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A damage model based on singular elastic fields

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Abstract

At re-entrant corners in elastic structures, the stress field is known to increase to infinity following a power law. From the material viewpoint it is paradoxical that it can locally sustain such an overburden. To avoid this paradox we propose a damage model where the Young's modulus of the material decreases (damage) also following a power law and such that the resulting stress field remains bounded. ***To cite this article: D. Leguillon, C. R. Mecanique 336 (2008).***

Résumé

Un modèle d'endommagement basé sur les champs elastiques singuliers. Aux points anguleux rentrants d'une structure élastique, il est bien connu que les contraintes deviennent infinies en suivant une loi puissance. Du point de vue du matériau, il est paradoxal que celui-ci puisse supporter localement une telle surcharge. Pour éviter ce paradoxe, nous proposons un modèle d'endommagement dans lequel le module d'Young du matériau décroît (endommagement) en suivant également une loi puissance telle que le champ de contraintes résultant reste borné. ***Pour citer cet article : D. Leguillon, C. R. Mecanique 336 (2008).***

Keywords: Damage; Elasticity; Singularities

Mots-clés : Endommagement ; Elasticité ; Singularités

1. Introduction

Within the plane elasticity framework (however, the generalization to 3D cases can also be considered without new difficulties), in homogeneous materials, singularities occur at re-entrant corners, the limit case being the crack where the opening vanishes. In composite structures, the elastic mismatch between components plays an additional role like at a point where an interface breaks a free surface. In the vicinity of these points the elastic displacement, strain and stress fields expand as

$$\begin{cases} \underline{U}(r,\theta) = \underline{U}(0,0) + k_0 r^{\lambda_0} \underline{u}_0(\theta) + \cdots & \text{with } 0 < \lambda_0 \leqslant 1 \\ \underline{\underline{\varepsilon}}(r,\theta) = k_0 r^{\lambda_0 - 1} \underline{\underline{e}}_0(\theta) + \cdots \to \infty & \text{as } r \to 0 \\ \underline{\underline{\sigma}}(r,\theta) = k_0 r^{\lambda_0 - 1} \underline{\underline{s}}_0(\theta) + \cdots \to \infty & \text{as } r \to 0 \end{cases} \tag{1}$$

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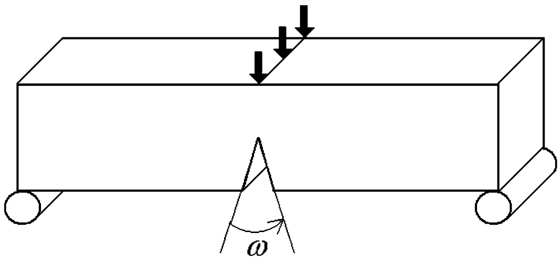

Fig. 1. The 3-point bending test.

Fig. 1. L'essai de flexion 3-points.

where r and θ are the polar coordinates with the origin at the singular point. In $(1)_1$, the leading term in the right hand side member is constant; it is used for consistency. The meaningful term is the next one as can be seen in the strain and stress fields expansions, $(1)_2$ and $(1)_3$, respectively. It is clear that these fields increase to infinity when approaching the singular point. This is the reason why such points are called singular. The coefficient k_0 is the generalized stress intensity factor (GSIF, $\text{MPa}\,\text{m}^{1-\lambda_0}$). The dimensionless shape functions $\underline{u}_0$, $\underline{\underline{e}}_0$ and $\underline{\underline{s}}_0$ are related to each other through the compatibility equations and the constitutive law. The characteristic exponent λ_0, together with $\underline{u}_0$, are a solution to an eigenvalue problem [1]

$$\left(-(\lambda)^2 A + \lambda(B - B^T) + C\right)\underline{u} = 0 \tag{2}$$

where A and C are symmetric operators and $(B - B^T)$ is skew symmetric (the details can be found in [1]). Typically λ_0 equals $1/2$ for a crack in a homogeneous material. The exponent λ_0 is the smallest positive solution to (2) and $\underline{u}_0$ the associated eigenfunction.

