Available online at www.sciencedirect.com
ScienceDirect
C. R. Mecanique 336 (2008) 428–433
COMPTES RENDUS
MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

Nonlocal discrete p-Laplacian driven image and manifold processing ☆

Abderrahim Elmoataz, Olivier Lezoray *, Sébastien Bougleux

Université de Caen Basse-Normandie and ENSICAEN, GREYC UMR CNRS 6072, 6, boulevard Maréchal Juin, 14050 Caen, France

Received 30 July 2007; accepted 14 February 2008

Available online 10 March 2008

Presented by Évariste Sanchez-Palencia

Abstract

A framework for nonlocal discrete p-Laplacian regularization on image and manifold represented by weighted graphs of the arbitrary topologies is proposed. The proposed discrete framework unifies the local and nonlocal regularization for image processing and extends them to the processing of any discrete data living on graphs. ***To cite this article: A. Elmoataz et al., C. R. Mecanique 336 (2008).***

Résumé

Traitement d'images et de données basé sur le p-Laplacian discret non local. Un cadre genéral pour la régularisation basée sur le p-Laplacien discret pour le traitement d'images et de données représentés par des graphes pondérés de topologies arbitraires est proposé dans cet article. Ce cadre unifie la régularisation locale ou non locale sur les images et l'étend naturellement au traitement de données discrètes sur graphes. ***Pour citer cet article : A. Elmoataz et al., C. R. Mecanique 336 (2008).***

Keywords: Computer science; Nonlocal discrete regularization; Weighted graph; p-Laplacian

Mots-clés : Informatique, algorithmique ; Régularisation discrète non locale ; Graphe pondéré ; p-Laplacien

Version française abrégée

La régularisation d'images par méthodes variationelles a été utilisée avec succès pour résoudre un grand nombre d'applications en traitement et analyse d'images [1]. Cependant, les fonctionnelles de régularisation usuelles utilisent des opérateurs différentiels qui sont par nature locaux. Depuis les travaux de Buades et Morel [2], sur le filtrage d'image par moyennes non locales, plusieurs travaux récents ont montré l'intérêt d'introduire des fonctionnelles de régularisation non locales pour prendre en compte des interactions plus complexes et introduire plus de flexibilité

☆ This research work was partially supported by the ANR foundation under grant ANR-06-MDCA-008-01/FOGRIMMI.

* Corresponding author.

E-mail addresses: abder.elmoataz@greyc.ensicaen.fr (A. Elmoataz), olivier.lezoray@unicaen.fr (O. Lezoray), sebastien.bougleux@greyc.ensicaen.fr (S. Bougleux).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2008.02.008

dans les fonctionnelles de régularisation. Kindermann, Osher et Jones [3] ont été les premiers à interpréter le filtre à moyennes non locales ainsi que d'autres filtres de voisinage sous forme d'une régularisation d'image basée sur des fonctionnelles non locales. Récemment, Gilboa et Osher [4] ont proposé une fonctionnelle quadratique non locale pour la régularisation d'image et la segmentation semi-supervisée qui se veut plus efficace et plus flexible que les méthodes de régularisation locales. Ces travaux peuvent être considérés comme des analogues non locaux des modèles de régularisation basés sur la variation totale. Cependant, ces méthodes ont été développées en supposant que les images sont continues dans un domaine continu, et la discrétisation des équations aux dérivées partielles utilisées pour les résoudre n'est pas sans poser de problèmes lorsque les images sont définies sur des domaines irréguliers ou sur des espaces de grandes dimensions. Dans cet article, nous proposons un cadre genéral pour la régularisation basée sur le p-Laplacien discret pour le traitement d'images et de données représentés par des graphes pondérés de topologies arbitraires. Ce cadre unifie la régularisation locale ou non locale sur les images et l'étend naturellement au traitement de données discrètes sur graphes. Toutes les méthodes de régularisation continues locales ou non locales avec une discrétisation donnée peuvent s'interpréter comme des cas particuliers de notre cadre général discret. Comme dans ce cadre la régularisation est directement exprimée en discret, cela ne nécéssite aucune résolution d'équations aux dérivées partielles. Nous considérons une image, ou toute fonction définie sur un domaine discret, comme une fonction définie sur un graphe pondéré $G_w = (V, E)$ où V est un ensemble de sommets, $E \subset V \times V$ un ensemble d'arêtes et w une fonction de similarité définie sur l'ensemble des arêtes. La régularisation discrète non locale de $f^0 \in \mathcal{H}(V)$ est définie par la minimisation suivante :

