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External estimate of the yield surface of an arbitrary ellipsoid containing a confocal void

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Abstract

This work provides an external estimate, based on limit-analysis, of the yield surface of an arbitrary (non-spheroidal) ellipsoid made of ideal-plastic von Mises material and containing a confocal ellipsoidal void, under conditions of homogeneous boundary strain rate. The upper estimate of the overall plastic dissipation is based on consideration of incompressible velocity fields satisfying conditions of homogeneous strain rate on all ellipsoids confocal with the void and the outer boundary. One establishes the existence, uniqueness and explicit expression of such a velocity field for every overall strain rate tensor imposed on this boundary. The estimate of the overall plastic dissipation obtained may be used either as a rigorous upper bound, to assess the quality of existing models for the overall behavior of porous ductile materials containing ellipsoidal voids, or as an approximation helpful in the development of new such models. ***To cite this article: J.-B. Leblond, M. Gologanu, C. R. Mecanique 336 (2008).***

Résumé

Estimation par l'extérieur de la surface de charge d'un ellipsoïde arbitraire contenant une cavité confocale. On définit, par analyse-limite, une « estimation extérieure » de la surface de charge d'un ellipsoïde arbitraire (non axisymétrique) constitué d'un matériau parfaitement plastique de von Mises et contenant un vide ellipsoïdal confocal, sous des conditions de taux de déformation homogène au bord. L'estimation par excès de la dissipation plastique globale est fondée sur la considération de champs de vitesse incompressibles et satisfaisant des conditions de taux de déformation homogène sur tous les ellipsoïdes confocaux avec le vide et le bord extérieur. On établit l'existence, l'unicité et l'expression explicite d'un tel champ pour chaque tenseur de taux de déformation global imposé sur ce bord. L'estimation de la dissipation plastique globale obtenue peut être utilisée soit comme une majoration rigoureuse, pour évaluer la qualité de modèles existants du comportement global de matériaux poreux ductiles contenant des vides ellipsoïdaux, soit comme une approximation utile dans le développement de nouveaux modèles de ce genre. ***Pour citer cet article : J.-B. Leblond, M. Gologanu, C. R. Mecanique 336 (2008).***

Keywords: Limit-analysis; External approach; Hollow ellipsoid; Ductile materials; Void shape effects

Mots-clés : Analyse-limite ; Approche par l'extérieur ; Ellipsoïde creux ; Matériaux ductiles ; Effets de forme des cavités

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1. Introduction

The first and most classical model of the overall behavior of porous plastic solids is due to Gurson [1]. This model was derived from limit-analysis of a hollow sphere loaded through conditions of homogeneous boundary strain rate. This sphere was considered to approximately represent some "elementary volume" in a porous material containing spherical voids.

However voids actually encountered in real materials are often non-spherical. This was the motivation for Gologanu et al.'s extension (named the *GLD model* hereafter) of Gurson's model to spheroidal voids [2–4]. Considering first prolate voids [2], then oblate ones [3], these authors again used limit-analysis to derive estimates of the yield surface of spheroidal domains containing such voids and confocal with them, under conditions of homogeneous boundary strain rate. The treatment was based on trial velocity fields satisfying conditions of homogeneous strain rate on every spheroid confocal with the void and the outer boundary.[1] The existence, uniqueness and explicit expression of such a field were established for every value of the overall strain rate tensor imposed on this boundary. Gologanu et al. [4] further improved the model a few years later, first by considering more trial velocity fields, second by using some results of the "nonlinear Hashin–Shtrikman (NLHS) theory" developed in recent years by Ponte-Castañeda [7], Willis [8] and Michel and Suquet [9].

Variants and extensions of the Gurson and GLD models have also been proposed by other authors, first through use of more sophisticated velocity fields (Garajeu [5]: exact elastic solution for a hollow sphere loaded through conditions of homogeneous boundary strain rate; Monchiet et al. [10]: exact elastic (Eshelby) solution for a spheroidal void in an infinite matrix), and also by considering matrices obeying Hill's anisotropic yield criterion instead of von Mises's isotropic one (Monchiet et al. [11], Keralavarma and Benzerga [12]). But more general (non-spheroidal) ellipsoidal voids have not been considered within the approach initiated by Gologanu et al., although such voids are quite common in practice, for instance in laminated plates.