The solutions enjoy a peculiar property: if $\lambda_0 > 0$ (required for solutions with finite strain energy) is solution then $-\lambda_0$ is also a solution (from the purely mathematical viewpoint), as well as the conjugates $\bar{\lambda}_0$ and $-\bar{\lambda}_0$ in the complex case. This property plays an important role in the GSIF computation (see Section 4).

The following sections are devoted to the description of the damage law in the particular case of a homogeneous isotropic V-notched specimen under symmetric loading as illustrated in Fig. 1. It is a generic case that can be extended without any difficulty to other situations involving singular points.

2. The damage model

Contrarily to classical approaches, we do not introduce in the elastic model a damage variable and do not postulate a priori an associated evolution law to express the way it varies as a function of other mechanical parameters [2]. We try to provide directly the damage state, i.e. the Young's modulus E variations, from arguments based on the presence of an infinitely large stress field following asymptotically a power law $(1)_3$. The damage law is then a posteriori derived (Section 5). The main assumption in the model is that the Young's modulus itself follows a power law of the form:

$$E = E_0\left(\frac{r}{a}\right)^{\beta} \text{ for } r \leqslant a \quad \text{and} \quad E = E_0 \text{ otherwise (with } \beta \geqslant 0) \tag{3}$$

where E_0 is the undamaged Young's modulus. The elastic solution expansion in power terms can be written:

$$\begin{cases} \underline{U}(r,\theta) = \underline{U}(0,0) + kr^{\lambda}\underline{u}(\theta) + \cdots \\ \underline{\underline{\varepsilon}}(r,\theta) = kr^{\lambda-1}\underline{\underline{e}}(\theta) + \cdots \\ \underline{\underline{\sigma}}(r,\theta) = kr^{\lambda+\beta-1}\underline{\underline{s}}(\theta) + \cdots \end{cases} \tag{4}$$

The exponent λ and the associated shape function $\underline{u}$ are now a solution to the eigenvalue problem:

$$\left(-\lambda(\lambda+\beta)A + \lambda B - (\lambda+\beta)B^T + C\right)\underline{u} = 0 \tag{5}$$

Using the change of variable suggested in [3], $\alpha = \lambda + \beta/2$, the new unknown exponent α is solution of

$$\left(-\alpha^2 A + \alpha(B - B^T) + D(\beta)\right)\underline{u} = 0 \quad \text{with } D(\beta) = C + \frac{\beta^2}{4}A - \frac{\beta}{2}(B + B^T) \tag{6}$$

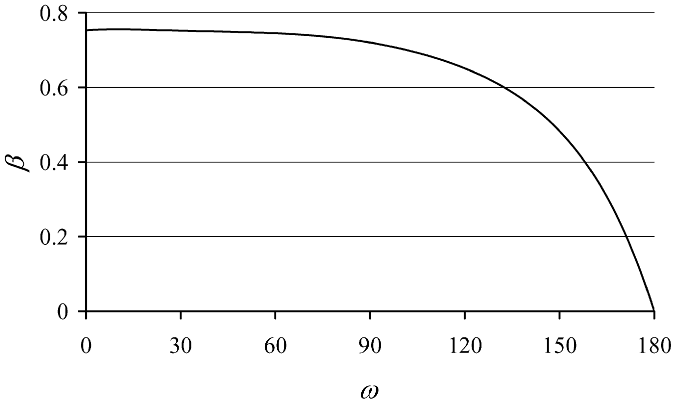

Fig. 2. The damage exponent β vs. the opening ω in a homogeneous isotropic material.

Fig. 2. L'exposant β de la loi d'endommagement vs. l'ouverture ω dans un matériau homogène isotrope.

The operator D is still symmetric, thus the previous property associating α and $-\alpha$ as solutions still holds; and as a consequence of the change of variable, it holds also for λ and $-\lambda-\beta$.

According to (6), α is a function of β: $\alpha=\alpha(\beta)$. The stress field expansion $(4)_3$ is rewritten

$$\underline{\underline{\sigma}}(r,\theta)=kr^{\alpha+\beta/2-1}\underline{\underline{s}}(\theta)+\cdots \tag{7}$$

and then β can be selected so that the stress field remains bounded, solving the equation

$$\alpha(\beta)+\beta/2-1=0 \tag{8}$$

For instance, in a homogeneous isotropic material, a notch opening $\omega=90°$ leads to $\lambda_0=0.545$, $\alpha=0.64$ and $\beta=0.72$. Fig. 2 plots the whole set of results for V-notch openings ω varying from 0° (crack) to 180° (straight edge).

