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Effect of couple stresses on the pulsatile flow through a constricted annulus

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Abstract

In this Note, the pulsatile flow of an incompressible couple stress fluid through an annulus with mild constriction at the outer wall is considered. This configuration is intended as a simple model for studying blood flow in a stenosed artery when a catheter is inserted into it. An analytical expression in terms of Bessel functions of the first and second kind is obtained for the velocity component. The impedance (resistance to the flow) and wall shear stress are calculated and their variation with respect to the couple stress fluid parameter, height of the constriction and size of the catheter on the impedance and wall shear stress is studied graphically. It is observed that increase in the catheter size increases the resistance to the flow as well as the wall shear stress while the trend is reversed in case of couple stress fluid parameter. ***To cite this article: D. Srinivasacharya, D. Srikanth, C. R. Mecanique 336 (2008).***

Résumé

Effet de couples de contrainte sur un flux pulsatoire à travers d'un tube annulaire contraint. Dans cette Note on considère le flux pulsatoire d'un fluide incompressible à couple de contrainte à travers d'un tube annulaire dont la paroi externe est légèrement étranglée. Cette configuration sert de modèle simplifié du flux sanguin dans une artère sténosée pendant l'insertion d'un catétère. La composante de vitesse a été obtenue sous la forme d'une expression analytique en termes des fonctions de Bessel de première et seconde espèce. L'impédance de résistance et la tension de cisaillement de la paroi est également évaluée, ainsi que la variation de ces paramètres en fonction du couple de contrainte relatif au fluide. On étudie graphiquement l'influence de la hauteur de la constriction et de la dimension du catétère sur l'impédance et le couple de cisaillement agissant sur la paroi. On observe que l'augmentation des dimensions du catétère fait augmenter la résistance au flux ainsi que la tension de la paroi ; cependant, la tendance est inverse dans le cas du paramètre du couple de cisaillement du fluide. ***Pour citer cet article : D. Srinivasacharya, D. Srikanth, C. R. Mecanique 336 (2008).***

Keywords: Fluid mechanics; Pulsatile blood flow; Catheterized artery; Constriction; Annulus; Couple stress fluid

Mots-clés : Mécanique des fluides ; Flux sanguin pulsatoire ; Artère catérisée ; Constriction ; Tube annulaire ; Fluide à couple de contrainte

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1. Introduction

The study of flow characteristics in a constricted tube has received much attention as it represents a mathematical model for the blood flow in an artery with stenosis (i.e. abnormal and unnatural growth, in the lumen of the artery). Several researchers [1–3] have studied the flow of blood in stenosed artery by considering it as a Newtonian fluid. It is well known that blood, at low shear rates and during its flow through narrow blood vessels, behaves like a non-Newtonian fluid. The couple stress fluid theory developed by Stokes [4] represents the simplest generalization of the classical viscous fluid theory that sustains couple stresses and the body couples. The important feature of these fluids is that the stress tensor is not symmetric and their accurate flow behavior cannot be predicted by the classical Newtonian theory. The main effect of couple stresses will be to introduce a size dependent effect that is not present in the classical viscous theories. This model has been widely used because of its relative mathematical simplicity compared with other models developed for the couple stress fluid. The fluids consisting of rigid, randomly oriented particles suspended in a viscous medium, such as blood fluids, lubricants containing small amount of high polymer additive, electro-rheological fluids and synthetic fluids are examples of these fluids. Chaturani has analyzed the problems of pulsatile flow of couple stress fluid with application to blood flow [5]. An analysis of the effects of couple stresses on the blood flow through thin artery with mild stenosis has been carried out by Sinha and Singh [6]. Srivastava [7] considered the flow of couple stress fluid through stenotic blood vessels.

