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An example of a quasi-trapped mode in a weakly non-linear elastic waveguide

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Abstract

This note generalizes an earlier suggested simple example of a trapped mode in a linearly elastic waveguide. A semi-infinite string with a point end mass is considered in the presence of a weakly non-linear support. The effect of non-linearity involves small amplitude non-localized disturbances resulting in a slow time-decay of the vibration amplitude. The rate of the decay is evaluated along with the correction to the vibration phase using the method of multiple scales. ***To cite this article: J. Kaplunov, E. Nolde, C. R. Mecanique 336 (2008).***

Résumé

Un exemple de mode quasi-piégé dans un guide d'ondes élastique faiblement non linéaire. Cette note généralise un exemple simple de mode piégé dans un guide d'ondes élastique linéaire, suggéré précédemment. Une corde semi-infinie terminée par une masse ponctuelle est considérée en présence d'un support faiblement non linéaire. Les effets non linéaires entraînent des composantes de petites amplitudes non localisées, qui conduisent à une décroissance lente de l'amplitude de vibration au cours du temps. Le taux de décroissance temporelle ainsi que la correction de phase sont évalués à partir de la méthode des échelles multiples. ***Pour citer cet article : J. Kaplunov, E. Nolde, C. R. Mecanique 336 (2008).***

Keywords: Dynamical systems; Trapped mode; Weak non-linearity; Multiple scale; Elasticity

Mots-clés : Systèmes dynamiques ; Mode piégé ; Faible non linéarité ; Echelle multiple ; Elasticité

1. Introduction

Apparently, the simplest example of a trapped mode in a linear elastic system may be observed in the case of an elastically supported infinite string with an attached point mass (see [1]). For the latter, a unique explicit solution exists for all values of the parameters. Other examples of trapped modes in inhomogeneous elastic waveguides may be found in [2–6].

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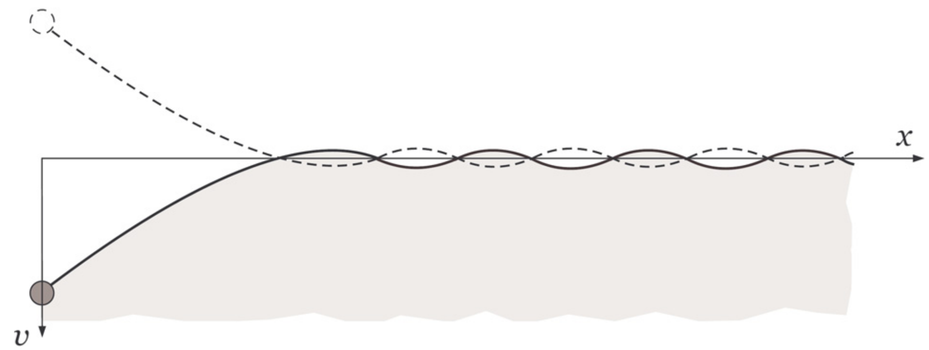

Fig. 1. Quasi-trapped mode.

In this paper we generalize the formulation of [1] by introducing a weak cubic non-linearity into the equation of string motion. In contrast to [1] we consider a semi-infinite waveguide. Asymptotic analysis based on the method of multiple scales reveals the presence of small amplitude non-localized disturbances along with a dominant trapped component. The lowest-order disturbance executes vibration with the triple frequency of the corresponding linear trapped mode. Below we recall the sought-for nearly localized mode, a "quasi-trapped" one.

The developed procedure is oriented to the elimination of the secular terms with respect to the time variable. At the same time it allows secular-type terms exponentially decaying with respect to the longitudinal coordinate.

It is shown that the assumed weak non-linearity leads to perturbations in a slow time to the phase and amplitude of the associated linear solution in the first and second order, respectively. As might be expected, the amplitude demonstrates a time decay due to the radiation to infinity with the aforementioned non-localized components.

We also remark that a similar problem was formulated in [7] within the context of the forced transient vibration under a point instantaneous excitation. However, our derivation is more complete concerning asymptotic consistency.

2. Statement of the problem

Consider a semi-infinite string resting on a weakly non-linear support with a point mass attached to its end, see Fig. 1. The governing equations of the problem can be written as

$$T\frac{\partial^2 v}{\partial x^2} - \rho\frac{\partial^2 v}{\partial t^2} - Kv - \eta v^3 = 0 \tag{1}$$

and

$$T\frac{\partial v}{\partial x} = M\frac{\partial^2 v}{\partial t^2} \quad \text{at } x = 0 \tag{2}$$

Here x is longitudinal coordinate ($0 \leqslant x < \infty$), t is time, v is lateral displacement, T is tension in the string, ρ is density, K and η are foundation parameters, and M is point mass.