However the problem of general ellipsoidal voids was attacked from another angle by Ponte-Castañeda and coworkers [13–15]. Kaisalam and Ponte-Castañeda [13] first used the NLHS theory to derive models based on the concept of linear comparison material. In principle, these models were applicable to porous plastic (and visco-plastic) materials containing arbitrary ellipsoidal voids and subjected to arbitrary loadings. However their use was hampered by the well-known inaccuracy of the NLHS upper bound for purely hydrostatic loadings. More recently, Ponte-Castañeda [14] devised a "second-order homogenization method" the results of which are much better for such loadings, although they no longer represent rigorous bounds. Using it, Danas and Ponte-Castañeda [15] very recently defined an accurate model for materials containing general ellipsoidal voids and loaded arbitrarily.

It would still be interesting, however, to develop some alternative GLD-type model for arbitrary ellipsoidal voids. Gologanu et al.'s approach indeed seems to somewhat more easily lead to explicit and basically simple[2] approximate expressions of the yield function.

This work stands as a first step in the development of such a model. It extends Gologanu et al.'s first works [2,3] by considering incompressible velocity fields satisfying conditions of homogeneous strain rate on an arbitrary family of confocal ellipsoids. One establishes the existence, uniqueness and explicit expression of such a field for every value of the overall strain rate tensor imposed on the outer boundary. This velocity field is used in a limit-analysis of an ellipsoidal domain containing a confocal ellipsoidal void and loaded through conditions of homogeneous boundary strain rate. The output is an external estimate of the yield surface of such a domain. Applications of this result are left for future work.

2. Geometric preliminaries

Let us first introduce general ellipsoidal coordinates, following Morse and Feshbach [16]. Let a, b, c denote arbitrary real numbers such that

$$a > b > c > 0 \tag{1}$$

[1] A variant using a velocity field orthogonal to each such spheroid was proposed by Garajeu et al. [5,6].

[2] In spite of the complex expressions of many of the coefficients involved.

One then defines associated ellipsoidal coordinates λ, μ, ν satisfying the inequalities

$$\lambda > -c^2 > \mu > -b^2 > \nu > -a^2 \tag{2}$$

Cartesian coordinates x, y, z are given in terms of λ, μ, ν by the following formulae:

$$\begin{cases} x = \pm\sqrt{\frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}} \\ y = \pm\sqrt{\frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)}} \\ z = \pm\sqrt{\frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}} \end{cases} \tag{3}$$

Note that these equations leave the signs of x, y, z unspecified so that there are in fact 8 possible triplets (x, y, z) for each triplet (λ, μ, ν). Eqs. (3) readily imply that

$$\begin{cases} \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \\ \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1 \\ \frac{x^2}{a^2+\nu} + \frac{y^2}{b^2+\nu} + \frac{z^2}{c^2+\nu} = 1 \end{cases} \tag{4}$$

Eqs. (4) implicitly define λ, μ, ν in terms of x, y, z through third-degree polynomial equations. They also show (account being taken of inequalities (2)) that surfaces of constant λ are confocal ellipsoids of semi-axes $\sqrt{a^2+\lambda}$, $\sqrt{b^2+\lambda}$, $\sqrt{c^2+\lambda}$, whereas surfaces of constant μ and ν are hyperboloids of one and two sheets, respectively.