Finally the solution is expressed:

$$\begin{cases} \underline{U}(r,\theta) &= \underline{U}(0,0)+kr^{1-\beta}\underline{u}(\theta)+\cdots \\ \underline{\underline{\varepsilon}}(r,\theta) &= kr^{-\beta}\underline{\underline{e}}(\theta)+\cdots \\ \underline{\underline{\sigma}}(r,\theta) &= k\underline{\underline{s}}(\theta)+\cdots \end{cases} \tag{9}$$

The strain field remains singular, whereas the stress field is finite. A neighbouring feature is met in the HRR solution for power hardening materials [4,5] where, even if the two fields remain singular, the strain one is by far more singular than the stress one.

Of course, the circular shape of the damage zone is a major approximation that makes it possible to carry out very simple calculations. It remains admissible if its radius remains small compared to the size of the structure under consideration.

3. Numerical computations

Solving a FE problem on a $\omega=90°$ notched domain embedding a damage zone with more and more refined meshes near the notch root gives a check on the lack of convergence of the stress field for $\beta<0.72$ (see Section 2), whereas a good convergence is observed for $\beta\geqslant 0.72$. The prescribed displacement field on the remote boundary (i.e. the whole boundary except the notch faces which remain free) is the singular field $r^{\lambda_0}\underline{u}_0(\theta)$ (1) (this choice is convenient but plays no role). Results are illustrated in Fig. 3 where the equivalent Von Mises stress computed at the first node along the bisector is plotted versus the mesh size in the vicinity of the notch root (i.e. the distance between the notch root and the first node along the bisector). The first plot corresponds to $\beta=0$ and exhibits a strong divergence. The second one, $\beta=0.35$, still shows a divergence but 'weaker' than the previous one (keep in mind the vertical scale). In the third plot, $\beta=0.72$, the Von Mises stress remains constant as predicted by theory $(9)_3$. Finally, the fourth plot shows that the solution vanishes at the notch root for larger values of β ($\beta=2$).

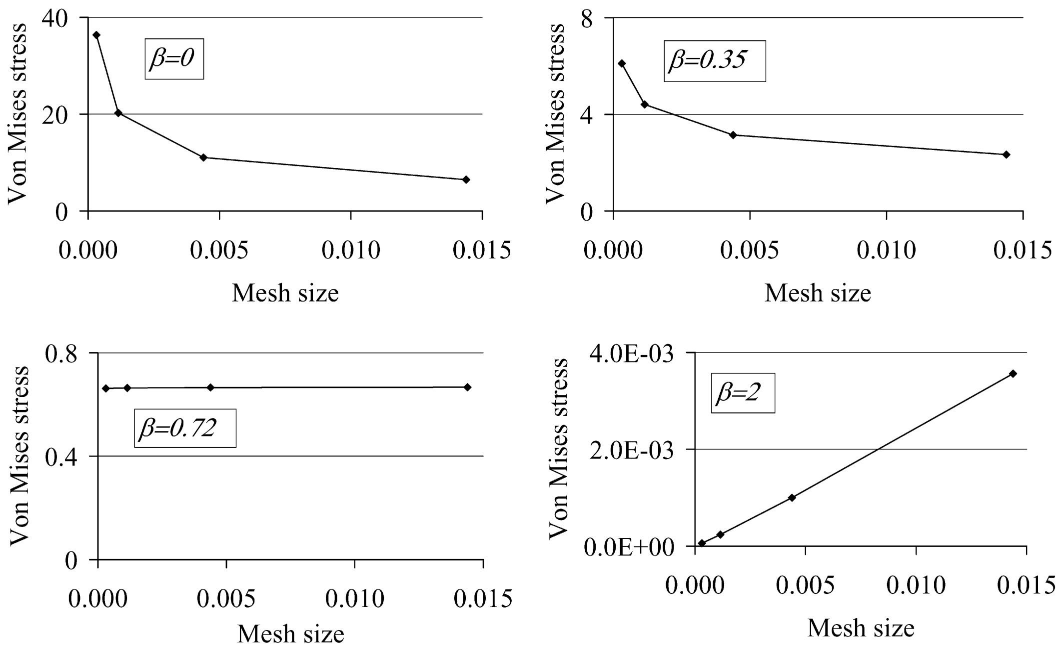

Fig. 3. Convergence of the stress field as a function of the mesh size in the vicinity of the notch root.