First, we scale the lateral displacement v and the independent variables x and t setting

$$v = w\sqrt{\frac{T}{K}}, \quad x = \xi\sqrt{\frac{T}{K}}, \quad t = \tau\sqrt{\frac{\rho}{K}}$$

and rewrite (1) and (2) in the form

$$\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial^2 w}{\partial \tau^2} - w - \varepsilon w^3 = 0 \tag{3}$$

$$\frac{\partial w}{\partial \xi} = m\frac{\partial^2 w}{\partial \tau^2} \quad \text{at } \xi = 0 \tag{4}$$

with two dimensionless problem parameters ε and m given by

$$\varepsilon = \frac{\eta T}{K^2}, \qquad m = \frac{M}{\rho}\sqrt{\frac{K}{T}}$$

where ε is assumed to be small ($\varepsilon \ll 1$).

At infinity we impose the Sommerfeld radiation condition. In addition, we require $w(\infty, t) = \mathrm{O}(\varepsilon)$ to bound the contribution of non-localized disturbances to the analyzed quasi-trapped mode.

3. Asymptotic analysis of the quasi-trapped mode

Let us define the time scales $T_0, T_1, T_2, \ldots (T_n = \varepsilon^n \tau)$, according to the method of multiple scales (e.g. see [8]), and introduce an asymptotic series in terms of the small parameter ε. Then, we have

$$w(\xi, \tau; \varepsilon) = w_0(\xi, T_0, T_1, T_2, \ldots) + \varepsilon w_1(\xi, T_0, T_1, T_2, \ldots) + \varepsilon^2 w_2(\xi, T_0, T_1, T_2, \ldots) + \cdots \tag{5}$$

Below we restrict ourselves to the scales T_0, T_1 and T_2, setting in particular

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2}$$

By substituting the three-scale expansion in (5) into the equation of motion (3) and boundary condition (4) we arrive at leading order

$$\frac{\partial^2 w_0}{\partial \xi^2} - \frac{\partial^2 w_0}{\partial T_0^2} - w_0 = 0 \tag{6}$$

$$\frac{\partial w_0}{\partial \xi} = m \frac{\partial^2 w_0}{\partial T_0^2} \qquad \text{at } \xi = 0 \tag{7}$$

The solution of this problem corresponds to a trapped mode in the associated linear waveguide (e.g. see [1]) and can be written as

$$w_0 = \mathrm{e}^{-\lambda \xi} \left(A \mathrm{e}^{\mathrm{i}\omega T_0} + \bar{A} \mathrm{e}^{-\mathrm{i}\omega T_0} \right) \tag{8}$$

where

$$\lambda = \sqrt{1 - \omega^2}, \qquad \omega = \sqrt{\frac{\lambda}{m}} \tag{9}$$

and therefore

$$\omega = \frac{1}{m} \sqrt{\sqrt{m^2 + \frac{1}{4}} - \frac{1}{2}} < 1 \tag{10}$$

In the above, $A = A(T_1, T_2)$; here and below the bar denotes a complex conjugate.

The first order problem takes the form

$$\frac{\partial^2 w_1}{\partial \xi^2} - \frac{\partial^2 w_1}{\partial T_0^2} - w_1 = 2 \frac{\partial^2 w_0}{\partial T_0 \partial T_1} + w_0^3 \tag{11}$$

$$\frac{\partial w_1}{\partial \xi} = m \left(\frac{\partial^2 w_1}{\partial T_0^2} + 2 \frac{\partial^2 w_0}{\partial T_0 \partial T_1} \right) \quad \text{at } \xi = 0 \tag{12}$$

with w_0 given by (8). Its solution is expressed as

$$\begin{aligned} w_1 = {} & \xi \mathrm{e}^{-\lambda \xi} \left(B_1 \mathrm{e}^{\mathrm{i}\omega T_0} + \bar{B}_1 \mathrm{e}^{-\mathrm{i}\omega T_0} \right) + \mathrm{e}^{-3\lambda \xi} \left(B_2 \mathrm{e}^{\mathrm{i}\omega T_0} + \bar{B}_2 \mathrm{e}^{-\mathrm{i}\omega T_0} + B_3 \mathrm{e}^{3\mathrm{i}\omega T_0} + \bar{B}_3 \mathrm{e}^{-3\mathrm{i}\omega T_0} \right) \\ & + C \mathrm{e}^{\mathrm{i}(3\omega T_0 - k\xi)} + \bar{C} \mathrm{e}^{-\mathrm{i}(3\omega T_0 - k\xi)} \end{aligned} \tag{13}$$

with

$$k = \sqrt{9\omega^2 - 1} \tag{14}$$

where the terms with coefficients $B_i = B_i(T_1, T_2)$ and $\bar{B}_i$, $i = 1, 2, 3$, are particular solutions of the inhomogeneous equation (11), whereas the terms with $C = C(T_1, T_2)$ and $\bar{C}$ satisfy the homogeneous equation (11) and are necessary for satisfying the boundary conditions (12). Below we consider the parameter domain $\omega > 1/3$. In this case the non-localized components in (13) (terms with C and $\bar{C}$) cause the loss of the vibration energy due to radiation to infinity.