Let $v(\rho)$ denote the function defined, for any real number ρ, by the formula

$$v(\rho) \equiv \sqrt{|a^2+\rho||b^2+\rho||c^2+\rho|} \tag{5}$$

Note that up to a factor of $\frac{4\pi}{3}$, $v(\lambda)$ represents the volume enclosed within the ellipsoidal surface of constant λ. (The interpretations of $v(\mu)$ and $v(\nu)$ are less straightforward.) This function is useful in various instances and notably to express the infinitesimal volume element $\mathrm{d}\Omega$:

$$\mathrm{d}\Omega = -\frac{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}{8v(\lambda)v(\mu)v(\nu)}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,\mathrm{d}\nu \tag{6}$$

The values of the derivatives $\frac{\partial\lambda}{\partial x}$, $\frac{\partial\lambda}{\partial y}$, $\frac{\partial\lambda}{\partial z}$ will be needed. They are found by differentiating equation $(4)_1$ at constant y and z, then at constant z and x, then at constant x and y. The results are as follows:

$$\begin{cases} \frac{\partial\lambda}{\partial x} = \frac{2x}{(a^2+\lambda)T} \\ \frac{\partial\lambda}{\partial y} = \frac{2y}{(b^2+\lambda)T} \\ \frac{\partial\lambda}{\partial z} = \frac{2z}{(c^2+\lambda)T} \end{cases}, \quad T \equiv \frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2} \tag{7}$$

3. Incompressible velocity fields satisfying conditions of homogeneous strain rate on confocal ellipsoids

The aim of this section is to investigate the following problem: *can one find incompressible velocity fields* $\mathbf{v}(\mathbf{r})$ *satisfying conditions of homogeneous strain rate on every ellipsoidal surface of constant* λ*?*

The velocity fields looked for must be of the form

$$\mathbf{v}(\mathbf{r}) = \mathbf{D}(\lambda)\cdot\mathbf{r} \quad \Leftrightarrow \quad \begin{cases} v_x(\mathbf{r}) = D_{xx}(\lambda)x + D_{xy}(\lambda)y + D_{xz}(\lambda)z \\ v_y(\mathbf{r}) = D_{yx}(\lambda)x + D_{yy}(\lambda)y + D_{yz}(\lambda)z \\ v_z(\mathbf{r}) = D_{zx}(\lambda)x + D_{zy}(\lambda)y + D_{zz}(\lambda)z \end{cases} \tag{8}$$

where $\mathbf{D}(\lambda)$ is a symmetric second-rank tensor depending on λ. Calculation of the divergence of such a field yields, upon use of Eqs. (7):

$$
\begin{aligned}
\operatorname{div}\mathbf{v}(\mathbf{r}) &= D_{xx}(\lambda)+D_{yy}(\lambda)+D_{zz}(\lambda)\\
&\quad+\frac{\mathrm{d}D_{xx}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial x}x+\frac{\mathrm{d}D_{xy}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial x}y+\frac{\mathrm{d}D_{xz}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial x}z\\
&\quad+\frac{\mathrm{d}D_{yx}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial y}x+\frac{\mathrm{d}D_{yy}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial y}y+\frac{\mathrm{d}D_{yz}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial y}z\\
&\quad+\frac{\mathrm{d}D_{zx}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial z}x+\frac{\mathrm{d}D_{zy}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial z}y+\frac{\mathrm{d}D_{zz}}{\mathrm{d}\lambda}(\lambda)\frac{\partial\lambda}{\partial z}z\\
&=\operatorname{tr}\mathbf{D}(\lambda)+\frac{2}{T}\left[\frac{\mathrm{d}D_{xx}}{\mathrm{d}\lambda}(\lambda)\frac{x^2}{a^2+\lambda}+\frac{\mathrm{d}D_{yy}}{\mathrm{d}\lambda}(\lambda)\frac{y^2}{b^2+\lambda}+\frac{\mathrm{d}D_{zz}}{\mathrm{d}\lambda}(\lambda)\frac{z^2}{c^2+\lambda}\right.\\
&\quad+\frac{\mathrm{d}D_{xy}}{\mathrm{d}\lambda}(\lambda)\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\right)xy+\frac{\mathrm{d}D_{yz}}{\mathrm{d}\lambda}(\lambda)\left(\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)yz\\
&\quad\left.+\frac{\mathrm{d}D_{zx}}{\mathrm{d}\lambda}(\lambda)\left(\frac{1}{c^2+\lambda}+\frac{1}{a^2+\lambda}\right)zx\right]
\end{aligned}
$$