The flow through an annulus with mild constriction at the outer wall can be used as a model for the blood flow through the catheterized stenotic artery. The insertion of a catheter (a long flexible cylindrical tube) into a constricted tube (i.e. stenosed artery) results in an annular region between the walls of the catheter and artery. This will alter the flow field, modify the pressure distribution and increase the resistance. Even though the catheter tool devices are used for the measurement of arterial blood pressure or pressure gradient and flow velocity or flow rate, X-ray angiography and intravascualar ultrasound diagnosis and coronary balloon angioplasty treatment of various arterial diseases, a little attention has been given in the literature to the flow in catheterized arteries. MacDonald [8] considered the pulsatile blood flow in a catheterized artery and obtained theoretical estimates for pressure gradient corrections for catheters, which are positioned eccentrically, as well as coaxially with the artery. The effect of catheterization on various flow characteristics in an artery with or without stenosis was studied by Karahalios [9]. Daripa and Dash [10] considered the steady and pulsatile flow of the Casson fluid in a narrow artery when a catheter is inserted into it and estimated the increase in frictional resistance in the artery due to catheterization.

In the present study, pulsatile flow of an incompressible couple stress fluid through a constricted annulus is investigated. The velocity, resistance to the flow (impedance) and shearing stress are calculated. The variation of impedance and shearing stress is analyzed for various values of couple stress and geometric parameters.

2. Formulation of the problem

Consider the flow of an incompressible couple stress fluid between an artery, an axisymmetric rigid tube of radius 'a' with a mild constriction (stenosis) and a catheter, a coaxial tube of radius 'ka' ($k \leqslant 1$). Let the length of the tube be L, the magnitude of the distance along the artery over which the stenosis is spread out be L_0, the location of the stenosis be indicated by d and the maximum height of the stenosis be h. The schematic diagram is shown in Fig. 1.

The equations governing the flow of an incompressible couple stress fluid in the absence of body force and body couple are [4]

$$\operatorname{div}\bar{q}^* = 0 \tag{1}$$

$$\rho\left[\frac{\partial \bar{q}^*}{\partial t} + (\bar{q}^*.\nabla)\bar{q}^*\right] = -\operatorname{grad} P^* - \mu \operatorname{curl}\operatorname{curl}\bar{q}^* - \eta \operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}\bar{q}^* \tag{2}$$

where ρ is the density, $\bar{q}^*$ is the velocity vector, η is the couple stress fluid parameter, P^* is the fluid pressure and μ is the fluid viscosity.

The force stress tensor τ and the couple stress tensor M that arises in the theory of couple stress fluids are given by

$$\tau = (-P^* + \lambda_1 \operatorname{div}\bar{q}^*)I + \mu\big[\operatorname{grad}\bar{q}^* + (\operatorname{grad}\bar{q}^*)^T\big] + \frac{1}{2}I \times [\operatorname{div} M + \rho C] \tag{3}$$

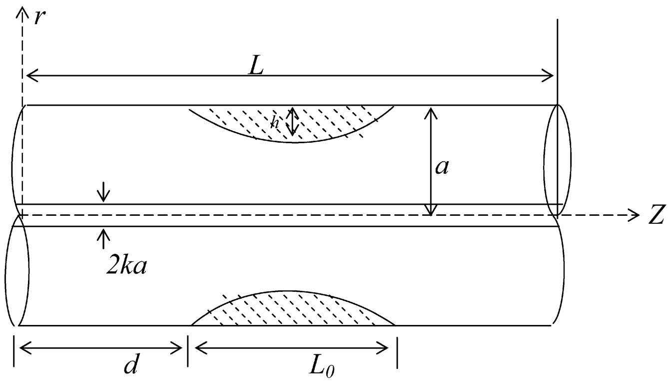

Fig. 1. Schematic diagram of catheterized stenosed artery.

