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On the dual variable of the Cauchy stress tensor in isotropic finite hyperelasticity

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Abstract

Elastic materials are governed by a constitutive law relating the second Piola–Kirchhoff stress tensor Σ and the right Cauchy–Green strain tensor $C = F^T F$. Isotropic elastic materials are the special cases for which the Cauchy stress tensor σ depends solely on the left Cauchy–Green strain tensor $B = FF^T$. In this Note we revisit the following property of isotropic hyperelastic materials: if the constitutive law relating Σ and C is derivable from a potential ϕ, then σ and $\ln B$ are related by a constitutive law derived from the compound potential $\phi \circ \exp$. We give a new and concise proof which is based on an explicit integral formula expressing the derivative of the exponential of a tensor. ***To cite this article: C. Vallée et al., C. R. Mecanique 336 (2008).***

Résumé

Variable duale du tenseur des contraintes de Cauchy dans le cas des matériaux hyperélastiques isotropes. Les matériaux élastiques sont régis par une loi de comportement reliant le second tenseur des contraintes de Piola–Kirchhoff Σ et le tenseur de Cauchy–Green droit $C = F^T F$. Les matériaux élastiques isotropes sont les seuls matériaux pour lesquels le tenseur des contraintes de Cauchy σ ne dépend que du tenseur des déformations $B = FF^T$. Dans cette Note nous revisitons la propriété suivante des matériaux isotropes hyperélastiques : si la loi de comportement reliant Σ et C dérive d'un potentiel ϕ, alors σ et $\ln B$ sont reliés par une loi de comportement dérivant du potentiel composé $\phi \circ \exp$. Nous donnons une preuve nouvelle et concise qui est basée sur une formule intégrale explicite exprimant la dérivée de l'exponentiel d'un tenseur. ***Pour citer cet article : C. Vallée et al., C. R. Mecanique 336 (2008).***

Keywords: Finite strain; Isotropic hyperelasticity; Dual variables; Logarithmic strain; Hencky strain tensor; Constitutive law

Mots-clés : Déformation finie ; Hyperélasticité isotrope ; Variables duales ; Déformation logarithmique ; Tenseur des déformations de Hencky ; Loi de comportement

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1. Introduction

According to the mass conservation principle, the mass density per unit volume ρ and its initial value ρ_0 are in the ratio

$$\frac{\rho_0}{\rho} = \det F = (\det C)^{\frac{1}{2}} = (\det B)^{\frac{1}{2}}$$

with F as the deformation gradient and $C = F^T F$ (respectively $B = FF^T$) as the right (respectively left) Cauchy–Green strain tensor.

The relation

$$(\det F)\sigma = F\Sigma F^T$$

between the Cauchy stress tensor σ and the second Piola–Kirchhoff stress tensor Σ can be rewritten

$$\frac{\sigma}{\rho} = F\frac{\Sigma}{\rho_0}F^T$$

Let us formulate the elastic materials constitutive laws as:

$$\frac{\Sigma}{\rho_0} = g(C)$$

The polar decomposition $F = RU$ of the deformation gradient (as the product of a rotation tensor R and a stretch tensor U, [1]) implies:

$$B = FF^T = RU^2R^T = RCR^T \quad \text{or} \quad C = R^T BR$$

This allows us to transform the above relation between Σ and C in a law satisfied by σ:

$$\frac{\sigma}{\rho} = RUg\big(R^T BR\big)UR^T = \big(RUR^T\big)Rg\big(R^T BR\big)R^T\big(RUR^T\big)$$

where we have separated the tensor RUR^T which is nothing else than the square root $B^{\frac{1}{2}}$ of the positive definite symmetric tensor B. A priori, for elastic materials, the tensor $\frac{\sigma}{\rho}$ is a function of B and R:

$$\frac{\sigma}{\rho} = B^{\frac{1}{2}}Rg\big(R^T BR\big)R^T B^{\frac{1}{2}}$$

It will depend solely on B in a single case: when the tensor $Rg(R^T BR)R^T$ does not depend on the rotation R. The rotations forming a group, the only possible tensorial functions g are those satisfying the relations of isotropy with respect to B:

$$Rg\big(R^T BR\big)R^T = g(B) \quad \text{or} \quad R^T g(B)R = g\big(R^T BR\big)$$