Writing then the incompressibility condition in the form $T\operatorname{div}\mathbf{v}(\mathbf{r})=0$, accounting for the expression $(7)_4$ of T and reordering terms, one gets the condition

$$
\begin{aligned}
&\left[\frac{2}{a^2+\lambda}\frac{\mathrm{d}D_{xx}}{\mathrm{d}\lambda}(\lambda)+\frac{\operatorname{tr}\mathbf{D}(\lambda)}{(a^2+\lambda)^2}\right]x^2+\left[\frac{2}{b^2+\lambda}\frac{\mathrm{d}D_{yy}}{\mathrm{d}\lambda}(\lambda)+\frac{\operatorname{tr}\mathbf{D}(\lambda)}{(b^2+\lambda)^2}\right]y^2\\
&\quad+\left[\frac{2}{c^2+\lambda}\frac{\mathrm{d}D_{zz}}{\mathrm{d}\lambda}(\lambda)+\frac{\operatorname{tr}\mathbf{D}(\lambda)}{(c^2+\lambda)^2}\right]z^2+2\frac{\mathrm{d}D_{xy}}{\mathrm{d}\lambda}(\lambda)\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\right)xy\\
&\quad+2\frac{\mathrm{d}D_{yz}}{\mathrm{d}\lambda}(\lambda)\left(\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)yz+2\frac{\mathrm{d}D_{zx}}{\mathrm{d}\lambda}(\lambda)\left(\frac{1}{c^2+\lambda}+\frac{1}{a^2+\lambda}\right)zx=0
\end{aligned}
\tag{9}
$$

In this equation x, y, z are arbitrary but tied to λ through Eq. $(4)_1$. However it is obvious that if condition (9) is satisfied for a given λ and a given triplet (x,y,z), it is also satisfied for the same λ and the triplet (kx,ky,kz) where k is an arbitrary number. Hence it must in fact be satisfied for *all* triplets (x,y,z), irrespective of whether Eq. $(4)_1$ holds or not. This means that the coefficients of the polynomial in x, y, z in the left-hand side must be zero:

$$
\begin{cases}
2\frac{\mathrm{d}D_{xx}}{\mathrm{d}\lambda}(\lambda)+\frac{\operatorname{tr}\mathbf{D}(\lambda)}{a^2+\lambda}=0\\
2\frac{\mathrm{d}D_{yy}}{\mathrm{d}\lambda}(\lambda)+\frac{\operatorname{tr}\mathbf{D}(\lambda)}{b^2+\lambda}=0;\\
2\frac{\mathrm{d}D_{zz}}{\mathrm{d}\lambda}(\lambda)+\frac{\operatorname{tr}\mathbf{D}(\lambda)}{c^2+\lambda}=0
\end{cases}
\qquad
\begin{cases}
\frac{\mathrm{d}D_{xy}}{\mathrm{d}\lambda}(\lambda)=0\\
\frac{\mathrm{d}D_{yz}}{\mathrm{d}\lambda}(\lambda)=0\\
\frac{\mathrm{d}D_{zx}}{\mathrm{d}\lambda}(\lambda)=0
\end{cases}
\tag{10}
$$

The solution of Eqs. $(10)_4$–$(10)_6$ is trivial:

$$
\begin{cases}
D_{xy}(\lambda)=\Delta_{xy}\equiv Cst\\
D_{yz}(\lambda)=\Delta_{yz}\equiv Cst\\
D_{zx}(\lambda)=\Delta_{zx}\equiv Cst
\end{cases}
\tag{11}
$$