Fig. 1. Schéma d'une artère sténosée avec catétère.

and

$$M = mI + 2\eta_1 \operatorname{grad}(\operatorname{curl} \bar{q}^*) + 2\eta_1 \big(\operatorname{grad}(\operatorname{curl} \bar{q}^*)\big)^T \tag{4}$$

where m is 1/3rd trace of M and ρC is the body couple tensor. The quantity λ_1 is the material constant and η' is the constant associated with couple stresses. The dimensions of the material constant λ_1 is that of viscosity whereas the dimensions of η and η' are those of momentum. These material constants are considered by the inequalities [4].

$$\mu \geqslant 0, \quad 3\lambda + 2\mu \geqslant 0, \quad \eta \geqslant 0, \quad \eta' \leqslant \eta \tag{5}$$

The problem has been studied in cylindrical coordinate system (r, θ, z). Since the flow is axisymmetric all the variables are independent of θ. Hence for this flow the velocity is given by $\bar{q} = (u^*(r,z,t), 0, w^*(r,z,t))$. Assume that the stenosis over a length of the artery being assumed to have developed in an axisymmetric manner. The stenosed wall of the artery is defined as [1]

$$r_s(z) = \begin{cases} a - \dfrac{h}{2}\left[1 + \cos\dfrac{2\pi}{L_0}\left(z - d - \dfrac{L_0}{2}\right)\right], & d \leqslant z \leqslant L_0 + d \\ a & \text{otherwise} \end{cases} \tag{6}$$

It can be shown that the radial velocity is negligibly small and can be neglected for a low Reynolds number flow in a tube with mild stenosis. In this case Eqs. (1) and (2) become

$$-\frac{\partial P^*}{\partial r} = 0 \tag{7}$$

$$\rho \frac{\partial w^*}{\partial t} = -\frac{\partial P^*}{\partial z} + \mu D^2 w^* - \eta D^4 w^* \tag{8}$$

where $D^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$.

Since the flow is pulsatile of circular frequency Ω, we seek a solution of the form $w^*(r,z,t) = w(r,z)e^{i\Omega t}$, $P^* = P(r,z)e^{i\Omega t}$.

Taking w_0 as a typical axial velocity and introducing the following non-dimensional variables

$$r = a\tilde{r}, \quad w = w_0\tilde{w}, \quad P = \frac{w_0 \mu \tilde{p}}{a}, \quad z = L\tilde{z}, \quad r_s = a\tilde{r}_s, \quad t = \frac{\tilde{t}}{\Omega} \tag{9}$$

into Eqs. (6)–(8) and dropping tildes, we get

$$r_s(z) = \begin{cases} 1 - \dfrac{\varepsilon}{2}\left[1 + \cos\dfrac{2\pi}{\gamma}\left(z - \dfrac{d_1}{L} - \dfrac{\gamma}{2}\right)\right], & d_1 \leqslant z \leqslant d_1 + \gamma \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

$$\frac{\partial p}{\partial r} = 0 \tag{11}$$

$$\alpha^2 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + D^2 w - \frac{1}{\varphi^2} D^4 w \tag{12}$$

where $\varepsilon = h/a$, $\gamma = L_0/L$, $d_1 = d/L$, $\alpha^2 = a^2 \Omega \rho/\mu$ is the Womersely number and $\varphi = a(\mu/\eta)^{1/2}$ is the couple stress fluid parameter. The term $-\frac{1}{\varphi^2} D^4 w$ in Eq. (12) gives the effect of couple stresses. Hence, as φ increases, the effect of couple stresses decrease.

The corresponding non-dimensional boundary conditions

$$w = 0 \quad \text{at } r = r_s(z) \text{ and } r = k \tag{13}$$

$$\frac{\partial^2 w}{\partial r^2} - \frac{\sigma}{r} \frac{\partial w}{\partial r} = 0 \quad \text{at } r = r_s(z) \text{ and } r = k \tag{14}$$

where $\sigma = \eta'/\eta$, another couple stress fluid parameter. Boundary condition (14) shows that couple stresses (4) vanish at the tube wall and catheter wall. All the effects of couple stresses will be absent in a material for which $\eta = \eta'$ i.e., $\sigma = 1$. This is equivalent to requiring that the couple stress tensor be symmetric. If couple stress tensor is symmetric, then all its effects will be absent.

In addition to the above boundary conditions, we have the condition that the pressure p is p_1 at $z = 0$ and p_0 at $z = L$.