Because of the relation $C = R^T BR$, the isotropy of the function g can alternatively be expressed with respect to C:

$$Rg(C)R^T = g\big(RCR^T\big) \quad \text{or} \quad R^T g\big(RCR^T\big)R = g(C)$$

To summarize: *if the law $\frac{\Sigma}{\rho_0} = g(C)$ is isotropic, then $\frac{\sigma}{\rho}$ depends only on B, and it is the sole case*; *furthermore, under this isotropy condition*

$$\frac{\sigma}{\rho} = B^{\frac{1}{2}}g(B)B^{\frac{1}{2}}$$

In this Note, we revisit the property of isotropic hyperelastic materials for which the existence of a potential expressing the constitutive law between $\frac{\Sigma}{\rho_0}$ and C implies the existence of a potential relating $\frac{\sigma}{\rho}$ and $\ln B$.

2. Coaxiality of B and $g(B)$

Theorem 2.1. *The isotropy of g implies that the symmetric tensors B and $g(B)$ are coaxial* (*i.e. they have the same eigenvectors*).

Proof 2.2. Let n be an eigenvector of B chosen unitary, and let us consider the rotation of angle π around n:

$$S = (\cos\pi)I + (1-\cos\pi)nn^T = 2nn^T - I$$

with I as the identity tensor. Such a symmetry S leaves n unchanged and changes any orthogonal vector to n in its opposite. The tensor B being symmetric, its other eigenvectors are orthogonal to n, as a consequence $S^T BS = B$.

The isotropy condition implies $S^T g(B)S = g(S^T BS)$ or $g(B)S = Sg(B)$, therefore $g(B)[Sn] = S[g(B)n]$ or $S[g(B)n] = g(B)n$. Since the sole vectors unchanged by S are the vectors parallel to n, the last equality is possible only when the vector $g(B)n$ remains parallel to the vector n, that is to say when n is also an eigenvector for $g(B)$. □

We easily deduce from this coaxiality property the next corollary, which will reveal important in the following:

Corollary 2.3. *B and $g(B)$ commute. Moreover, for every real number s, $g(B)$ commutes with the power B^s of B.*

Corollary 2.4. *The expression $\frac{\sigma}{\rho} = B^{\frac{1}{2}} g(B) B^{\frac{1}{2}}$ simplifies in $\frac{\sigma}{\rho} = g(B)B$.*

Proof 2.5. It follows from Corollary 2.3 with $s = \frac{1}{2}$. □

3. Isotropy of the constitutive law relating $\frac{\sigma}{\rho}$ and B

Theorem 3.1. *The isotropy of the constitutive law relating $\frac{\Sigma}{\rho_0}$ and C is transferred to the constitutive law relating $\frac{\sigma}{\rho}$ and B.*

Proof 3.2. Let Ω be a rotation, if we change B into $\Omega^T B\Omega$, then $\frac{\sigma}{\rho}$ is changed in:

$$g(\Omega^T B\Omega)\Omega^T B\Omega = \Omega^T g(B)\Omega\Omega^T B\Omega = \Omega^T g(B)B\Omega = \Omega^T \frac{\sigma}{\rho}\Omega \qquad \square$$

4. Hyperelastic materials

4.1. Existence of a potential between the second Piola–Kirchhoff stress tensor Σ and the right Cauchy–Green strain tensor C

Let us consider a differentiable function ϕ of C, its Fréchet derivative $D\phi(C)$ is a linear mapping from the space of symmetric tensors to $\mathbb{R}$. Thus, there exists a symmetric tensor denoted $\frac{\partial\phi}{\partial C}$ such that for every variation δC of C:

$$D\phi(C)\delta C = \operatorname{tr}\left(\frac{\partial\phi}{\partial C}\delta C\right)$$

Hyperelastic materials are those for which there exists a function ϕ such that

$$\frac{\Sigma}{\rho_0} = \frac{\partial\phi}{\partial C}$$

In this assumption, we will say that the constitutive law relating the tensors $\frac{\Sigma}{\rho_0}$ and C is derivable from the potential ϕ.