$$\min_{f \in \mathcal{H}(V)} \left\{ E_w^p(f, f^0, \lambda) = \frac{1}{p} \sum_{v \in V} \|\nabla_w f\|^p + \frac{\lambda}{2} \|f - f^0\|^2_{\mathcal{H}(V)} \right\} \tag{1}$$

où $\mathcal{H}(V)$ est un espace de Hilbert défini sur les sommets de G_w, $p \in [1, +\infty)$ est le dégré de régularité, λ est un paramètre d'attache aux données et $\nabla_w f$ représente le gradient pondéré d'une fonction f sur le graphe. Nous introduisons alors des opérateurs différentiels sur graphe dont le gradient pondéré et le p-Laplacien (Éqs. (3) et (5)). Puis, nous montrons que la minimisation de (1) peut se traduire comme l'analogue discret des équations d'Euler–Lagrange et la solution peut être obtenue par un algorithme de type Gauss–Jacobi (Éq. (10)) ; ceci amène à des processus de filtrage simples et rapides paramétrés par la fonction de pondération w et le degré de régularité p. Ces équations peuvent s'appliquer de la même manière sur des images, des maillages ou des données de dimensions quelconques en adaptant simplement la topologie du graphe et la fonction de pondération (Figs. 1 et 2).

1. Introduction

Regularization by variational methods has shown its effectiveness for many applications. However, there are some limitations in the functionals used in regularization such as total variational models or active contour models (see [1] and references therein). Indeed, the latter are based on derivatives which only consider local features of the data. Since the advent of the nonlocal means filter [2], the use of nonlocal interactions, to capture the complex structures of the data, has received a lot of attention and has shown to be very effective and allows much more flexibility in the regularization process. Kindermann, Osher and Jones [3] were the first to interpret nonlocal means and neighborhood filters as regularization based on nonlocal functionals. Later, Gilboa and Osher [4] have proposed a nonlocal quadratic functional of weighted differences for image regularization and semi-supervised segmentation. These works can be considered as the nonlocal analogues of Total Variation models for image regularization. Most of the proposed regularization processes have been proposed in the context of image processing where images are considered as continuous functions on continuous domains. Then, one considers a continuous energy functional which is classically solved by the corresponding Euler–Lagrange equation or its associated flow. However, the discretization of the underlying differential operators is difficult for high dimensional data and for image and data defined on irregular domains.

In this Note, we propose a framework for nonlocal discrete p-Laplacian regularization on Image and Manifold represented by weighted graphs of the arbitrary topologies [5]. The proposed discrete framework unifies local and nonlocal regularization for image processing and extends them to the processing of any discrete data living on graphs. All the continuous regularization methods (local or nonlocal) with a given discretization scheme can be considered as particular cases of our proposed discrete regularization. Since the proposed framework is directly expressed in a discrete setting, no partial difference equations resolution is needed. The proposed regularization enables local or nonlocal regularization by using appropriated graphs topologies and edge weights. Let $G_w = (V, E)$ be a weighted graph consisting in a set of vertices V, a set of edges $E \subset V \times V$, and a similarity weight function w defined on edges. Let $\mathcal{H}(V)$ be a Hilbert space defined on the vertices of G_w. We formalize the discrete data regularization of a function $f^0 \in \mathcal{H}(V)$ by the following minimization problem:

$$\min_{f \in \mathcal{H}(V)} \left\{ E_w^p(f, f^0, \lambda) = \frac{1}{p} \sum_{v \in V} \|\nabla_w f\|^p + \frac{\lambda}{2} \|f - f^0\|^2_{\mathcal{H}(V)} \right\} \tag{2}$$

where $p \in [1, +\infty)$ is the smoothness degree, λ is the fidelity parameter, and $\nabla_w f$ represents the weighted gradient of the function f over the graph. The solution of problem (2) leads to a family of nonlinear processing methods, parameterized by the weight function, the degree of smoothness, and the fidelity parameter.

2. Discrete *p*-Laplacian regularization

In this section, we recall some basic definitions on graphs and we introduce discrete differential operators [6] and the p-Laplacian on graphs which can be considered as a discrete analogue of the continuous p-Laplacian.

2.1. Preliminary definitions

A *graph* $G_w = (V, E)$ consists in a finite set V of N vertices and a finite set $E \subseteq V \times V$ of edges. We assume G_w to be undirected, with no self-loops and no multiple edges. Let (u, v) be the edge that connects the vertices u and v. An undirected graph is *weighted* if it is associated with a weight function $w : E \to \mathbb{R}_+$ satisfying $w(u, v) = w(v, u)$, for all $(u, v) \in E$, and $w(u, v) = 0$ if $(u, v) \notin E$. The weight function represents a similarity measure between two vertices of the graph. We use the notation $u \sim v$ for two adjacent vertices. Let $\mathcal{H}(V)$ denote the Hilbert space of real-valued functions on vertices. A function $f : V \to \mathbb{R}^m$ in $\mathcal{H}(V)$ assigns a vector $f(v)$ to each vertex $v \in V$. Clearly, f can be represented by a column vector of $\mathbb{R}^N$, $f = [f_1, \ldots, f_N]^T$. By analogy with functional space we define $\int_V f = \sum_V f(u)$. The function space $\mathcal{H}(V)$ is endowed with the usual inner product $\langle f, h \rangle_{\mathcal{H}(V)} := \sum_{v \in V} f(v)h(v)$, where $f, h \in \mathcal{H}(V)$. Similarly, one can define $\mathcal{H}(E)$, the space of real-valued functions on edges.

2.2. Weighted gradient and divergence operators

Let $G_w = (V, E)$ denote a weighted graph. The *difference operator* $d : \mathcal{H}(V) \to \mathcal{H}(E)$ of a function $f \in \mathcal{H}(V)$ on an edge $(u, v) \in E$, is defined by: $(df)(u, v) := \sqrt{w(u, v)}(f(v) - f(u))$, $\forall (u, v) \in E$. The *directional derivative* (or *edge derivative*) of a function $f \in \mathcal{H}(V)$ at a vertex v along an edge $e = (u, v)$, is defined as $\partial_v f_u := (df)(u, v)$. The *weighted gradient operator* ∇_w of a function $f \in \mathcal{H}(V)$ at a vertex v is the vector operator defined by $\nabla_w f(v) = (\partial_u f(v) : u \sim v)^T$. The *local variation* of f at v, is defined to be:

$$\|\nabla_w f(v)\| := \sqrt{\sum_{u \sim v} \bigl(\partial_u f(v)\bigr)^2} = \sqrt{\sum_{u \sim v} w(u, v)\bigl(f(u) - f(v)\bigr)^2} \tag{3}$$

It can be viewed as a measure of the regularity of a function around a vertex. The *adjoint operator* of the difference operator, denoted by $d^* : \mathcal{H}(E) \to \mathcal{H}(V)$, is defined by $\langle df, h \rangle_{\mathcal{H}(E)} := \langle f, d^*h \rangle_{\mathcal{H}(V)}$, with $f \in \mathcal{H}(V)$ and $h \in \mathcal{H}(E)$. Using the definitions of the inner products in $\mathcal{H}(V)$ and $\mathcal{H}(E)$, and of the difference operator, we obtain the expression of d^* at a vertex v:

$$(d^*h)(v) = \sum_{u \sim v} \sqrt{w(u, v)}\,\bigl(h(u, v) - h(v, u)\bigr) \tag{4}$$

The *divergence operator*, defined by $-d^*$, measures the net outflow of a function in $\mathcal{H}(E)$ at each vertex of V.