3. Solution of the problem

It can be noted from (11) that p is a function of z only. Eq. (12) can be simplified to the form

$$\left(D^2 - \alpha_1^2\right)\left(D^2 - \alpha_2^2\right) w = -\varphi^2 \frac{\mathrm{d}p}{\mathrm{d}z} \tag{15}$$

where

$$\alpha_1^2 + \alpha_2^2 = \varphi^2 \quad \text{and} \quad \alpha_1^2 \alpha_2^2 = i\alpha^2 \Omega \varphi^2 \tag{16}$$

Using the separation of variables, the solution of (15) is

$$w(r,z) = C_1(z) I_0(\alpha_1 r) + C_2(z) K_0(\alpha_1 r) + C_3(z) I_0(\alpha_2 r) + C_4(z) K_0(\alpha_2 r) - \frac{1}{i\alpha^2 \Omega} \frac{\mathrm{d}p}{\mathrm{d}z} \tag{17}$$

where $I_0(\alpha_i r)$ and $K_0(\alpha_i r)$ for $i = 1, 2$ are the modified Bessel functions of the zeroth order first and second kind respectively. $C_1(z)$, $C_2(z)$, $C_3(z)$ and $C_4(z)$ are arbitrary functions of z. Using the non-dimensional boundary conditions (12) and (13), we can obtain the values of $C_1(z)$, $C_2(z)$, $C_3(z)$, and $C_4(z)$.

The dimensionless flux, defined as $Q = \int_k^{r_s(z)} 2rw\,\mathrm{d}r$ can be obtained from Eq. (17) in the following form

$$Q = 2\frac{\mathrm{d}p}{\mathrm{d}z} F\left[r_s(z), k\right] \tag{18}$$

where

$$\begin{aligned} F\left[r_s(z), k\right] = {} & \frac{d_1(z)}{\alpha_1}\left[r_s(z) I_1\left(\alpha_1 r_s(z)\right) - k I_1(\alpha_1 k)\right] - \frac{d_2(z)}{\alpha_1}\left[r_s(z) K_1\left(\alpha_1 r_s(z)\right) - k K_1(\alpha_1 k)\right] \\ & + \frac{d_3(z)}{\alpha_2}\left[r_s(z) I_1\left(\alpha_2 r_s(z)\right) - k I_1(\alpha_2 k)\right] - \frac{d_4(z)}{\alpha_2}\left[r_s(z) K_1\left(\alpha_2 r_s(z)\right) - k K_1(\alpha_2 k)\right] \\ & - \frac{((r_s(z))^2 - k^2)(\alpha_1^2 + \alpha_2^2)}{\alpha_1^2 \alpha_2^2} \end{aligned} \tag{19}$$

with $\frac{\mathrm{d}p}{\mathrm{d}z} d_i(z) = C_i(z)$, $i = 1, 2, 3, 4$.

The pressure drop Δp across the tube is obtained from Eq. (18) as

$$\Delta p = \frac{Q}{2} \int_0^1 \frac{\mathrm{d}z}{F[r_s(z), k]} \tag{20}$$

The dimensionless resistance to the flow (resistive impedance), λ is given by

$$\lambda = \frac{\Delta p}{Q} = \frac{1}{2}\int_0^1 \frac{dz}{F[r_s(z),k]} \tag{21}$$

which can be written as

$$\lambda = \frac{1}{2}\left[\int_0^{d_1} \frac{dz}{F[r_s(z),k]} + \int_{d_1}^{d_1+\gamma} \frac{dz}{F[r_s(z),k]} + \int_{d_1+\gamma}^{1} \frac{dz}{F[r_s(z),k]}\right] \tag{22}$$

Since $r_s(z) = 1$, in the regions $0 \leqslant z \leqslant d_1$ and $d_1 + \gamma \leqslant z \leqslant 1$, the resistance to the flow (λ) simplifies to

$$\lambda = \frac{1-\gamma}{F[r_s(z),k]_{r_s(z)=1}} + \gamma \int_0^1 \frac{d\xi}{F[r_s(\xi),k]} \tag{23}$$

where $\xi = (z - d_1)/\gamma$.