4.2. Derivative of the exponential of a matrix

Let us consider a square matrix A and a real number t, the exponential $\exp(tA)$ is the unique solution of the matricial ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA) = A\exp(tA)$$

which is equal to I at $t = 0$. The exponential exp thus defined is an invertible mapping. Its inverse is known as logarithmic mapping and denoted ln. Hence, we have

$$\exp(\ln B) = (\exp \circ \ln)(B) = B$$

This relation will be used later on.

Let δA be a variation of A, in the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} D(\exp)(tA)(t\delta A) = \delta A\big[\exp(tA)\big] + AD(\exp)(tA)(t\delta A)$$

let us introduce the square matrix $M(t)$ defined by

$$D(\exp)(tA)(t\delta A) = \big[\exp(tA)\big]M(t)$$

The above equation becomes

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\exp(tA)\right]M(t) + \big[\exp(tA)\big]\frac{\mathrm{d}M}{\mathrm{d}t} = \delta A\big[\exp(tA)\big] + A\big[\exp(tA)\big]M(t)$$

and simplifies itself into the ordinary differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \big[\exp(-tA)\big]\delta A\big[\exp(tA)\big]$$

which can be integrated by quadrature. Because $M(0)$ vanishes, we easily deduce from it the value of $M(1)$ and thereafter the variation of the exponential of a matrix [2]:

$$D(\exp)(A)(\delta A) = \big[\exp(A)\big]\int_0^1 \big[\exp(-sA)\big]\delta A\big[\exp(sA)\big]\,\mathrm{d}s$$

In the special case where A is the logarithm of the positive definite tensor B, this formula allows us to claim that for every variation δB of B:

$$D(\exp)(\ln B)\delta B = B\int_0^1 B^{-s}\delta B B^{s}\,\mathrm{d}s$$

4.3. Existence of a potential between the Cauchy stress tensor and the Hencky strain tensor

Theorem 4.1. *If the tensor $\frac{\Sigma}{\rho_0}$ is derivable from a potential ϕ of the tensor C, then the tensor $\frac{\sigma}{\rho}$ is derivable from the potential $2\phi \circ \exp$ of the Hencky strain tensor $h = \frac{1}{2}\ln B$.*

Proof 4.2. By deriving the compound function $\phi \circ \exp$, we find successively:

$$\begin{aligned} D(\phi \circ \exp)(\ln B)\delta B &= D\phi(B)\big(D(\exp)(\ln B)\delta B\big) \\ &= \mathrm{tr}\left(\frac{\partial \phi}{\partial B}\big[D(\exp)(\ln B)\delta B\big]\right) = \mathrm{tr}\left(g(B)B\int_0^1 B^{-s}\delta B B^{s}\,\mathrm{d}s\right) \\ &= \int_0^1 \mathrm{tr}\big[g(B)BB^{-s}\delta B B^{s}\big]\,\mathrm{d}s \end{aligned}$$

To simplify the last integral, it is necessary to pay attention on the switchings because the matrix δB does not commute with the others. However, under the trace, we can make cross at the beginning the last term of the product of 5 matrices. Then from Corollary 2.3, we can switch this term B^s with $g(B)$ and afterwards with B, it ends up just before B^{-s}. The product of the two matrices B^s and B^{-s} reduces to the identity tensor I, and the integral simplifies itself into

$$\mathrm{tr}\big(g(B)B\delta B\big) = \mathrm{tr}\left(\frac{\sigma}{\rho}\delta B\right)$$

The final value of the integral allows to conclude to the constitutive law:

$$\frac{\sigma}{\rho} = \frac{\partial(\phi \circ \exp)}{\partial(\ln B)} = \frac{\partial(2\phi \circ \exp)}{\partial h} \qquad \square$$

5. Conclusion

Without resorting to the Taylor expansion of the logarithm [3] or of the exponential [4] of a symmetric tensor, nor to its spectral decomposition [5], we have given an intrinsic proof of the existence of the compound potential $\phi \circ \exp$ between $\frac{\sigma}{\rho}$ and $\ln B$. Numerous isotropic hyperelastic constitutive laws expressing directly σ in term of $\ln B$ have been proposed [6–10] and numerically implemented [11].

When the compound potential $\phi \circ \exp$ is convex, the consideration of its Legendre–Fenchel–Moreau transform is a tool to perform the inversion of the constitutive law [12–15], i.e. to express the Hencky logarithmic strain tensor h in term of the Cauchy stress tensor σ.

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