-\xi_{j+1}^h)\|_Y^2+\|\varphi_0-\varphi_0^{hk}\|_W^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^{hk}\|_V^2\\
&\qquad+\max_{1\leqslant n\leqslant N}\left\{\|\boldsymbol{u}_n-\boldsymbol{w}_n^h\|_V^2+\|\varphi_n-\psi_n^h\|_W^2+\|\zeta_n-\xi_n^h\|_Y^2\right\}\Bigg)
\end{aligned}
\tag{9}
$$

where $\boldsymbol{u}_0^{hk}$ and φ_0^{hk} are the unique solutions to discrete problems (7) *and* (8)*, respectively.*

The proof of Proposition 2.1 is done by using a discrete version of Gronwall's inequality and after some tedious algebraic manipulations.

Error estimates (9) are the basis for the analysis of the convergence rate of the algorithm, which we now present. Let $\overline{\Omega}$ be a polyhedral domain and denote by $\mathcal{T}^h$ a regular triangulation of $\overline{\Omega}$ compatible with the partition of the boundary $\Gamma=\partial\Omega$ into Γ_D and Γ_N and into Γ_A and Γ_S.

The linear convergence of the algorithm with respect to $h+k$ is established in the following:

Theorem 2.1. *Let the assumptions of Proposition* 2.1 *hold and denote by $\{\boldsymbol{u},\varphi,\zeta\}$ and $\{\boldsymbol{u}^{hk},\varphi^{hk},\zeta^{hk}\}$ the respective solutions to problems VP and VP^{hk}. Let the finite element spaces V^h, W^h and E^h be composed of continuous and piecewise affine functions and assume that the discrete initial condition ζ_0^h is given by $\zeta_0^h=\pi^h\zeta_0$, where $\pi^h:C(\overline{\Omega})\to E^h$ is the standard finite element interpolation operator.*

Under the additional regularity conditions $\boldsymbol{u}\in C^1([0,T];V)\cap C([0,T];[H^2(\Omega)]^d)$, $\varphi\in C([0,T];H^2(\Omega))$ and $\zeta\in H^2(0,T;Y)\cap H^1(0,T;H^1(\Omega))$, the numerical algorithm introduced in Problem VP^{hk} is linearly convergent; that is, there exists $c>0$, independent of h and k, such that,

$$
\max_{0\leqslant n\leqslant N}\left\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\varphi_n-\varphi_n^{hk}\|_W+\|\zeta_n-\zeta_n^{hk}\|_Y\right\}\leqslant c(h+k)
$$

3. Numerical results

First, the "discrete initial conditions" for the displacements and the electric potential, $\boldsymbol{u}_0^{hk}$ and φ_0^{hk}, are obtained by solving (7) and (8), respectively. We notice that these two coupled linear equations lead to a nonsymmetric linear system in terms of a product variable $\mathbf{x}_0=(\boldsymbol{u}_0^{hk},\varphi_0^{hk})$, which is solved by using LU method.

Secondly, let the solution $(\boldsymbol{u}_{n-1}^{hk},\varphi_{n-1}^{hk},\zeta_{n-1}^{hk})$ at time t_{n-1} be known. We then obtain the discrete damage field at time t_n, ζ_n^{hk}, from the discrete linear variational equation (4), which leads to a symmetric linear system and Cholesky's method is applied for its resolution.

Finally, the discrete displacement field and the discrete electric potential are obtained from the coupled linear equations (5) and (6), respectively. Again, it can be written in terms of a product variable $\mathbf{x}_n=(\boldsymbol{u}_n^{hk},\varphi_n^{hk})$ and the resulting nonsymmetric linear system is solved by using LU method.

The numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical 2D run took about 10 minutes of CPU time.

As a two-dimensional example, we consider a three-dimensional piezoelectric body with a symmetry property on the X_1 direction, in such a way that a plane problem can be assumed on the plane X_2X_3. The two-dimensional resulting body is clamped on its left vertical boundary and we assume that no mechanical forces act in the body and that no electric charges are applied in the body.

The body is assumed PZT-5A, a piezoceramic material of 6 mm symmetry class with coefficients and notations detailed in [5].

The following data were employed in these simulations:

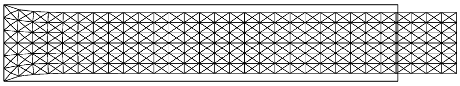

Fig. 1. Deformed mesh (×100) at final time and initial configuration.

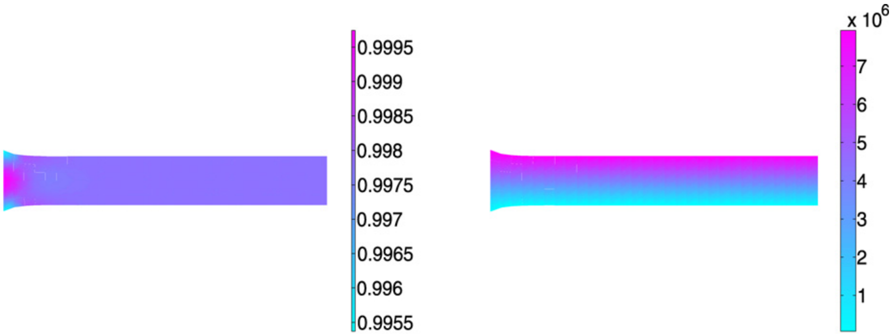

Fig. 2. Damage field and electric potential at final time on the deformed mesh.

$$\Omega = (0,6) \times (0,1.2), \quad \Gamma_D = \{0\} \times [0,1.2], \quad \Gamma_N = \partial\Omega - \Gamma_D, \quad \Gamma_A = [0,6] \times \{0,1.2\}, \quad \Gamma_S = \partial\Omega - \Gamma_A,$$
$$T = 1\ \mathrm{s}, \quad \kappa = 10^{-2}, \quad \boldsymbol{f}_B = \boldsymbol{0}\ \mathrm{N/m^3}, \quad \boldsymbol{f}_N = \boldsymbol{0}\ \mathrm{N/m^2}, \quad q_B = 0\ \mathrm{C/m^3}, \quad q_S = 0\ \mathrm{C/m^2}, \quad \zeta_0 = 1.$$

The damage source function is given by $\phi(\boldsymbol{\varepsilon}, \zeta, \nabla\varphi) = -\max\{\zeta - \zeta_*, 0\}(\lambda_D \frac{1-\zeta}{\zeta} + \lambda_U R(\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u})))$, with the process parameters $\lambda_D = 0.01$ and $\lambda_U = 10^3$, and R being an appropriate truncation operator. Moreover, we assume that $\varphi_A = 0$ on $[0,6] \times \{0\}$ and $\varphi_A = 8 \times 10^6 V$ on $[0,6] \times \{1.2\}$.

Taking $k = 0.01$ as the time discretization parameter, the deformed mesh (amplified by 100) at final time and the initial configuration are shown in Fig. 1. The piezoelectric effect is clearly observed: a deformation is produced because of the piezoelectric effect since no mechanical forces were applied. As a reaction to the electrical forces, the body has an extension in the horizontal direction and a compression in the vertical one. In Fig. 2 the damage field (left-hand side) and the electric potential (right-hand side) are plotted on the deformed mesh. Because of the extension movement and the clamping conditions, the more damaged areas concentrate on the left part of the body. Moreover, since a large difference of electric potential is assumed, the electric potential seems to be constant through the horizontal direction.

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