Fig. 3. Convergence du champ de contraintes en fonction de la taille de maille au voisinage du fond d'entaille.

4. Generalized stress intensity factor (GSIF) computation

In (9) β and the shape function $\underline{u}$ are known and depend only on the local geometry, the notch opening ω in the present case. They are independent of the global geometry of the structure as well as of the applied load. The roles of the geometry and the load occur in the expansions only through the GSIF k. The path independent integral Ψ [1] can still be used to compute this GSIF, noting that the dual function to $r^{\lambda}\underline{u}(\theta)$ is now $r^{-\lambda-\beta}\underline{u}^{-}(\theta)$ which reduces in the present case to $r^{-1}\underline{u}^{-}(\theta)$

$$k = \frac{\Psi(\underline{U}_h, r^{-1}\underline{u}^{-}(\theta))}{\Psi(r^{\lambda}\underline{u}(\theta), r^{-1}\underline{u}^{-}(\theta))} \quad \text{with } \Psi(\underline{U}, \underline{V}) = \frac{1}{2}\int_{\Gamma} \left[\underline{\underline{\sigma}}(\underline{U}).\underline{n}.\underline{V} - \underline{\underline{\sigma}}(\underline{V}).\underline{n}.\underline{U}\right] \mathrm{d}s \tag{10}$$

The function $\underline{U}_h$ is the FE approximation of $\underline{U}$. The contour Γ can be any line embedded in the damage zone, beginning and finishing on the free faces of the notch, $\underline{n}$ is the normal to Γ pointing toward the notch root (the origin of the polar coordinates).

The shape function $\underline{\underline{s}}$ is normalized in such a way that the hoop stress equals 1 ($s_{\theta\theta} = 1$) along the bisector (the symmetry axis). Thus, according to $(9)_3$, the GSIF k must coincide with the FE value of $\sigma_{\theta\theta}$ at the origin, the higher order terms (higher powers) vanishing at the origin. By FE computations one gets: $\sigma_{\theta\theta} \simeq 1.69$, whereas the path integral (10) calculation gives: $k = 1.67$ ($\omega = 90°$) as shown in Fig. 4.

It can be pointed out that there is a lack of accuracy in the very close vicinity of the notch root. Nevertheless, the visible stress drop remains closely related to the mesh size.

5. Damage growth

Obviously, no damage law is postulated in Section 2. Following a classical definition of the damage variable $d = E/E_0$ and according to (3), the two variables d and a are equivalent. Thus, the damage law can be described by the growth of the damage zone and can be derived by matched asymptotic expansions with the assumption that the radius a remains small compared to the size of the structure under consideration.

The outer expansion or far field ($a \to 0$), expressed in the physical Cartesian space variables x_i is written:

$$\underline{U}^a(x_1, x_2) = \underline{U}^0(x_1, x_2) + f_1(a)\underline{U}^1(x_1, x_2) + \cdots \quad \text{with } f_1(a) \to 0 \text{ as } a \to 0 \tag{11}$$

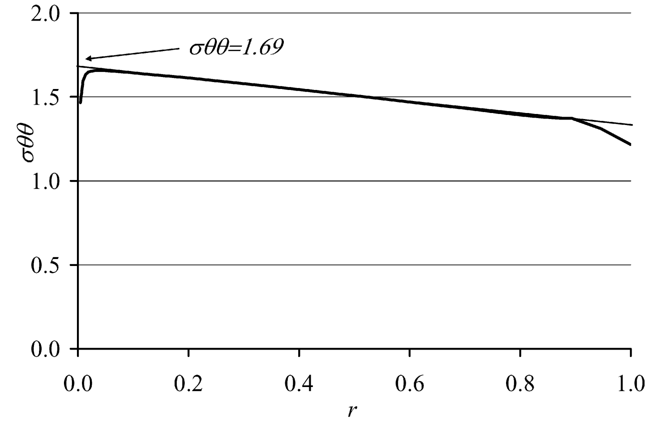

Fig. 4. The hoop stress $\sigma_{\theta\theta}$ along the bisector function of the distance r to the notch root.