The stress T_{rz} can be written as

$$T_{rz} = T_{rz}^s + T_{rz}^A \tag{24}$$

where T_{rz}^s is the symmetric part and T_{rz}^A is anti-symmetric parts of the stress and are given by

$$T_{rz}^s = \mu\frac{\partial w}{\partial r} \quad \text{and} \quad T_{rz}^A = \frac{1}{2}\left(\frac{\partial m_{r\theta}}{\partial r} + \frac{m_{r\theta} + m_{\theta r}}{r}\right)$$

where $m_{r\theta} = \frac{1}{2}(-\eta\frac{\partial^2 w}{\partial r^2} + \frac{\eta'}{r}\frac{\partial w}{\partial r})$ and $m_{\theta r} = \frac{1}{2}(-\eta'\frac{\partial^2 w}{\partial r^2} + \frac{\eta}{r}\frac{\partial w}{\partial r})$.

Hence the total stress is given by

$$\begin{aligned} T_{rz} &= \alpha_1\left(1 - \frac{\alpha_1^2}{4(\alpha_1^2+\alpha_2^2)}\right)\left[C_1(z)I_1(\alpha_1 r) - C_2(z)K_1(\alpha_1 r)\right] \\ &\quad + \alpha_2\left(1 - \frac{\alpha_2^2}{4(\alpha_1^2+\alpha_2^2)}\right)\left[C_3(z)I_1(\alpha_2 r) - C_4(z)K_1(\alpha_2 r)\right] \end{aligned} \tag{25}$$

4. Results and discussions

The system of equations in terms of $C_1(z)$, $C_2(z)$, $C_3(z)$, and $C_4(z)$ obtained by using the boundary conditions (13) and (14), the dimensionless impedance (23) and the shearing stress (25) are evaluated numerically using MATHEMATICA for various values non-dimensional parameters like ε (the maximum height of the stenosis, which is very small because of the mild stenosis), k (the catheter size) and φ and σ (the parameters which represent the effect of couple stresses) and the results are graphically presented in Figs. 2–7.

It is interesting to notice from these figures that the impedance and stress distribution curves are intersecting at a particular value of t. Further, it is observed that the impedance or the stress distributions are decreasing as t increases and then starts increasing after certain value of t. The distributions are negative for some period of time and become positive. The time for which this change takes place corresponds to separation of flow [3].

Under a given pressure gradient a greater resistance implies less flow of fluid. Thus the impedance gives the measure of the volume of the fluid transported by the artery. Fig. 2 shows the variation of impedance with t for different values of the catheter size k when $\varphi = 1.5$, $\gamma = 0.5$, $\alpha = 1.5$, $\sigma = 0.5$, $\varepsilon = 0.1$. $k \to 0$ corresponds to the case when there is no catheter. It can be observed from this figure that the presence of catheter in a stenosed artery increases the impedance. Further, as the size of the catheter (k) increases the impedance increases significantly. Fig. 3 shows the effect of φ, the couple stress parameter, on the impedance for the fixed values of $k = 0.1$, $\gamma = 0.5$, $\alpha = 1.5$, $\sigma = 0.5$, $\varepsilon = 0.15$. As φ increases impedance decreases rapidly. $\varphi \to \infty$ corresponds to Newtonian fluid case. It can be noted that the impedance in case of couple stress fluid is more than that of a Newtonian fluid case i.e., the presence of couple stresses in the fluid increases the impedance. Fig. 4 shows the effect of σ on impedance. Here it is worth

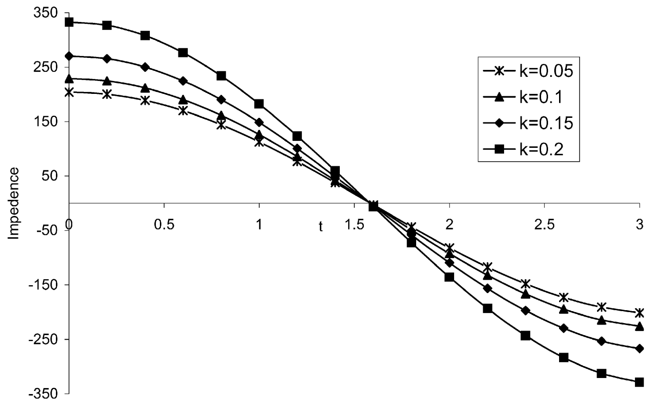

Fig. 2. Effect of k on impedence with $\varphi = 1.5$, $\gamma = 0.5$, $\alpha = 1.5$; $\sigma = 0.5$, $\varepsilon = 0.1$.