To solve Eqs. $(10)_1$–$(10)_3$, take their sum to get the following differential equation on $\operatorname{tr}\mathbf{D}(\lambda)$:

$$
2\frac{\mathrm{d}(\operatorname{tr}\mathbf{D})}{\mathrm{d}\lambda}(\lambda)+\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)\operatorname{tr}\mathbf{D}(\lambda)=0
\tag{12}
$$

and integrate it into

$$
\operatorname{tr}\mathbf{D}(\lambda)=\frac{A}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}=\frac{A}{v(\lambda)}
\tag{13}
$$

where A is an arbitrary constant. Then reinsert this expression into Eqs. $(10)_1$–$(10)_3$ and integrate to get the following expressions of $D_{xx}(\lambda)$, $D_{yy}(\lambda)$, $D_{zz}(\lambda)$:

$$
\begin{cases}
D_{xx}(\lambda)=A\int_\lambda^{+\infty}\frac{\mathrm{d}\rho}{2(a^2+\rho)v(\rho)}+\Delta_{xx}\\
D_{yy}(\lambda)=A\int_\lambda^{+\infty}\frac{\mathrm{d}\rho}{2(b^2+\rho)v(\rho)}+\Delta_{yy}\\
D_{zz}(\lambda)=A\int_\lambda^{+\infty}\frac{\mathrm{d}\rho}{2(c^2+\rho)v(\rho)}+\Delta_{zz}
\end{cases}
\tag{14}
$$

In these expressions Δ_{xx}, Δ_{yy}, Δ_{zz} are *a priori* arbitrary constants. However the expression of $\operatorname{tr}\mathbf{D}(\lambda)$ deduced from there,

$$\begin{aligned}\operatorname{tr}\mathbf{D}(\lambda) &= A\int_{\lambda}^{+\infty}\left(\frac{1}{a^2+\rho}+\frac{1}{b^2+\rho}+\frac{1}{b^2+\rho}\right)\frac{\mathrm{d}\rho}{2v(\rho)}+\Delta_{xx}+\Delta_{yy}+\Delta_{zz}\\ &= -A\int_{\lambda}^{+\infty}\frac{\mathrm{d}(1/v)}{\mathrm{d}\rho}(\rho)\,\mathrm{d}\rho+\Delta_{xx}+\Delta_{yy}+\Delta_{zz}\\ &= \frac{A}{v(\lambda)}+\Delta_{xx}+\Delta_{yy}+\Delta_{zz}\end{aligned}$$

must be identical to that provided by Eq. (13). Thus the condition

$$\Delta_{xx}+\Delta_{yy}+\Delta_{zz}=0 \tag{15}$$

must be fulfilled, implying that the symmetric second-rank tensor $\mathbf{\Delta}$ of components Δ_{xx}, Δ_{yy}, Δ_{zz}, Δ_{xy}, Δ_{yz}, Δ_{zx} must be traceless.

In conclusion, the solution of the problem (provided by Eqs. (11), (14) and (15)) can be summarized as follows: *the velocity fields looked for are of the form* (8)*, where* $\mathbf{D}(\lambda)$ *is itself of the form*

$$\mathbf{D}(\lambda)=A\mathbf{D}^0(\lambda)+\mathbf{\Delta} \tag{16}$$

A being an arbitrary constant, $\mathbf{\Delta}$ *an arbitrary traceless symmetric second-rank tensor independent of* λ*, and* $\mathbf{D}^0(\lambda)$ *the symmetric second-rank tensor depending on* λ *given by*

$$\begin{cases} D^0_{xx}(\lambda)=\int_{\lambda}^{+\infty}\frac{\mathrm{d}\rho}{2(a^2+\rho)v(\rho)}\\ D^0_{yy}(\lambda)=\int_{\lambda}^{+\infty}\frac{\mathrm{d}\rho}{2(b^2+\rho)v(\rho)}\\ D^0_{zz}(\lambda)=\int_{\lambda}^{+\infty}\frac{\mathrm{d}\rho}{2(c^2+\rho)v(\rho)}\\ D^0_{xy}(\lambda)=D^0_{yz}(\lambda)=D^0_{zx}(\lambda)=0\end{cases} \tag{17}$$

The integrals defining the diagonal components of $\mathbf{D}^0(\lambda)$ are of elliptic type, but the trace of this tensor has a simple expression:

$$\operatorname{tr}\mathbf{D}^0(\lambda)=\frac{1}{v(\lambda)} \tag{18}$$

Also, note that up to some arbitrary multiplicative constant, the velocity field $\mathbf{v}^0(\mathbf{r})$ corresponding to the tensor $\mathbf{D}^0(\lambda)$ is the only one of the type considered which vanishes at infinity.