It is also remarkable that we allow in (13) the terms with $\xi e^{-\lambda\xi}$. These, however, are not of great interest at large values of the longitudinal coordinate ξ. Therefore, they do not require a special care characteristic of secular terms.

As a result, we have from (11) and (12)

$$B_1 = -\frac{i\omega}{\lambda}\frac{\partial A}{\partial T_1}, \quad B_2 = \frac{3}{8\lambda^2}A^2\bar{A}, \quad B_3 = \frac{1}{8}A^3, \quad C = -\frac{3\lambda(9\lambda + ik)}{32(10 - 9\omega^2)}A^3$$

In addition, by canceling out the secular terms in time, resulting from the summands with $e^{\pm i\omega T_0}$ in the right-hand sides of (11) and (12), we arrive at the equation

$$\frac{\partial A}{\partial T_1} - 4i\delta A^2\bar{A} = 0 \tag{15}$$

with

$$\delta = \frac{3\omega}{16(2 - \omega^2)} \tag{16}$$

By taking

$$A = \frac{1}{2}\alpha e^{i\beta} \tag{17}$$

where $\alpha = \alpha(T_1, T_2)$ and $\beta = \beta(T_1, T_2)$ are real quantities, we obtain

$$\frac{\partial \alpha}{\partial T_1} = 0, \qquad \frac{\partial \beta}{\partial T_1} - \delta\alpha^2 = 0 \tag{18}$$

Thus,

$$\alpha = \alpha_0(T_2), \qquad \beta = \delta\alpha_0^2 T_1 + \beta_0(T_2) \tag{19}$$

and

$$w_0 = \alpha_0 e^{-\lambda\xi}\cos\left(\omega T_0 + \delta\alpha_0^2 T_1 + \beta_0\right) \tag{20}$$

Next, we investigate a second order problem to establish an explicit time dependence for the real amplitude factor α_0 in (20). It takes the form

$$\frac{\partial^2 w_2}{\partial\xi^2} - \frac{\partial^2 w_2}{\partial T_0^2} - w_2 = \frac{\partial^2 w_0}{\partial T_1^2} + 2\frac{\partial^2 w_0}{\partial T_0\partial T_2} + 2\frac{\partial^2 w_1}{\partial T_0\partial T_1} + 3w_0^2 w_1 \tag{21}$$

$$\frac{\partial w_2}{\partial\xi} = m\left(\frac{\partial^2 w_2}{\partial T_0^2} + \frac{\partial^2 w_0}{\partial T_1^2} + 2\frac{\partial^2 w_0}{\partial T_0\partial T_2} + 2\frac{\partial^2 w_1}{\partial T_0\partial T_1}\right) \quad \text{at } \xi = 0 \tag{22}$$

with w_0 and w_1 given by (8) and (13), respectively. The sought-for solution can be written as

$$\begin{aligned} w_2 = {} & e^{-\lambda\xi}\left[(D_1\xi + D_2\xi^2)e^{i\omega T_0} + (\bar{D}_1\xi + \bar{D}_2\xi^2)e^{-i\omega T_0}\right] \\ & + e^{-3\lambda\xi}\left[(D_3 + D_4\xi)e^{i\omega T_0} + (\bar{D}_3 + \bar{D}_4\xi)e^{-i\omega T_0} + (D_5 + D_6\xi)e^{3i\omega T_0} + (\bar{D}_5 + \bar{D}_6\xi)e^{-3i\omega T_0}\right] \\ & + e^{-5\lambda\xi}\left[D_7 e^{i\omega T_0} + \bar{D}_7 e^{-i\omega T_0} + D_8 e^{3i\omega T_0} + \bar{D}_8 e^{-3i\omega T_0} + D_9 e^{5i\omega T_0} + \bar{D}_9 e^{-5i\omega T_0}\right] \\ & + e^{-2\lambda\xi}\left[D_{10}e^{i(\omega T_0 - k\xi)} + \bar{D}_{10}e^{-i(\omega T_0 - k\xi)} + D_{11}e^{i(3\omega T_0 - k\xi)}\right. \\ & \left. + \bar{D}_{11}e^{-i(3\omega T_0 - k\xi)} + D_{12}e^{i(5\omega T_0 - k\xi)} + \bar{D}_{12}e^{-i(5\omega T_0 - k\xi)}\right] \\ & + (C_1 + C_2\xi)e^{i(3\omega T_0 - k\xi)} + (\bar{C}_1 + \bar{C}_2\xi)e^{-i(3\omega T_0 - k\xi)} + C_3 e^{i(5\omega T_0 - \chi\xi)} + \bar{C}_3 e^{-i(5\omega T_0 - \chi\xi)} \end{aligned} \tag{23}$$