2.3. A family of weighted p-Laplace operators

The *weighted p-Laplace operator*, $\Delta_w^p : \mathcal{H}(V) \to \mathcal{H}(V)$ with $1 \leqslant p < +\infty$, is defined by

$$\Delta_w^p f := \frac{1}{p} d^* \big(\|\nabla_w f\|^{p-2} df \big)$$

Substituting the difference operator and (4) into the definition of $\Delta_w^p f$, we obtain the expression of Δ_w^p at a vertex v:

$$\Delta_w^p f(v) = \frac{1}{p} \sum_{u \sim v} \gamma(u, v) \big(f(v) - f(u) \big) \tag{5}$$

where γ is the function defined by $\gamma(u,v) := w_{uv}(\|\nabla_w f(v)\|^{p-2} + \|\nabla_w f(u)\|^{p-2})$. The operator Δ_w^p is nonlinear, with the exception of $p = 2$. Furthermore, Δ_w^p is positive semi-definite:

$$\begin{aligned} \langle f, p\Delta_w^p f \rangle_{\mathcal{H}(V)} &= \big\langle f, d^* \big(\|\nabla_w f\|^{p-2} df \big) \big\rangle_{\mathcal{H}(V)} = \big\langle df, \|\nabla_w f\|^{p-2} df \big\rangle_{\mathcal{H}(E)} \\ &= \sum_{v \in V} \big\| \nabla_w f(v) \big\|^{p-2} \sum_{u \sim v} \big((df)(u,v) \big)^2 = \sum_{v \in V} \big\| \nabla_w f(v) \big\|^p \geqslant 0 \end{aligned} \tag{6}$$

The definition of Δ_w^p can be considered as the discrete analogue of the p-Laplacian in the continuous case. When $p = 2$, Δ_w^2 represents the *weighted Laplace operator* on G_w, and (5) reduces to: $\Delta f(v) := \Delta_w^2 f(v) = \sum_{u \sim v} w(u,v)(f(v) - f(u))$. When $p = 1$, Δ_w^1 represents the *weighted curvature operator* on G_w, and expression (5) reduces to: $\kappa f(v) := \Delta_w^1 f(v) = \frac{1}{2} \sum_{u \sim v} w(u,v)(\frac{1}{\|\nabla_w f(v)\|} + \frac{1}{\|\nabla_w f(u)\|})(f(v) - f(u))$. In practice, to avoid zero denominator in the curvature computation, the local variation (3) is replaced by its regularized version: $\|\nabla_w f(v)\|_\epsilon := \sqrt{\|\nabla_w f(v)\|^2 + \epsilon^2}$, with $\epsilon \to 0$ a fixed small constant.

3. *p*-Laplace regularization on weighted graphs

In this Section, one considers a general function $f^0 : V \subset \mathbb{R}^n \to \mathbb{R}^m$ defined on graphs of the arbitrary topologies and we want to regularize this function. In a given context, the function f^0 represents an observation of a clean function $g : V \to \mathbb{R}$ corrupted by a given noise n such that $f^0 = g + n$. Such noise is assumed to have zero mean and variance σ^2, which usually corresponds to observation errors. The regularization of such a function corresponds to an optimization problem which can be formalized by the minimization of a weighted sum of two energy terms:

$$f^* = \min_{f \in \mathcal{H}(V)} \left\{ E_w^p(f, f^0, \lambda) := \frac{1}{p} \sum_{v \in V} \big\| \nabla_w f(v) \big\|^p + \frac{\lambda}{2} \|f - f^0\|^2_{\mathcal{H}(V)} \right\} \tag{7}$$