4. Upper estimate of the plastic dissipation of a hollow ellipsoid

Consider now an arbitrary ellipsoid of volume $\frac{4\pi}{3}\Omega$, made of some ideal-plastic von Mises material with yield stress σ_0 in simple tension, and containing a confocal ellipsoidal void of semi-axes a, b, c and volume $\frac{4\pi}{3}\omega$ ($\omega=abc$). Use the ellipsoidal coordinates λ, μ, ν associated to a, b, c. Then the value of λ on the inner ellipsoid is 0, that on the outer one is Λ, the values of $v(0)$ and $v(\Lambda)$ are ω and Ω respectively, and Λ is determined by the following third-degree polynomial equation obtained by equating the ratio ω/Ω to the given porosity (void volume fraction) f:

$$\frac{\omega}{\Omega}=\frac{v(0)}{v(\Lambda)}=f\quad\Rightarrow\quad(a^2+\Lambda)(b^2+\Lambda)(c^2+\Lambda)-\frac{a^2b^2c^2}{f^2}=0 \tag{19}$$

Now load this ellipsoid through conditions of homogeneous boundary strain rate. Then there is a unique incompressible velocity field of the type studied in Section 3 compatible with the value of the overall strain rate tensor $\mathbf{D}$ imposed on the outer boundary; that is, the equation

$$A\mathbf{D}^0(\Lambda)+\mathbf{\Delta}=\mathbf{D} \tag{20}$$

has a unique solution. Indeed taking the trace of both sides, one gets, since $\mathbf{\Delta}$ is traceless:

$$A\,\mathrm{tr}\,\mathbf{D}^0(\Lambda) = \frac{A}{v(\Lambda)} = \frac{A}{\Omega} = \mathrm{tr}\,\mathbf{D} \quad \Rightarrow \quad A = \Omega\,\mathrm{tr}\,\mathbf{D} \tag{21}$$

and then

$$\mathbf{\Delta} = \mathbf{D} - A\mathbf{D}^0(\Lambda) = \mathbf{D} - \Omega(\mathrm{tr}\,\mathbf{D})\mathbf{D}^0(\Lambda) \tag{22}$$

One may use the thus unambiguously defined velocity field in a limit-analysis of the ellipsoidal domain considered. The components of the associated strain rate $\mathbf{d}(\mathbf{r})$ are needed for this purpose. These components are easily deduced from the elements derived so far:

$$\begin{cases} d_{xx}(\mathbf{r}) = A\big(D^0_{xx}(\lambda) - \frac{x^2}{(a^2+\lambda)^2 T v(\lambda)}\big) + \Delta_{xx} \\ d_{yy}(\mathbf{r}) = A\big(D^0_{yy}(\lambda) - \frac{y^2}{(b^2+\lambda)^2 T v(\lambda)}\big) + \Delta_{yy} \\ d_{zz}(\mathbf{r}) = A\big(D^0_{zz}(\lambda) - \frac{z^2}{(c^2+\lambda)^2 T v(\lambda)}\big) + \Delta_{zz} \\ d_{xy}(\mathbf{r}) = -A\frac{xy}{(a^2+\lambda)(b^2+\lambda)T v(\lambda)} + \Delta_{xy} \\ d_{yz}(\mathbf{r}) = -A\frac{yz}{(b^2+\lambda)(c^2+\lambda)T v(\lambda)} + \Delta_{yz} \\ d_{zx}(\mathbf{r}) = -A\frac{zx}{(c^2+\lambda)(a^2+\lambda)T v(\lambda)} + \Delta_{zx}, \end{cases} \tag{23}$$

and the equivalent strain rate is then given by

$$d_{\mathrm{eq}}(\mathbf{r}) = \sqrt{\frac{2}{3}\big[d^2_{xx}(\mathbf{r}) + d^2_{yy}(\mathbf{r}) + d^2_{zz}(\mathbf{r}) + 2d^2_{xy}(\mathbf{r}) + 2d^2_{yz}(\mathbf{r}) + 2d^2_{zx}(\mathbf{r})\big]} \tag{24}$$