Fig. 4. La contrainte ortho-radiale $\sigma_{\theta\theta}$ le long de l'axe de symétrie en fonction de la distance r au fond d'entaille.

It is valid in the whole structure except in a vicinity of the damage zone. The leading term $\underline{U}^0$ is solution to the unperturbed problem (the damage zone is not visible), as a consequence its behaviour is known when approaching the notch root $(1)_1$ (Cartesian and polar coordinates are mixed without confusion)

$$\underline{U}^0(x_1,x_2)=\underline{U}^0(0,0)+k_0 r^{\lambda_0}\underline{u}_0(\theta)+\cdots \tag{12}$$

The inner expansion or near field is expressed in terms of stretched dimensionless variables $y_i = x_i/a$, $a \to 0$, it is valid in the complementary part, i.e. in the vicinity of the damage zone

$$\underline{U}^a(x_1,x_2)=\underline{U}^a(dy_1,dy_2)=\underline{U}^0(0,0)+k_0 a^{\lambda_0}\underline{V}^1(y_1,y_2)+\cdots \tag{13}$$

This form of the expansion is due to the matching conditions which prescribe in addition the behaviour at infinity of the function $\underline{V}^1$

$$\underline{V}^1(y_1,y_2)\sim\rho^{\lambda_0}\underline{u}_0(\theta)\quad \text{as } \rho\to\infty\ (\sim \text{ means "behaves like" and } \rho=r/a) \tag{14}$$

As a consequence the change in potential energy due to this damage zone is, from [6]:

$$\delta W_p = k_0^2 a^{2\lambda_0} L+\cdots \tag{15}$$

The scaling coefficient L (MPa^{-1}) is positive and depends only on the local geometry, i.e. on the notch opening. Let us now assume that the energy required to decrease the Young's modulus (i.e. to damage the material) by ΔE in a surface dS (in 2D) is $D_c \Delta E/E_0\, dS$, where D_c (MPa) is assumed to be a material parameter [7]. Then, in the present case the energy balance leads to (since $dS \approx a^2$)

$$k_0^2 a^{2\lambda_0-2} L \geqslant \kappa D_c \quad \text{where } \kappa=\frac{\beta}{2(\beta+2)}\omega \text{ is a dimensionless parameter} \tag{16}$$

Relation (16) gives an upper bound for a since $2\lambda_0 - 2 < 0$; the damage zone growth is a stable mechanism. Its radius a increases with the load (through k_0)

$$a^{2(1-\lambda_0)} \leqslant \frac{k_0^2 L}{\kappa D_c} \tag{17}$$

leading to a damage law in the following form:

$$\frac{\mathrm{d}a}{\mathrm{d}t}=\frac{1}{1-\lambda_0}\left(\frac{L}{\kappa D_c}\right)^{\frac{1}{2(1-\lambda_0)}}|k_0|^{\frac{2\lambda_0-1}{(1-\lambda_0)}}k_0\frac{\mathrm{d}k_0}{\mathrm{d}t}\quad \text{if } \frac{\mathrm{d}a}{\mathrm{d}t}\geqslant 0;\qquad \frac{\mathrm{d}a}{\mathrm{d}t}=0 \text{ otherwise} \tag{18}$$

In case of a V-notch, it seems reasonable to restrict this law to opening deformations ($k_0 > 0$), nevertheless, this assumption seems not so clear in more entangled situations like composite materials for instance, so we keep it in the above general form.

6. Conclusion

This model has been developed for a future application to fatigue crack initiation at corners and more generally at singular points in complex structures. We aim at providing a more rigorous justification to the empirical use of the GSIF in fatigue laws [8,9].

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