The first term in (7) is the smoothness term or regularizer, meanwhile the second is the fitting term. The parameter $\lambda \geqslant 0$ is a fidelity parameter, called the Lagrange multiplier, which specifies the trade-off between the two competing terms. Both terms of the energy E_w^p are strictly convex functions of f. In particular, by standard arguments in convex analysis, the problem (7) has a unique solution, for $p = 1$ and $p = 2$, which satisfies:

$$\left. \frac{\partial E_w^p(f, f^0, \lambda)}{\partial f} \right|_v = \frac{1}{p} \frac{\partial}{\partial f} \big\| \nabla_w f(v) \big\|^p + \lambda \big(f(v) - f^0(v) \big) = \Delta_w^p f(v) + \lambda \big(f(v) - f^0(v) \big) = 0, \quad \forall v \in V \tag{8}$$

The solution of problem (7) is also the solution of the system of Eqs. (8). This is a nonlinear system, with the exception of $p = 2$. Substituting the expression of the p-Laplace operator into (8), we obtain:

$$\left(\lambda + \frac{1}{p} \sum_{u \sim v} \gamma(u,v) \right) f(v) - \frac{1}{p} \sum_{u \sim v} \gamma(u,v) f(u) = \lambda f^0(v), \quad \forall v \in V \tag{9}$$

Among the existing methods to solve the system of Eq. (9), we use the Gauss–Jacobi iterative algorithm. Let t be an iteration step, and let $f^{(t)}$ be the solution of Eq. (9) at the step t. The corresponding linearized Gauss–Jacobi algorithm is given by:

$$\begin{cases} f^{(0)} = f^0 \\ \gamma^{(t)}(u,v) = w(u,v)\left(\left\|\nabla_w f^{(t)}(v)\right\|^{p-2} + \left\|\nabla_w f^{(t)}(u)\right\|^{p-2}\right), \quad \forall (u,v) \in E \\ f^{(t+1)}(v) = \left(p\lambda + \sum_{u\sim v}\gamma^{(t)}(u,v)\right)^{-1}\left(p\lambda f^0(v) + \sum_{u\sim v}\gamma^{(t)}(u,v) f^{(t)}(u)\right), \quad \forall v \in V \end{cases} \tag{10}$$

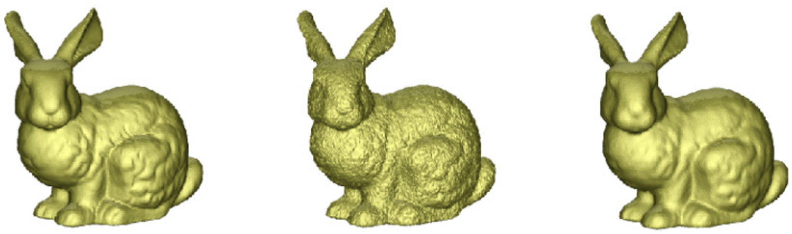
where $\gamma^{(t)}$ is the function γ at the step t. The weights $w(u,v)$ are computed from f^0, or can be given a priori. We define the function φ at an iteration t of algorithm (10) by:

$$\varphi^{(t)}(v,u) = \frac{\gamma^{(t)}(u,v)}{p\lambda + \sum_{u\sim v}\gamma^{(t)}(u,v)} \quad \text{if } u \neq v, \quad \text{and} \quad \varphi^{(t)}(v,v) = \frac{p\lambda}{p\lambda + \sum_{u\sim v}\gamma^{(t)}(u,v)}$$

Then, an iteration of the regularization algorithm (10) is rewritten as:

$$f^{(t+1)}(v) = \varphi^{(t)}(v,v) f^0(v) + \sum_{u\sim v}\varphi^{(t)}(v,u) f^{(t)}(u), \quad \forall v \in V \tag{11}$$

(a) Noisy image ($\sigma = 15$). (b) $p = 2$, local processing. (c) $p = 1$, local processing. (d) $p = 1$, nonlocal processing (3×3 patch). (e) $p = 1$, nonlocal processing (5×5 patch).