Fig. 2. Effet de k sur l'impédance, avec $\varphi = 1,5$; $\gamma = 0,5$; $\sigma = 0,5$; $\varepsilon = 0,1$.

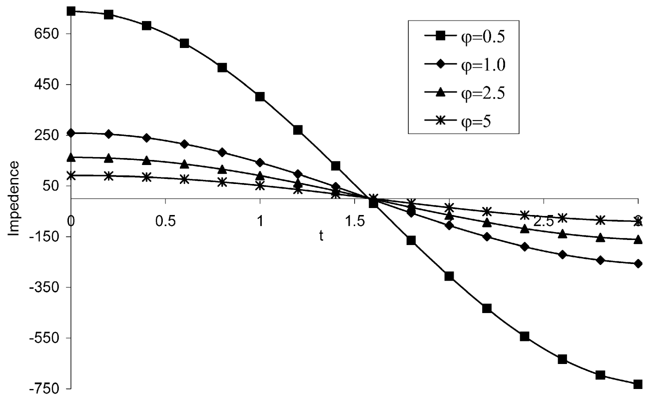

Fig. 3. Effect of φ on impedence with $k = 0.1$, $\gamma = 0.5$, $\alpha = 1.5$, $\sigma = 0.5$, $\varepsilon = 0.15$.

Fig. 3. Effet de φ sur l'impédance avec $k = 0,1$, $\gamma = 0,5$; $\alpha = 1,5$; $\sigma = 0,5$; $\varepsilon = 0,15$.

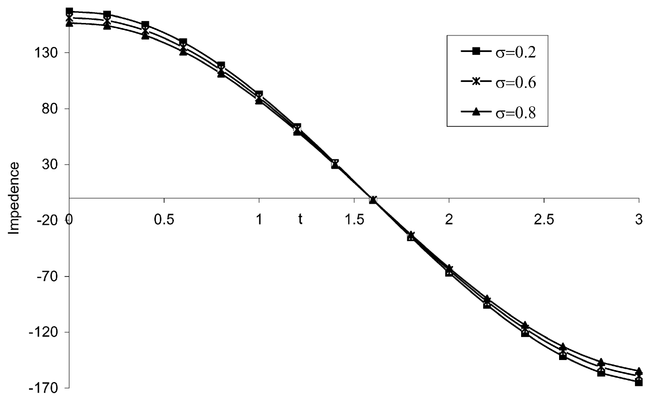

Fig. 4. Effect of σ on impedence with $k = 0.1$, $\varphi = 2.5$, $\gamma = 0.5$, $\alpha = 1.5$, $\varepsilon = 0.15$.

Fig. 4. Effet de σ sur l'impédance avec $k = 0,1$, $\varphi = 2,5$; $\gamma = 0,5$; $\alpha = 1,5$; $\varepsilon = 0,15$.

noting that as σ increases impedance also increases. The couple stresses will be absent for the case $\sigma = 1$. From this figure also it can be observed that the impedance is more for couple stress fluids.

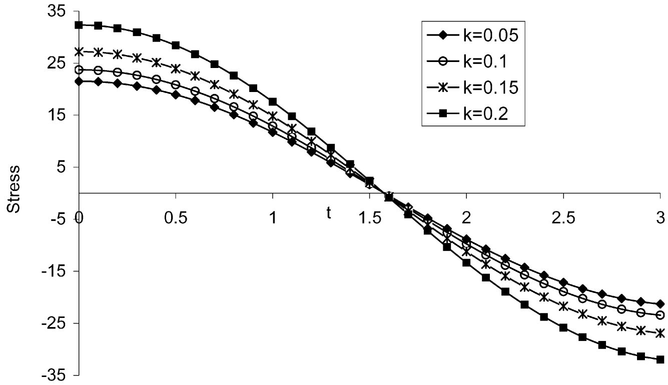

Fig. 5. Effect of k on stress with $\varphi = 2$, $\alpha = 2$, $\sigma = 0.5$, $\varepsilon = 0.1$.