with $\chi = \sqrt{25\omega^2 - 1}$, $C_j = C_j(T_1, T_2)$ $(j = 1, 2, 3)$ and $D_i = D_i(T_1, T_2)$ $(i = 1, \ldots, 12)$. The terms with the coefficients D_i, $\bar{D}_i$, C_2 and $\bar{C}_2$ are particular solutions of inhomogeneous equation (21), whereas the terms with C_j and $\bar{C}_j$ $(j = 1, 3)$ satisfy related homogeneous equations. For the sake of brevity, we omit here the explicit expressions for these coefficients.

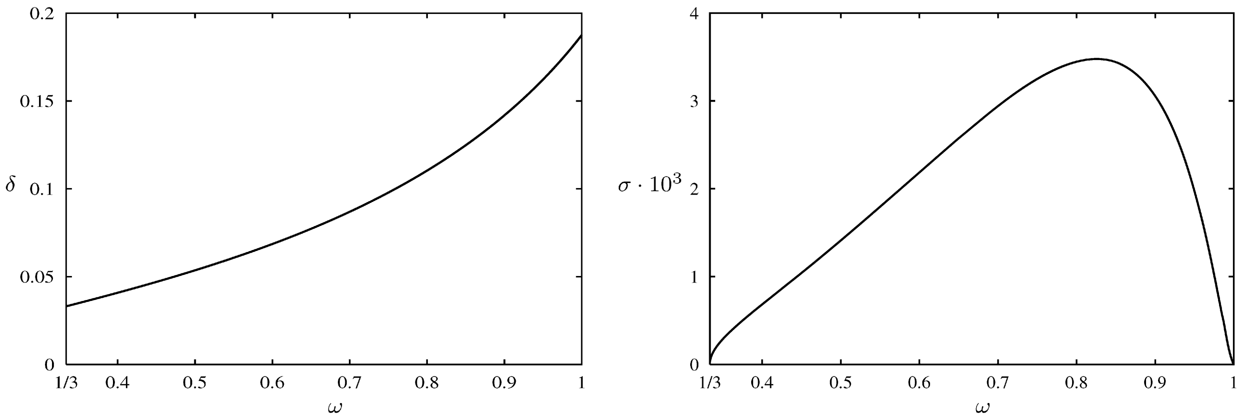

Fig. 2. Coefficients δ and σ in (16) and (27) vs. ω.

The presence of non-localized terms with the time-dependence $e^{\pm i\omega T_0}$ in (23) leads to the decay of the complex vibration amplitude A in the time scale T_2. It satisfies the equation

$$\frac{\partial A}{\partial T_2} + \mu A^3 \bar{A}^2 = 0 \tag{24}$$

with a complex coefficient expressed as

$$\mu = \frac{3i\delta}{2}\left[\frac{3\omega^4 - 3\omega^2 - 2}{(2-\omega^2)^2\lambda^2} - \frac{2}{9} + \lambda^2 \frac{13 - 9\omega^2 - 3ik\lambda}{4(10 - 9\omega^2)}\right] \tag{25}$$

where λ, k and δ are defined by (9), (14) and (16). As before, Eq. (24) follows from the requirement of the absence of secular (in time) terms associated with the components with $e^{\pm i\omega T_0}$ in the right-hand sides of (21) and (22).

Now, assuming (17) with α and β given by (19), we have for the real amplitude factor α_0

$$\frac{\partial \alpha_0}{\partial T_2} + \frac{1}{4}\sigma \alpha_0^5 = 0 \tag{26}$$

where

$$\sigma = \frac{9k\lambda^3\delta}{32(10 - 9\omega^2)} \tag{27}$$

Then, we obtain the explicit formula describing a slow time-decay in (20). It is

$$\alpha_0 = \frac{\alpha_*}{\sqrt{\alpha_*^4 \sigma T_2 + 1}} = \frac{\alpha_*}{\sqrt{\alpha_*^4 \sigma \varepsilon^2 \tau + 1}} \tag{28}$$

where α_* is an arbitrary constant. Thus, the problem parameters come to the expression of the quasi-trapped mode (20) with (19) and (28) through the dimensionless coefficients δ and σ. These depend only on the natural frequency ω given by (10) and are depicted in Fig. 2.

In conclusion we mention that a weak non-linearity should have a similar effect on mode trapping in more complicated elastic waveguides studied in [2–6].

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