(f) Original Bunny (g) Noisy Bunny (h) Regularized ($p = 2$)

Fig. 1. Processing examples for different values of p for local and nonlocal image and mesh processing.

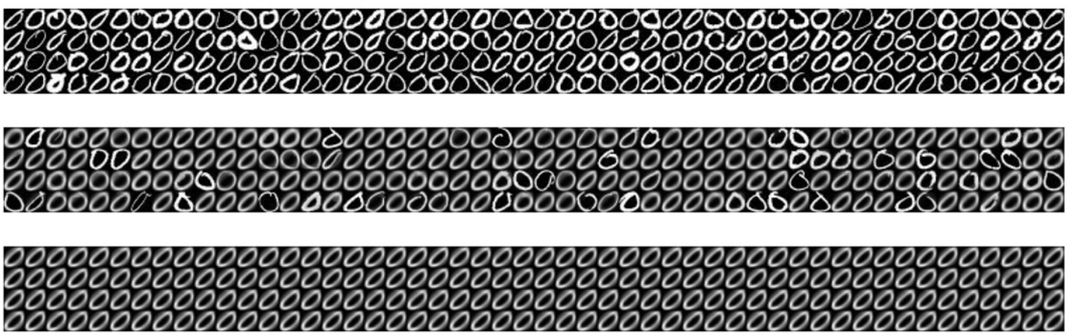


Fig. 2. Nonlocal manifold regularization on USPS handwritten digit 0 database (first line). Second line: regularization with $\lambda = 0.01$. Third line: regularization with $\lambda = 0$.

At each iteration, the new value $f^{(t+1)}$, at a vertex v, depends on two quantities, the original value $f^0(v)$, and a weighted average of the existing values in a neighborhood of v. This shows that the proposed filter, obtained by iterating (11), is a low-pass filter which can be adapted to many graph structures and weight functions.

4. Examples

In this section, we present some examples of the proposed method on images (Fig. 1(a)–(e)), meshes (Fig. 1(f)–(h)) and data base images (i.e. manifolds, see Fig. 2). A grid graph is considered for image processing ($f^0 : V \subset \mathbb{R}^2 \to \mathbb{R}^3$), a neighborhood graph is considered for mesh processing ($f^0 : V \subset \mathbb{R}^3 \to \mathbb{R}^3$) and a complete graph is considered for data processing ($f^0 : V \to \mathbb{R}^{16\times16}$).

References

[1] T.F. Chan, J. Shen, Image Processing and Analysis – Variational, PDE, Wavelets, and Stochastic Methods, SIAM, 2005.
[2] A. Buades, B. Coll, J.-M. Morel, A review of image denoising algorithms, with a new one, Multiscale Modeling and Simulation 4 (2) (2005) 490–530.
[3] S. Kindermann, S. Osher, P.W. Jones, Deblurring and denoising of images by nonlocal functionals, Multiscale Modeling and Simulation 4 (4) (2005) 1091–1115.
[4] G. Gilboa, S. Osher, Nonlocal linear image regularization and supervised segmentation, Multiscale Modeling and Simulation (MMS) 6 (2007) 595–630.
[5] O. Lezoray, A. Elmoataz, S. Bougleux, Graph regularization for color image processing, Computer Vision and Image Understanding (CVIU) 107 (1–2) (2007) 38–55.
[6] A. Bensoussan, J.-L. Menaldi, Difference equations on weighted graphs, Journal of Convex Analysis 12 (1) (2005) 13–44.