Fig. 5. Effet de k sur la tension avec $\varphi = 2$; $\alpha = 2$; $\sigma = 0{,}5$; $\varepsilon = 0{,}1$.

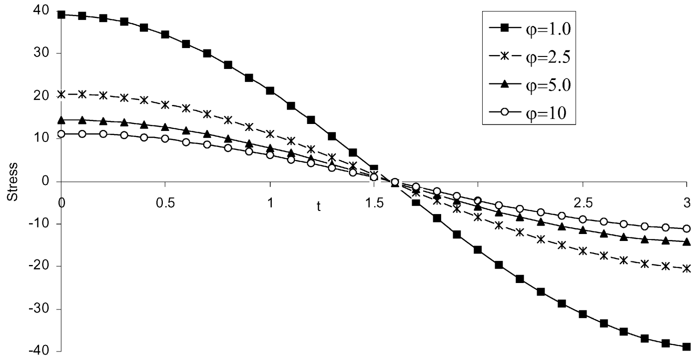

Fig. 6. Effect of φ on stress with $k = 0.1$, $\alpha = 2$, $\sigma = 0.5$, $\varepsilon = 0.1$.

Fig. 6. Effet de φ sur la tension avec $k = 0{,}1$, $\alpha = 2$; $\sigma = 0{,}5$; $\varepsilon = 0{,}1$.

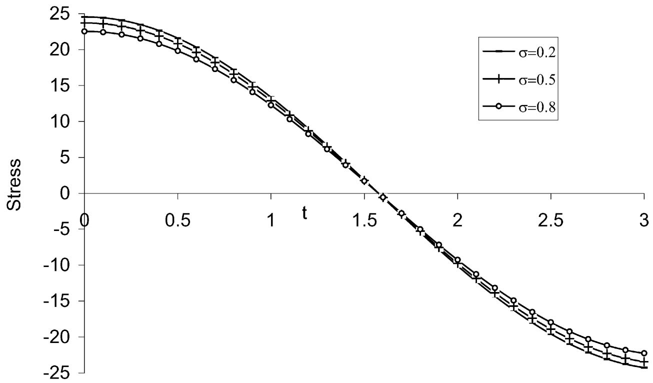

Fig. 7. Effect of σ on stress with $k = 0.1$, $\varphi = 2$, $\alpha = 2$, $\varepsilon = 0.1$.

Fig. 7. Effet de σ sur la tension avec $k = 0{,}1$, $\varphi = 2$; $\alpha = 2$; $\varepsilon = 0{,}1$.

Figs. 5–7 depict the variation of shear stress at the maximum height of the stenosis for various values of the fluid and geometric parameters. The effect of the catheter size on the shear stress with $\varphi = 2$, $\alpha = 2$, $\sigma = 0.5$, $\varepsilon = 0.1$ is shown in Fig. 5. It can be observed that in the absence of the catheter, shear stress is very low and when the catheter is introduced and its size increases from 0.05 to 2, the shear stress increases by about 50%. From Fig. 6, it is seen that as the couple stress parameter φ increases stress decreases. For the fixed values of $k = 0.1$, $\varphi = 2$, $\alpha = 2$, $\varepsilon = 0.1$ the effect of σ on stress is given in Fig. 7. It can be noted from this figure that as σ increases stress decreases.

5. Conclusions

This work presents an analytical solution for the couple stress fluid through an annulus with mild constriction at the outer wall. The resistive impedance and wall shear stress are calculated and the effects of geometric parameters and couple stress fluid parameters on the impedance and wall shear stresses are studied. The presence of catheter in a stenosed artery and the couple stresses in the fluid increases the impedance and shear stress.

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