The upper estimate $\Pi^+(\mathbf{D})$ of the overall plastic dissipation is identical to the average value of the local plastic dissipation $\sigma_0 d_{\mathrm{eq}}(\mathbf{r})$ corresponding to the trial velocity field considered over the ellipsoidal domain; that is,

$$\Pi^+(\mathbf{D}) \equiv \frac{\sigma_0}{\frac{4\pi}{3}\Omega} \int_{\lambda=0}^{\lambda=\Lambda} \int_{\mu=-b^2}^{\mu=-c^2} \int_{\nu=-a^2}^{\nu=-b^2} \left(\sum_{i=1}^{8} d^{(i)}_{\mathrm{eq}}(\lambda,\mu,\nu)\right) \mathrm{d}\Omega \tag{25}$$

where $\mathrm{d}\Omega$ is given by Eq. (6). The $d^{(i)}_{\mathrm{eq}}(\lambda,\mu,\nu)$ $(i = 1\ldots 8)$ here denote the values of $d_{\mathrm{eq}}(\mathbf{r})$ for the 8 triplets (x, y, z) corresponding to the triplet (λ,μ,ν).

Once the function $\Pi^+(\mathbf{D})$ is known, the external estimate of the overall yield surface of the ellipsoid is given by the equation

$$\mathbf{\Sigma} = \frac{\partial \Pi^+}{\partial \mathbf{D}}(\mathbf{D}) \tag{26}$$

where the components of $\mathbf{D}$ act as parameters. (This equation defines a 5-dimensional surface in the 6-dimensional space of overall stresses $\mathbf{\Sigma}$ since the 6 components of $\mathbf{\Sigma}$ depend only on the ratios of 5 components of $\mathbf{D}$ to the last one, the function $\frac{\partial \Pi^+}{\partial \mathbf{D}}(\mathbf{D})$ being positively homogeneous of degree 0.)

5. Perspectives

Although the shape of the domain considered (ellipsoidal and confocal with the void) is somewhat arbitrary, it is commonly considered as an acceptable approximation of the actual shape of some representative volume element in a porous material.[3] The conditions of homogeneous boundary strain rate imposed on it are also commonly accepted as representative, at least prior to the onset of strain localization phenomena. It is therefore reasonable to use the external

[3] Danas and Ponte-Castañeda [15] have argued on theoretical grounds that for small porosities, as usually encountered, elementary cells of *any* shape are in fact acceptable, provided that they respect the given value of the porosity and the shape of the voids.

estimate of the yield surface just derived as an aid in the definition of models for plastic porous materials containing ellipsoidal voids.

This may be done in two ways. One possibility is to accept the estimate of the overall plastic dissipation $\Pi^+(\mathbf{D})$ as it stands, without any simplification, calculate it numerically for various values of the porosity and the two aspect ratios of the void, and use the yield surface thus determined as a reference to assess the validity of existing models, such as that of Danas and Ponte-Castañeda [15]. Since $\Pi^+(\mathbf{D})$ represents a rigorous upper bound of the true overall plastic dissipation, the yield surface it defines must be exterior to that proposed by any acceptable model.

In a more ambitious perspective, one may accept $\Pi^+(\mathbf{D})$ as a good approximation of the true overall plastic dissipation, and look for simplifications of its expression allowing to calculate it analytically, so as to derive an explicit approximate expression of the yield surface. Doing this will mean extending Gologanu et al.'s first works [2,3] on spheroidal voids to general ellipsoidal voids. Because of the rusticity of the trial velocity field used, it will be necessary in a second step to similarly extend Gologanu et al.'s subsequent work [4], which used other velocity fields plus some results of the NLHS theory to improve the expressions of some coefficients.

In the search of this extended model, approximate formulae avoiding the appearance of elliptic integrals (like in the preceding formulae) as far as possible will have to be looked for, in order to facilitate its use.

Note added in proof

Since this Note was written, an alternative method of construction of the velocity field defined in Section 3 was suggested to the author by Djimédo Kondo. This method consists of adding to Eshelby's solution for an ellipsoidal stress-free void in an infinite elastic matrix, that generated by a uniform "free strain" imposed in the void and adjusted so as to satisfy conditions of homogeneous strain on the outer boundary. In essence, this idea was suggested in Chapters 5 and 6 of Monchiet's [17] thesis, in the case of spheroidal voids.

References

[1] A.L. Gurson, Continuum theory of ductile rupture by void nucleation and growth: Part I – Yield criteria and flow rules for porous ductile media, ASME J. Engrg. Materials Technol. 99 (1977) 2–15.

[2] M. Gologanu, J.B. Leblond, J. Devaux, Approximate models for ductile metals containing non-spherical voids – Case of axisymmetric prolate ellipsoidal cavities, J. Mech. Phys. Solids 41 (1993) 1723–1754.

[3] M. Gologanu, J.B. Leblond, J. Devaux, Approximate models for ductile metals containing non-spherical voids – Case of axisymmetric oblate ellipsoidal cavities, ASME J. Engrg. Materials Technol. 116 (1994) 290–297.

[4] M. Gologanu, J.B. Leblond, G. Perrin, J. Devaux, Recent extensions of Gurson's model for porous ductile metals, in: P. Suquet (Ed.), Continuum Micromechanics, Springer-Verlag, New York, 1997, pp. 61–130.

[5] M. Garajeu, Contribution à l'étude du comportement non linéaire de milieux poreux avec ou sans renfort, PhD thesis, Université de la Méditerranée, Marseille, France, 1995.

[6] M. Garajeu, J.C. Michel, P. Suquet, A micromechanical approach of damage in viscoplastic materials by evolution in size, shape and distribution of voids, Comput. Meth. Appl. Mech. Engrg. 183 (2000) 223–246.

[7] P. Ponte Castañeda, The effective mechanical properties of nonlinear isotropic materials, J. Mech. Phys. Solids 39 (1991) 45–71.

[8] J. Willis, On methods for bounding the overall properties of nonlinear composites, J. Mech. Phys. Solids 39 (1991) 73–86.

[9] J.C. Michel, P. Suquet, The constitutive law of nonlinear viscous and porous materials, J. Mech. Phys. Solids 40 (1992) 783–812.

[10] V. Monchiet, E. Charkaluk, D. Kondo, An improvement of Gurson-type models of porous materials by using Eshelby-like trial velocity fields, C. R. Mecanique 335 (2007) 32–41.

[11] V. Monchiet, O. Cazacu, E. Charkaluk, D. Kondo, Macroscopic yield criteria for plastic anisotropic materials containing spheroidal voids, Int. J. Plasticity 24 (2008) 1158–1189.

[12] S.M. Keralavarma, A. Benzerga, An approximate yield criterion for anisotropic porous media, C. R. Mecanique 336 (2008) 685–692.

[13] M. Kaisalam, P. Ponte-Castañeda, A general constitutive theory for linear and nonlinear particulate media with microstructure evolution, J. Mech. Phys. Solids 46 (1998) 427–465.

[14] P. Ponte-Castañeda, Second-order homogenization estimates for nonlinear composites incorporating field fluctuations, I. Theory, J. Mech. Phys. Solids 50 (2002) 737–757.

[15] K. Danas, P. Ponte-Castañeda, A finite-strain model for anisotropic viscoplastic porous media: I – Theory; II – Applications, Eur. J. Mech. A/Solids, in press.

[16] P.M. Morse, H. Feshbach, Methods of Theoretical Physics, Part I, McGraw-Hill, New York, 1953.

[17] V. Monchiet, Contribution à la modélisation micromécanique de l'endommagement et de la fatigue des métaux ductiles, PhD thesis, Université de Lille, France, 2006.